Part III

Interaction with Single Electrons - Plane Wave Orbits

Single electron motion in EM plane wave

- Electron momentum in electromagnetic wave with fields **E** and **B** given by Lorentz equation (cgs units):

$$\frac{d\mathbf{p}}{dt} = -e(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}), \tag{22}$$

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- Energy equation (taking dot product of **v** with Eq. (22))

$$\frac{d}{dt}\left(\gamma mc^2\right) = -e(\mathbf{v} \cdot \mathbf{E}), \tag{23}$$

where $\mathbf{p} = \gamma m\mathbf{v}$, and $\gamma = (1 + p^2/m^2c^2)^{\frac{1}{2}}$ is the relativistic factor.

Electromagnetic plane wave

- Consider elliptically polarized plane-wave $\mathbf{A}(\omega, \mathbf{k})$ travelling in the positive x-direction. Vector potential

$$\mathbf{A} = (0, \delta a_0 \cos\phi, (1-\delta^2)^{\frac{1}{2}} a_0 \sin\phi), \tag{24}$$

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- Polarization parameter δ:
  - $\delta = \{\pm 1, 0\} \rightarrow$ linearly polarized wave
  - $\delta = \pm 1/\sqrt{2} \rightarrow$ circular wave.

Normalized (dimensionless) units

$$
\begin{aligned}
t &\rightarrow \omega t \\
x &\rightarrow kx \\
\mathbf{v} &\rightarrow \mathbf{v}/c \\
\mathbf{p} &\rightarrow \mathbf{p}/mc \\
\mathbf{A} &\rightarrow e\mathbf{A}/mc^2 \\
\mathbf{E} &\rightarrow e\mathbf{E}/m\omega c \\
\mathbf{B} &\rightarrow e\mathbf{B}/m\omega c
\end{aligned}
$$

Equivalent to setting $\omega = k = c = e = m = 1$ (cf. atomic units)

Perpendicular (canonical) momentum

Using the relations

$$\mathbf{E} = -\partial \mathbf{A}/\partial t$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} = (0, -\partial A_z/\partial x, \partial A_y/\partial x)$$

the perpendicular component of momentum Eq. (22) becomes:

$$\frac{d\mathbf{p}_\perp}{dt} = \frac{\partial \mathbf{A}}{\partial t} + v_x \frac{\partial \mathbf{A}}{\partial x},$$

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which after integrating gives *canonical momentum*:

$$\mathbf{p}_\perp - \mathbf{A} = \mathbf{p}_{\perp 0}, \tag{25}$$

where $\mathbf{p}_{\perp 0}$ is a constant of motion.

Longitudinal momentum

The longitudinal components of Eq. (22) and Eq. (23) yield a pair of equations which can be subtracted from each other thus:

$$\frac{dp_x}{dt} - \frac{d\gamma}{dt} = -v_y \left(\frac{\partial A_y}{\partial t} + \frac{\partial A_y}{\partial x} \right) - v_z \left(\frac{\partial A_z}{\partial t} + \frac{\partial A_z}{\partial x} \right).$$

Because the EM wave is a function of $t - x$ only, the terms on the RHS vanish identically, so we can immediately integrate the RHS to get:

$$\gamma - p_x = \alpha,$$

where α is a constant of motion still to be determined.

General solution

Use identity

$$\gamma^2 - p_x^2 - p_\perp^2 = 1$$

and choose $\mathbf{p}_{\perp 0} = 0$ to get relationship between the parallel and perpendicular momenta:

$$p_x = \frac{1 - \alpha^2 + p_\perp^2}{2\alpha}. \quad (26)$$

– independent of polarization δ.

Change of variables: wave phase

Need to specify α and integrate Eq. (25) and Eq. (26). This is simplified by changing variables. Noting that

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \frac{p_x}{\gamma}\frac{\partial\phi}{\partial x} = \frac{\alpha}{\gamma},$$

we have

$$\mathbf{p} = \gamma\frac{d\mathbf{r}}{dt} = \gamma\frac{d\phi}{dt}\frac{d\mathbf{r}}{d\phi} = \alpha\frac{d\mathbf{r}}{d\phi}. \tag{27}$$

Special cases: (1) laboratory frame

- Lab frame: the electron initially at rest before the EM wave arrives, so that at $t = 0$, $p_x = p_y = 0$ and $\gamma = 1$.
- From Eq. (26) it follows that $\alpha = 1$.

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- Lab frame: the electron initially at rest before the EM wave arrives, so that at $t = 0$, $p_x = p_y = 0$ and $\gamma = 1$.
- From Eq. (26) it follows that $\alpha = 1$. This leads to the following expression for the momenta in the lab frame:

$$\begin{aligned} p_x &= \frac{a_0^2}{4}\left[1 + (2\delta^2 - 1)\cos 2\phi\right], \\ p_y &= \delta a_0 \cos\phi, \\ p_z &= (1-\delta^2)^{1/2} a_0 \sin\phi. \end{aligned} \tag{28}$$

Laboratory frame: solution

With Eq. (27) we can integrate expressions Eqs. (28a–28c) to obtain the lab-frame orbits valid for arbitrary polarization δ:

$$\begin{aligned} x &= \frac{1}{4} a_0^2 \left[\phi + \frac{2\delta^2 - 1}{2} \sin 2\phi \right], \\ y &= \delta a_0 \sin \phi, \\ z &= -(1-\delta^2)^{1/2} a_0 \cos \phi. \end{aligned} \qquad (29)$$

NB: solution is self-similar in the variables $(x/a_0^2, y/a_0)$

Laboratory frame: linearly polarized wave ($\delta = 1$)

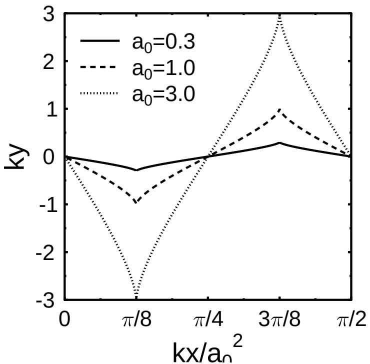


Figure: For a 1 μm laser wavelength, the pump strengths a_0 correspond roughly to intensities of 10^{17}, 10^{18} and 10^{19} Wcm^{-2} respectively.

Laboratory frame: drift velocity

- Longitudinal motion has a *secular* component which will grow in time or with propagation distance.
- Electron starts to *drift* with an average momentum

$$p_D \equiv \overline{p_x} = \frac{a_0^2}{4}$$

corresponding to a velocity:

$$\frac{v_D}{c} = \overline{v_x} = \frac{\overline{p_x}}{\overline{\gamma}} = \frac{a_0^2}{4 + a_0^2}, \tag{30}$$

where the overscore denotes averaging over the rapidly varying EM phase ϕ.

Laboratory frame: circularly polarized light ($\delta = \pm 1/\sqrt{2}$)

- Longitudinal oscillating component at 2ϕ vanishes identically
- Transverse motion is *circular* with radius a_0 and momentum $p_\perp = a_0/\sqrt{2}$.
- Combines with the linear drift in Eq. (30) to give a *helical* orbit with pitch angle

$$\theta_p = p_\perp / p_D = \sqrt{8} a_0^{-1}. \tag{31}$$

Special cases: (2) Average rest frame

- Require drift velocity to vanish: $\overline{p_x} = 0$ in Eq. (26), then:

$$1 + \overline{A^2} - \alpha^2 = 0.$$

- Average over a laser cycle to remove rapidly varying terms, and noting that $\overline{\cos^2 \phi} = 1/2$ gives:

$$\alpha = \left(1 + \frac{a_0^2}{2}\right)^{1/2} \equiv \gamma_0. \qquad (32)$$

Rest frame: solution

Plugging α from Eq. (32) back into Eq. (26) gives the momenta:

$$\begin{aligned} p_x &= \left(2\delta^2 - 1\right) \frac{a_0^2}{4\gamma_0} \cos 2\phi, \\ p_y &= \delta a_0 \cos\phi, \\ p_z &= (1-\delta^2)^{\frac{1}{2}} a_0 \sin\phi. \end{aligned} \tag{33}$$

Noting that in this case, $\mathbf{p} = \gamma_0 d\mathbf{r}/d\phi$, we can integrate again to get the orbits:

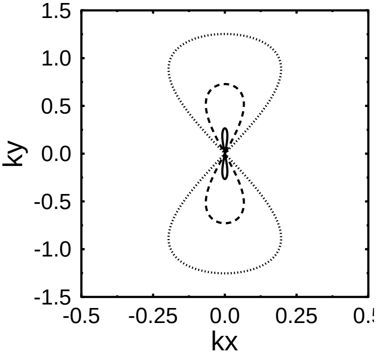
$$\begin{aligned} x &= \left(\delta^2 - \frac{1}{2}\right) q^2 \sin 2\phi, \\ \mathbf{r}_\perp &= 2(\delta q \sin\phi, -(1-\delta^2)^{\frac{1}{2}} q \cos\phi), \end{aligned} \tag{34}$$

where $q = a_0/2\gamma_0$.

Rest frame: linearly polarized wave ($\delta = 1$)

- Eliminate ϕ for linear polarization ($\delta = 1$), to obtain *figure-of-eight* orbit.

$$16x^2 = y^2(4q^2 - y^2). \tag{35}$$



Rest frame: limits

- Extreme intensity limit, $a_0 \gg 1$:

$$\begin{aligned} \gamma_0 &\rightarrow a_0/\sqrt{2} \\ q &\rightarrow 1/\sqrt{2} \end{aligned}$$

- 'fat-8' orbit with

$$\mid x \mid_{\max} = 1/4, \ \mid y \mid_{\max} = \sqrt{2}.$$

- Circularly polarized light: $\delta = \pm 1/\sqrt{2}$; $p_x = 0$

$\Rightarrow$ circle with radius

$$\frac{a_0}{\sqrt{2}\gamma_0}$$

.

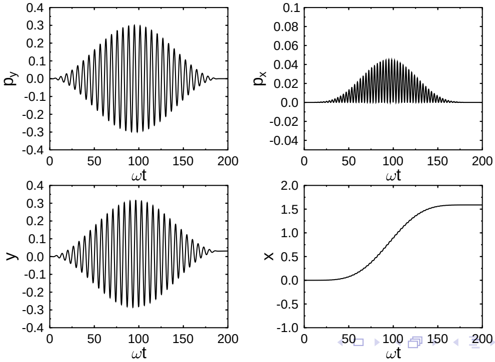
Finite pulse duration

$$\mathbf{A}(x,t) = a_0 f(t)\cos\phi, \tag{36}$$

Adiabatic approximation: $df/dt \ll \omega f$.

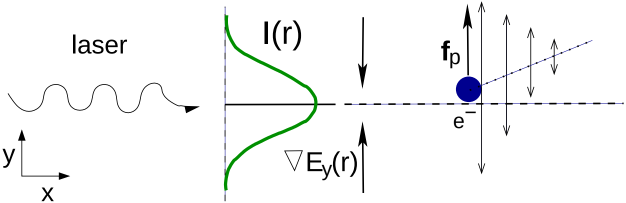
Example

$f(t) = \sin(\pi t/2t_L)$, with $\omega t_L = 600/\pi$.



Ponderomotive force

- Single electron oscillating slightly off-centre of focused laser beam:



- After 1st quarter-cycle, sees *lower* field
- Doesn't quite return to initial position

$\Rightarrow$ Accelerated away from axis

Ponderomotive force: non-relativistic

In the limit $v/c \ll 1$, the equation of motion (22) for the electron becomes:

$$\frac{\partial v_y}{\partial t} = -\frac{e}{m} E_y(\mathbf{r}). \tag{37}$$

Taylor expanding electric field:

$$E_y(\mathbf{r}) \simeq E_0(y)\cos\phi + y\frac{\partial E_0(y)}{\partial y}\cos\phi + ...,$$

where $\phi = \omega t - kx$ as before.
To lowest order, we therefore have

$$v_y^{(1)} = -v_{\rm os}\sin\phi; \quad y^{(1)} = \frac{v_{\rm os}}{\omega}\cos\phi,$$

where $v_{\rm os} = eE_L/m\omega$.

Ponderomotive force: non-relativistic (contd.)

Substituting back into Eq. (37) gives

$$\frac{\partial v_y^{(2)}}{\partial t} = -\frac{e^2}{m^2\omega^2} E_0 \frac{\partial E_0(y)}{\partial y} \cos^2\phi.$$

Multiplying by m and taking the cycle-average yields the *ponderomotive force* on the electron:

$$f_p \equiv m\overline{\frac{\partial v_y^{(2)}}{\partial t}} = -\frac{e^2}{4m\omega^2}\frac{\partial E_0^2}{\partial y}. \quad (38)$$

Ponderomotive force: relativistic

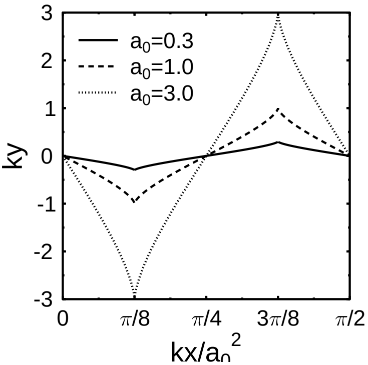
Lorentz equation (22) in terms of the vector potential **A**:

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{p} = \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} - \frac{e}{c}\mathbf{v}\times\nabla\times\mathbf{A}. \quad (39)$$

Separate the timescales of the electron motion into slow and fast components $\mathbf{p} = \mathbf{p}^s + \mathbf{p}^f$ and average over a laser cycle,

$$\mathbf{f}_p = \frac{dp^s}{dt} = -mc^2\nabla\overline{\gamma}, \quad (40)$$

where $\gamma = \left(1 + \frac{p_s^2}{m^2c^2} + \frac{1}{2}a_0^2\right)^{1/2}$.



Solution in terms of wave phase $\phi = \omega t - kx$:

$$x = \frac{1}{4} a_0^2 \left(\phi + \frac{1}{2} \sin 2\phi \right),$$

$$y = a_0 \sin \phi$$

Drift velocity:

$$\frac{v_D}{c} = \frac{a_0^2}{4 + a_0^2}$$