Part VII

Propagation of Finite-Width Laser Pulses: Relativistic Self-Focussing and Channelling

3D propagation effects

Have so far assumed laser to act like ‘photon bullet’
Real laser pulses are created with focusing optics & are subject to:

- diffraction due to finite σ_L:

$$Z_R = k\sigma_L^2/\lambda$$

- refraction due to density gradients
- refraction due to self-nonlinearity

Nonlinear refraction effects

- Ionizing plasma: radial density gradients
- Uniform, fully ionized plasma:
  1. relativistic self-focusing. Power threshold:

$$P_c \simeq 17 \left(\frac{\omega_0}{\omega_p}\right)^2 \text{ GW}, \quad (107)$$

  2. ponderomotive channelling
- Effects important for $P_L > 2TW$

Relativistic self-focussing: Geometric optics

Consider laser beam with a radial profile

$$a(r) = a_0 \exp(-r^2/2\sigma_0^2),$$

spot size σ_0 just inside a region of uniform, underdense plasma, see Fig. 10.

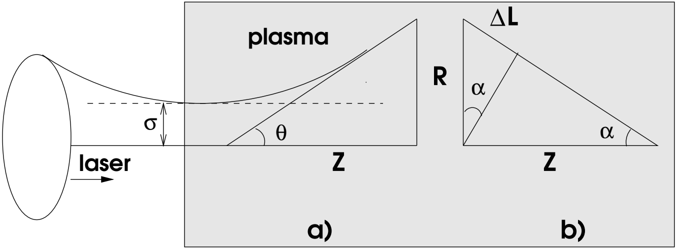


Figure: a) diffraction, b) self-focusing

Geometric optics picture: vacuum behaviour

In the absence of nonlinear effects, the beam will diffract with a divergence angle (**?**)

$$\theta_d = \frac{dR}{dZ} = \frac{\sigma_0}{Z_R} = \frac{1}{k\sigma_0}, \qquad (108)$$

where Z_R is the Rayleigh length (half the confocal parameter), defined here by:

$$Z_R = \frac{2\pi\sigma_0^2}{\lambda}. \qquad (109)$$

Geometric optics picture: high intensities

Dispersion relation of EM wave is altered due to the effective relativistic mass increase of the electrons:

$$\omega^2 = c^2 k^2 + \omega_p'^2, \tag{110}$$

where $\omega_p'^2 = \omega_p^2/\gamma_0$ is the effective plasma frequency for the EM wave.

$$\gamma_0 = (1 + a_0^2)^{1/2}$$

. Pump amplitude

$$a_0 = \frac{eE_L}{m_e \omega c} = 0.85\sqrt{I_{18}\lambda_\mu^2}$$

Geometric optics picture: nonlinear refractive index

The corresponding refractive index is:

$$\eta(r) \equiv \frac{ck}{\omega} = \left\{ 1 - \frac{\omega_p^2}{\omega^2 \left[1 + a(r)^2/2\right]^{1/2}} \right\}^{1/2} . \qquad (111)$$

For a beam with a profile $a(r)$ as above, $\eta(r)$ is *peaked* on axis ($d\eta/dr < 0$), which in optics terminology represents a 'positive', or focusing, lens.
Contrast with *divergent* refractive index ($d\eta/dr > 0$) created by field ionization:

$$\eta(r) \simeq 1 - \frac{1}{2}\frac{n_e(r)}{n_c}.$$

Geometric optics picture: phase fronts I

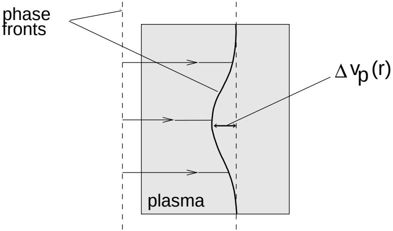
Phase velocity of the wave fronts:

$$\frac{v_p(r)}{c} = \frac{1}{\eta} \simeq 1 + \frac{\omega_p^2}{2\omega^2}\left(1 - \frac{a^2(r)}{4}\right). \tag{112}$$

Looking across the beam profile (see Fig. 6), the phase fronts will travel more slowly at the center than at the edge, giving a velocity difference:

$$\frac{\Delta v_p(r)}{c} = \frac{\omega_p^2}{8\omega^2} a_0^2 e^{-r^2/\sigma_0^2}.$$

Geometric optics picture: phase fronts II



Maximum path difference:

$$\begin{aligned}\Delta L &= |\Delta v_p|_{\max} t = \left|\frac{\Delta v_p}{c}\right|_{\max} Z \\ &= \alpha R = \alpha^2 Z\end{aligned}$$

Geometric optics picture: refraction angle

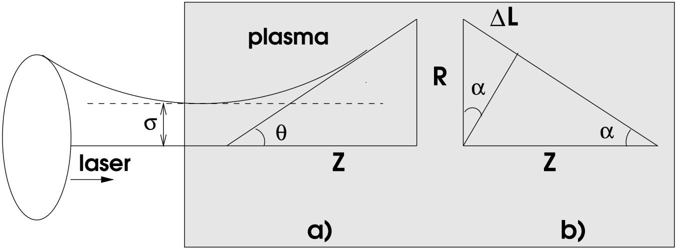


Figure: a) diffraction, b) self-focusing

Maximum focusing angle of the beam is given by

$$\alpha^2 = \left|\frac{\Delta v_p}{c}\right|_{\max} = \frac{\omega_p^2 a_0^2}{8\omega^2}. \qquad (113)$$

Geometric optics picture: power threshold

Beam spreading due to diffraction will be cancelled by self-focusing effects if $\theta = \alpha$, or setting Eq. (108) and Eq. (113) equal,

$$a_0^2 \left(\frac{\omega_p \sigma_0}{c} \right)^2 \geq 8. \tag{114}$$

This represents a *power* threshold, since the laser power $P_L \propto a_0^2 \sigma_0^2$. In numbers:

$$P_L > 9 \left(\frac{\omega}{\omega_p} \right)^2 \text{ GW}$$

– not accurate ($\sim \times 2$ too low) because we didn't take beam profile into account.

Nonlinear Schrödinger equation

In general, laser pulse envelope varies slowly compared to the laser (or plasma) frequency – *envelope approximation* often justified. Start from EM wave equation for dimensionless vector potential A:

$$\frac{\partial^2 A}{\partial t^2} - c^2 \nabla^2 A = -\omega_p^2 \frac{nA}{\gamma}. \tag{115}$$

n is electron density; γ relativistic 'quiver' factor. Now separate rapid variations in the laser field phase, $\psi = \omega t - kz$, from its amplitude $a(r, z, t)$, by applying the *slowly-varying envelope approximation* (SVEA):

$$A = \frac{1}{2}\left(a e^{i\psi} + a^* e^{-i\psi}\right). \tag{116}$$

SVEA: space & time derivatives

The time- and space-derivatives are easily obtained by observing that $\partial\psi/\partial t = i\omega$ and $\partial\psi/\partial z = -ik$, so that:

$$
\begin{aligned}
\frac{\partial A}{\partial t} &= i\omega\frac{1}{2}ae^{i\psi} + \frac{1}{2}\frac{\partial a}{\partial t}e^{i\psi} + c.c. \\
\frac{\partial^2 A}{\partial t^2} &= \left(-\omega^2\frac{a}{2} + i\omega\frac{\partial a}{\partial t} + \frac{1}{2}\frac{\partial^2 a}{\partial t^2}\right)e^{i\psi} + c.c. \\
\frac{\partial^2 A}{\partial z^2} &= \left(-k^2\frac{a}{2} - ik\frac{\partial a}{\partial z} + \frac{1}{2}\frac{\partial^2 a}{\partial z^2}\right)e^{i\psi} + c.c. \\
\nabla_\perp^2 A &= \frac{1}{2}\nabla_\perp^2 ae^{i\psi} + c.c.
\end{aligned}
$$

SVEA: wave equation

Simplifications:

- Neglect 2nd derivatives of the envelope amplitude a
- Apply linear dispersion relation $\omega^2 = \omega_p^2 + c^2 k^2$
- Transforming time- and space-variables to a window moving with the group velocity of the pulse (cf wakefield analysis):
  $v_g = c^2 k/\omega \quad \tau = t, \quad \xi = z - v_g t$

$$i\omega \frac{\partial a}{\partial \tau} = c^2 \nabla_\perp^2 \frac{a}{2} + \omega_p^2 (1 - \frac{n}{\gamma}) \frac{a}{2}. \tag{117}$$

Paraxial (envelope) equation – tidy form

Wave equation can be simplified by using the normalizations:

$$\tilde{\tau} = \frac{\omega_p^2 \tau}{\omega} \quad \text{and} \quad \tilde{r} = k_p r,$$

The envelope (or paraxial) equation in dimensionless form:

$$\frac{\partial a}{\partial \tau} + \frac{i}{2}\nabla_\perp^2 a + \frac{i}{2}\left(1 - \frac{n}{\gamma}\right) a = 0. \tag{118}$$

Nonlinear Schrödinger equation (NLSE)

For weakly nonlinear, circularly polarized pump, we can expand the relativistic factor:

$$\gamma^{-1} \simeq 1 - \frac{|\, a \,|^2}{4}.$$

Paraxial equation then becomes: **(n=1)**

$$\frac{\partial a}{\partial t} = -\frac{i}{2}\nabla_\perp^2 a - \frac{i}{8} \,|\, a \,|^2\, a. \tag{119}$$

The NLSE exhibits a number of *conservation properties*, which can be obtained by taking spatial moments. For example, multiplying both sides by a^* and adding the complex conjugate yields:

$$\frac{\partial}{\partial t} \mid a \mid^2 = \frac{i}{2} a \nabla_\perp^2 a^* - \frac{i}{2} a^* \nabla_\perp^2 a.$$

Integrating over the focal area of the pulse in cylindrical geometry, we find:

$$\int_0^\infty \frac{\partial}{\partial t} \mid a \mid^2 \, d^2 r dr = 0.$$

Identify

$$P \equiv \int \mid a \mid^2 \, d^2 r \tag{120}$$

with the wave action, photon number, or normalized beam power –*conserved*.

Beam 'Hamiltonian'

Other conservation relations can be similarly derived, eg:
Hamiltonian:

$$\mathfrak{H} \equiv \frac{1}{2} \mid \nabla_\perp a \mid^2 - \frac{1}{16} \mid a \mid^4, \qquad (121)$$

which obeys:

$$\frac{\partial}{\partial t} \int_0^\infty \mathfrak{H} d^2 r \equiv \frac{\partial H}{\partial t} = 0,$$

To focus or not to focus: RMS beam width

To determine whether a beam will focus, we take the r^2 moment of Eq. (119) – radial beam 'acceleration'

$$< \delta r^2 > \equiv < r^2 > - < r >^2 .$$

so that

$$\begin{aligned} \frac{\partial^2}{\partial t^2} < \delta r^2 > &= \frac{\partial^2}{\partial t^2} < r^2 > \\ &= \frac{1}{P} \int_0^\infty r^2 \frac{\partial \mid a \mid^2}{\partial t^2} d^2 r \\ &= \frac{4H}{P}. \end{aligned} \tag{122}$$

Beam equation

P and H are consvered, so integrating Eq. (122) twice then gives:

$$< r^2 >= \frac{2H}{P} t^2 + Ct + D, \tag{123}$$

where C and D are constants which depend on the focusing optics and the initial spot size respectively.

$$P = \int |a|^2 d^2 r, \quad H = \int \left(\frac{1}{2} |\nabla_\perp a|^2 - \frac{1}{16} |a|^4 \right) d^2 r.$$

Assumption: beam does not deviate from the axis: $\partial < r > / \partial t = 0$ – no *hosing*

Relativistic focussing threshold

Assume wavefronts are initially parallel ($C = 0$), which will be approximately the case for a Gaussian beam focused to its diffraction limit just inside a region of plasma, Fig. 11.

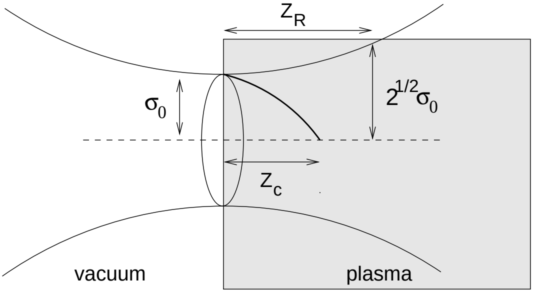


Figure: Geometry for relativistic whole-beam focusing.

Gaussian optics

For non-relativistic pump strengths at the focus $a \ll 1$, we can set

$$\mathfrak{H} = \frac{1}{2} \mid \nabla a \mid^2$$

Assume Gaussian beam profile $a(r) = a_0 \exp(-r^2/2\sigma_0^2)$, where σ_0 is the nominal vacuum focal spot size.
Beam power and global Hamiltonian simplify to:

$$P = \pi a_0^2 \sigma_0^2; \quad H = \frac{\pi a_0^2}{2}, \tag{124}$$

Hence:

$$\begin{aligned} < r^2 > (t=0) &= P^{-1} \int r^2 \mid a \mid^2 d^2 r \\ &= \sigma_0^2. \end{aligned}$$

Beam diffraction in vacuum

Beam width simplifies to

$$< r^2 >= \sigma_0^2 + \frac{t^2}{\sigma_0^2}. \qquad (125)$$

Substitute back physical units: $t \to \omega_p^2/\omega_0 t$; $r \to \omega_p r/c$ and write $z = ct$:

$$\sigma(z) \equiv \sqrt{< r(z)^2 >} = \sigma_0 \left(1 + \frac{z^2}{z_R^2}\right)^{1/2}, \qquad (126)$$

where $z_R = k\sigma_0^2$ is the Rayleigh length.
NB: beam waist sometimes defined as $r_0 = \sqrt{2}\sigma_0$, so that $z_R = kr_0^2/2 = \pi r_0^2/\lambda$. Rayleigh length is the distance at which the focal area $\pi\sigma^2$ doubles in size.

Focussing threshold

So how big does the pump strength have to be before diffraction is balanced by focussing?
In terms of the beam-moment analysis here, this is equivalent to requiring the global Hamiltonian to vanish; i.e. $H = 0$. For a Gaussian beam, we have (relativistic focusing balance the diffraction)

$$H = \frac{\pi a_0^2}{2}\left(1 - \frac{a_0^2\sigma_0^2}{16}\right) = 0,$$

which is satisfied for $a_0^2\sigma_0^2 = 16$.
Critical power in normalized units:

$$P_c = 16\pi. \tag{127}$$

Beam power – normalized & practical units

Recall the expression for the beam power P_L in watts:

$$\begin{aligned} P_L &= \int_S I d^2 r \\ &= \pi c \epsilon_0 \int_0^\infty E^2(r) r dr. \end{aligned}$$

The electric field E is related to the pump strength $a(r)$ by

$$E(r) = \frac{m\omega c}{e} a(r),$$

so we can write P_L in terms of the normalized power

$$\tilde{P} = \int_0^\infty a^2(r) d^2 r,$$

Focussing threshold – practical units

Litvak, 1970; Max *et al.*1974, Sprangle *et al.*1988

Relation between laser power and critical power:

$$
\begin{aligned}
P_L &= \left(\frac{m\omega c}{e}\right)^2 \left(\frac{c}{\omega_p}\right)^2 \frac{c\epsilon_0}{2} \int_0^\infty 2\pi r a^2(r) dr \\
&= \frac{1}{2}\left(\frac{m}{e}\right)^2 c^5 \epsilon_0 \left(\frac{\omega}{\omega_p}\right)^2 \tilde{P} \\
&\simeq 0.35 \left(\frac{\omega}{\omega_p}\right)^2 \tilde{P} \text{ GW}.
\end{aligned}
$$

The critical power $\tilde{P}_c = 16\pi$ thus corresponds to a physical threshold:

$$
P_c \simeq 17.5 \left(\frac{\omega}{\omega_p}\right)^2 \text{GW}, \qquad (128)
$$

Focussing threshold – example

Critical power

$$P_c \simeq 17.5 \left(\frac{\omega}{\omega_p}\right)^2 \text{GW}, \qquad (129)$$

Example

$$\lambda_L = 0.8\mu m, \quad n_e = 10^{19} \quad \text{cm}^{-3}$$

$$\Rightarrow \frac{n_e}{n_c} = \left(\frac{\omega_p}{\omega}\right)^2 = \frac{10^{19}}{1.6 \times 10^{21}} = 6 \times 10^{-3}$$

$$\Rightarrow P_c = 2.6\text{TW}$$

Beam radius in relativistic regime

The beam radius defined by Eq. (126) can now be written

$$
\begin{aligned}
< r^2 > &= \sigma_0^2 + \frac{t^2}{\sigma_0^2}\left(1 - \frac{P}{P_c}\right) \\
&= \sigma_0^2\left[1 + \frac{z^2}{z_R^2}\left(1 - \frac{P}{P_c}\right)\right].
\end{aligned} \tag{130}
$$

Beam radius in relativistic regime: scenarios

3 regimes:

- $P/P_c < 1$: beam spread angle $\theta \equiv < r^2 >^{1/2} /z$ is reduced by an amount $(1 - P/P_c)^{1/2}$
- $P = P_c$: beam should propagate indefinitely with constant radius
- $P > P_c$: beam will collapse to zero radius in a distance:

$$z_c = \frac{z_R}{(P/P_c - 1)^{1/2}}. \tag{131}$$

Ponderomotive channel formation

How does beam focussing affect the plasma?
Consider the electron fluid to be adiabatic, or 'frozen' on the timescale of the EM envelope.

$$\frac{d}{dt}(\gamma \mathbf{v}) = c\nabla\phi - \frac{c}{2\gamma}\nabla a^2, \tag{132}$$

where we now assume that v varies much more slowly than the laser field $a\cos\omega_0 t$. Let $\mathbf{v} \to 0$, and take divergence, to get:

$$\nabla^2\phi = \frac{1}{2\gamma}\nabla^2 a^2 = \nabla^2\gamma.$$

The relativistic factor is just a function of the pulse amplitude: $\gamma = (1 + a^2)^{1/2}$.

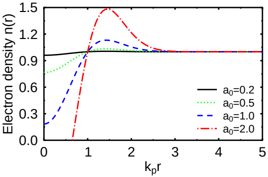
Transverse plasma response

Cigar-shaped pulse: take $\nabla = \nabla_\perp$, and apply Poisson's equation (ϕ and n normalized as before)

$$\nabla_\perp^2 \phi = k_p^2 (n - 1),$$

to obtain density perturbation:

$$n = 1 + k_p^{-2} \nabla_\perp^2 \gamma. \tag{133}$$



Cavitation condition

Consider Gaussian pulse profile $a(r) = a_0 \exp(-r^2/2\sigma^2)$. After time-averaging over the laser period, the density depression term in cylindrical coordinates can be written:

$$\nabla_\perp^2 \gamma = \frac{1}{4\gamma} \nabla_\perp^2 a^2 = \frac{1}{4\gamma} \frac{4a_0^2}{\sigma^2} \left(\frac{r^2}{\sigma^2} - 1 \right) \exp(-r^2/\sigma^2).$$

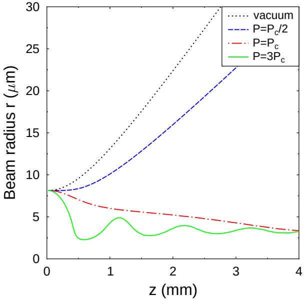
The deepest depression is on the laser axis at $r = 0$, so we obtain a 'cavitation' condition, $n = 0$, for:

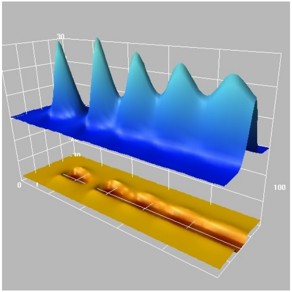
$$I_{18} \lambda_\mu^2 > \frac{1}{20} n_{18} \sigma_\mu^2, \tag{134}$$

– quite easily fulfilled with TW lasers.

Relativistic beam propagation

Numerical solution of NLSE with an initial radial Gaussian beam profile with $\sigma_0 = 7.5\ \mu$m and pump strengths a_0.
Beam powers P/P_c: $0.5, 1.0$ and 3.0 respectively.





Relativistic beam propagation: 2D particle-in-cell simulations

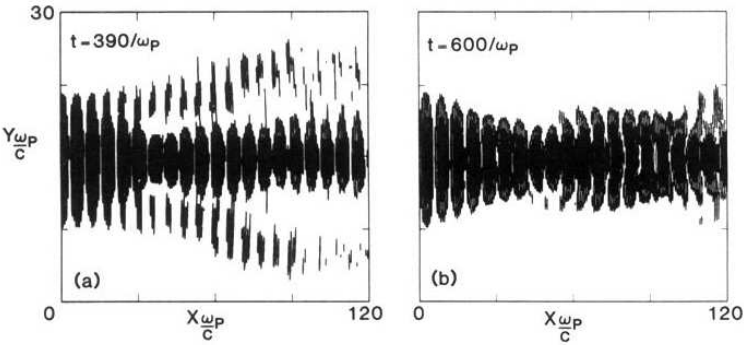


Figure: Two-dimensional PIC simulation of relativistic self-focusing. At the earlier time ($390/\omega_p$), only the central portion of the pulse is focused: the wings are scattered away from the axis. At $600/\omega_p$, after the ions have been pushed away from the center to form a channel, nearly all the light is focused, subsequently exhibiting radial oscillations. (W. B. Mori, 1988)

Experiments on relativistic beam propagation
Chicago (1992)

C. K. Rhodes, Borisov *et al.* (1992)

- Laser: 500 fs KrF* laser ($\lambda = 248$ nm), peak power 0.3 TW, 8×10^{17} Wcm^{-2}
- Target: *gas-fill* chamber, e.g. N_2
- Electron density needed to trigger self-focusing from Eq. (107): $n_e \simeq 10^{21}$ cm^{-3}.
- Possible problems with ionization-induced defocussing.
- Imaging resolution only 40 μm!

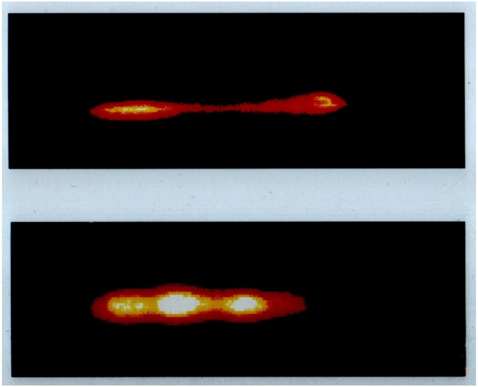
Experiments on relativistic beam propagation
Saclay (1994)

T. Auguste, P. Monot, P. Gibbon *et al.* (1994)

- T. Auguste, P. Monot *et al.* (1994)
- Laser: 1 ps Nd-glass laser ($\lambda = 1\ \mu$m), peak power 15 TW, 1×10^{19} Wcm^{-2}
- Target: H_2 *gas-jet*
- Electron densities: $n_e \simeq 10^{18} - 10^{19}$ cm^{-3}.
- $P/P_c \simeq 0.5 - 5$
- Imaging via 90^o Thomson-scattering, resolution 2–3 μm

Saclay experiment – Thomson scatter images

Top: $P/P_c = 2$, bottom: $P/P_c = 5$



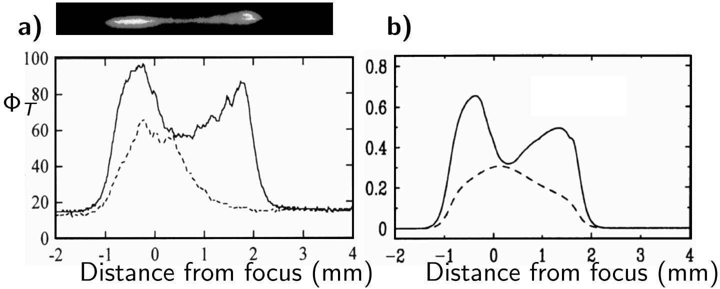


Figure: a) Experimental Thomson scattering of relativistic self-channeling at two laser powers $P/P_c = 2$ (CCD image and solid lines), and $P/P_c = 0.125$ (dashed lines). The curves in b) are theoretical predictions from a wave-envelope model. The Rayleigh length, $Z_R = 300\ \mu$m.

Saclay experiment – summary

- Lengthening of the propagation channel with laser power
- Infer channels of many Rayleigh lengths
- Qualitative agreement with Thomson images reconstructed from a paraxial propagation model

Relativistic self-focussing in pre-formed plasmas

Borghesi, Willi *et al.* (1997):
Preformed plasma created using two-pulse interaction. Critical power P_c is reduced to 100-200 GW owing to densities $O(n_c)$

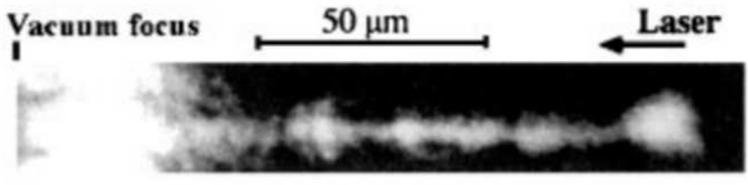


Figure: Self-emission at 0.527 μm from a TW-laser produced channel in a preformed plasma created from a plastic film situated to the left of the vacuum focus.

Bench-Top Particle Accelerators

Standard electron acceleration in fast plasma wave (recap):

- Acceleration length

$$L_a \simeq 3.2\, n_{18}^{-3/2} \lambda_{\mu\mathrm{m}}^{-2} \ \text{cm}$$

- Energy gain

$$\Delta U \simeq 3.2\, n_{18}^{-1} \lambda_{\mu\mathrm{m}}^{-2} \ \text{GeV}$$

Acceleration mechanisms

Large variety of attributed acceleration mechanisms in experiments (long and short pulse):

- self-modulated
- forced-wave
- wave-breaking
- guided
- bubble-regime

GeV milestone reached September 2006 (LBL)

Livingstone chart for laser-plasma electron accelerators

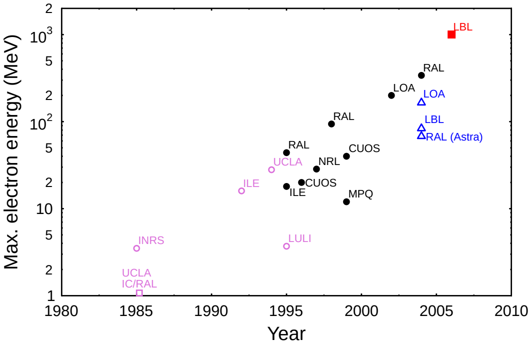


Figure: Open circles represent early beat-wave experiments; filled circles single pulse wakefield experiments; triangles quasi-*monoenergetic* electron beams