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A SIMPLE FAST METHOD IN FINDING PARTICULAR SOLUTIONS OF SOME NONLINEAR PDE*

LIU Shi-kuo (刘式适)[1], FU Zun-tao (付遵涛)[1,2],
LIU Shi-da (刘式达)[1,2], ZHAO Qiang (赵 强)[1]

(1. Department of Geophysics, Peking University, Beijing 100871, P R China;
2. SKLTR, Peking University, Beijing 100871, P R China)

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Abstract: The "trial function method" (TFM for short) and a routine way in finding traveling wave solutions to some nonlinear partial differential equations (PDE for short) wer explained. Two types of evolution equations are studied, one is a generalized Burgers or KdV equation, the other is the Fisher equation with special nonlinear forms of its reaction rate term. One can see that this method is simple, fast and allowing further extension.

Key words: trial function method; nonlinear PDE; shock wave solution; solitary wave solution

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1 A Trial Function and a Routine to Find Analytical Solution of Two Types of Nonlinear PDE

We treat the nonlinear evolution equation, which is formed by adding high order derivative terms and nonlinear terms to the Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \cdots + u^p\left[\frac{\partial u}{\partial x}\right]^q + \alpha_1\frac{\partial u}{\partial x} + \cdots + \alpha_n\frac{\partial^n u}{\partial x^n} = 0, \tag{1}$$

which p, q, n and $\alpha_i (i = 1, 2, \cdots, n)$ being parameters independent of x and t. Although these equations[1-10] have been studying for decades, we have succeeded in some extension, particularly in the cases $n = 3$, 4[11 and 12] and even $n = 5$. We limit us to study the traveling wave solution of Eq. (1) in the following form (trial function solution)

$$u = u_0 + \frac{B e^{b\xi}}{(1 + e^{a\xi})^d}, \tag{2}$$

with $u = u(\xi)$, $\xi = x - ct$, $u_0 = 0$ or $u_0 = c \pm\sqrt{c^2 + 2A}$; where c, B, a, b, d and A are constants to be determined.

First, we determined the exponent d by substituting Eq. (2) into Eq. (1) and partially

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Biography: LIU Shi-kuo (1938 -), Professor, Supervisor of Ph D Candidate equalizing the highest order derivative term and nonlinear term

$$d = \frac{n - q}{p + q - 1}. \tag{3}$$

Next, one can select the constants c, B, a, b and A and determine the relation between α_i by substituting (3) into (1) and (2) and making some elementary manipulations. We mention that in [10] d is chosen as 2 and we generalize it to other values. In the next sections, the trial function and its corresponding solutions to two kinds of nonlinear evolution equations are illustrated.

2 The Solutions to Nonlinear Equation Composed of Derivative Terms

Burgers, KdV, KdV-Burgers, Benney[13] equations can be generalized to the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} + \cdots = 0, \tag{4}$$

with $p = 1$, $q = 1$, so

$$d = n - 1, \tag{5}$$

with nth order derivative as the highest one.

(A) Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} = 0, \tag{6}$$

then $d = 1$, so its trial function is

$$u = u_0 + \frac{Be^{b\xi}}{1 + e^{a\xi}} \qquad (u_0 = c + \sqrt{c^2 + 2A}), \tag{7}$$

with its corresponding solutions of the following two cases:

1) when $b = 0$, $B = -2\sqrt{c^2 + 2A}$, $a = \sqrt{c^2 + 2A}/\alpha$, its shock wave solution is written as

$$u = c + \sqrt{c^2 + 2A}\tanh\frac{\sqrt{c^2 + 2A}}{2\alpha}\xi; \tag{8}$$

2) when $b = a$, $B = -2\sqrt{c^2 + 2A}$, $a = -\sqrt{c^2 + 2A}/\alpha$, the shock wave solution is just the same as (8).

(B) KdV equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \tag{9}$$

then $d = 2$, its trial function is

$$u = u_0 + \frac{Be^{b\xi}}{(1 + e^{a\xi})^2} \qquad (u_0 = c - \sqrt{c^2 + 2A}), \tag{10}$$

when $b = a$, $B = 12\sqrt{c^2 + 2A}$, $a = \sqrt{\sqrt{c^2 + 2A}/\beta}$, its solitary wave solution is

$$u = c - \sqrt{c^2 + 2A} + 3\sqrt{c^2 + 2A}\,\mathrm{sech}^2\sqrt{\sqrt{c2 + 2A}/\beta}\xi. \tag{11}$$

(C) KdV-Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \tag{12}$$

then $d = 2$, its trial function can be written as

$$u = u_0 + \frac{Be^{b\xi}}{(1+e^{a\xi})^2} \qquad (u_0 = c + \sqrt{c^2+2A}). \tag{13}$$

Actually, its two solutions can be written as:

1) when $b = 0$, $B = -\frac{12\alpha^2}{25\beta}$, $a = \frac{\alpha}{5\beta}$, $\sqrt{c^2+2A} = \frac{6\alpha^2}{25\beta}$ $(\beta > 0)$, its solution is

$$u = c + \sqrt{c^2+2A} - \frac{3\alpha^2}{25\beta}\left(1 - \tanh\frac{\alpha}{10\beta}\xi\right)^2; \tag{14}$$

2) when $b = 2a$, the shock wave solution is just the same as (14).

(D) Benney equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} = 0, \tag{15}$$

then $d = 3$, its trial function is with the following form

$$u = u_0 + \frac{Be^{b\xi}}{(1+e^{a\xi})^3} \qquad (u_0 = c + \sqrt{c^2+2A}), \tag{16}$$

from which the following solution can be got:

1) when $b = 0$, the shock wave solution is of the following form

$$\left.\begin{aligned} u &= u_0 + \frac{15\alpha}{47\sqrt{47}}\sqrt{\frac{\alpha}{\gamma}}\left[1 + \tanh\frac{1}{2\sqrt{47}}\sqrt{\frac{\alpha}{\gamma}}\xi\right]^3 \qquad (\alpha < 0, \gamma < 0), \\ u &= u_0 - \frac{15\alpha}{47\sqrt{47}}\sqrt{\frac{\alpha}{\gamma}}\left[1 - \tanh\frac{1}{2\sqrt{47}}\sqrt{\frac{\alpha}{\gamma}}\xi\right]^3 \qquad (\alpha > 0, \gamma > 0). \end{aligned}\right\} \tag{17}$$

under the condition that

$$\beta > 0, \quad \beta^2 = \frac{144\alpha\gamma}{47}; \tag{18}$$

2) when $b = a$, the solitary wave solution is written as

$$\left.\begin{aligned} u &= u_0 + 15\alpha\sqrt{\frac{\alpha}{\gamma}}\operatorname{sech}^2\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\xi\left[1 - \tanh\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\xi\right] \qquad (\alpha < 0, \gamma < 0), \\ u &= u_0 - 15\alpha\sqrt{\frac{\alpha}{\gamma}}\operatorname{sech}^2\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\xi\left[1 + \tanh\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\xi\right] \qquad (\alpha > 0, \gamma > 0) \end{aligned}\right\} \tag{19}$$

under the condition that

$$\beta < 0, \quad \beta^2 = 16\alpha\gamma; \tag{20}$$

3) when $b = 2a$, the same results as (19) can be obtained.

Similarly, we can solve the equations when the highest derivative is of order 5. This is partially accomplished, we omit here the long derivation.

3 The Solutions to a Kind of Reaction-Diffusion Equation or Fisher Equation

We study the following reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2} + f(u), \tag{21}$$

whose nonlinear reaction term has the following four forms

$$f_1(u) = ku(1-u), \tag{22}$$

$$f_2(u) = ku(1-u^m) \qquad (m > 0), \tag{23}$$

$$f_3(u) = ku(1-u)(u-r) \qquad \left(0 \le r \le \frac{1}{2}\right), \tag{24}$$

and

$$f_4(u) = ku(1 - u^m)(u^m - r) \qquad (0 \le r \le \frac{1}{2}). \tag{25}$$

Substituting (2) into (21) and equalizing the diffusion term and the highest order nonlinear term, we have

$$d = \frac{2}{p-1}, \tag{26}$$

with p to be the order of the highest order nonlinear term in Eq. (21).

(A) When the reaction term is (22), then $p = 2$, $d = 2$; and thus the trial function of Eq. (21) is

$$u = \frac{Be^{b\xi}}{[1 + e^{a\xi}]^2}, \tag{27}$$

its corresponding solutions are

1) when $b = 0$, $B = 1$, $a = \frac{c}{5\nu}$, $c^2 = \frac{25}{6}k\nu > 4k\nu$, the solution turns to be

$$u = \frac{1}{(1 + \exp[-c\xi/(5\nu)])^2} = \left[\frac{1}{2}\left(1 - \tanh\frac{c}{10\nu}\xi\right)\right]^2; \tag{28}$$

2) when $b = 2a$, $B = 1$, $a = -\frac{c}{5\nu}$, $c^2 = \frac{25}{6}k\nu > 4k\nu$, the solution differs only by two signs

$$u = \frac{1}{(1 + \exp[-c\xi/(5\nu)])^2} = \left[\frac{1}{2}\left(1 + \tanh\frac{c}{10\nu}\xi\right)\right]^2; \tag{29}$$

3) when $b = a$, $c = 0$, $B = 6$, $a = \mathrm{i}\sqrt{\frac{k}{\nu}}$, it is a stationary solution

$$u = \frac{3}{2}\sec^2\sqrt{\frac{k}{4\nu}}x \qquad \left(-\frac{\pi}{2} < \sqrt{\frac{k}{4\nu}}x < \frac{\pi}{2}\right). \tag{30}$$

(B) When the reaction term is (23), then $p = m + 1$, $d = \frac{2}{m}$; the trial function of Eq. (21) is of the following form

$$u = \frac{Be^{b\xi}}{[1 + e^{a\xi}]^{2/m}}, \tag{31}$$

its corresponding solutions are

1) when $b = 0$, $B = 1$, $a^2 = \frac{km^2}{2(m+2)\nu}$, $c = \frac{3km}{2a}$ $(c^2 > 4k\nu)$, the solution is

$$u = \frac{1}{\left[1 + \exp\left[\pm m\left(\frac{k\xi}{2(m+2)\nu}\right)^{1/2}\right]\right]^{2/m}} = \left[\frac{1}{2}\left(1 \mp \tanh\frac{m}{2}\sqrt{\frac{k}{2(m+2)\nu}}\xi\right)\right]^{2/m}; \tag{32}$$

2) when $mb = 2a$, $B = 1$, $a^2 = \frac{km^2}{2(m+2)\nu}$, $c^2 = (m+4)^2\frac{k\nu}{2(+2)}$ $(c^2 > 4k\nu)$, the solution becomes

$$u = \frac{1}{\left[1 + \exp\left[\pm m\left(\frac{k\xi}{2(m+2)\nu}\right)^{1/2}\right]\right]^{2/m}} = \left[\frac{1}{2}\left(1 \mp \tanh\frac{m}{2}\sqrt{\frac{k}{2(m+2)\nu}}\xi\right)\right]^{2/m}; \tag{33}$$

3) when $mb = a$, $c = 0$, $B = [2(m+2)]^{1/m}$, $a = \mathrm{i}m\sqrt{k/\nu}$, the solution changes to

$$u = \frac{[2(m+2)]^{1/m}\mathrm{e}^{\pm\sqrt{k/\nu}\xi}}{[1+\mathrm{e}^{\pm m\sqrt{k/\nu}\xi}]^{2/m}} \quad \left(m\sqrt{\frac{k}{\nu}}\xi \neq \pi, 3\pi, \cdots\right). \tag{34}$$

(C) When the reaction term is (24), then $p = 3$, $d = 1$; so the trial function of Eq. (21) can be written as

$$u = \frac{B}{1+\mathrm{e}^{a\xi}}, \tag{35}$$

its corresponding solutions are

1) when $B = 1$, $a = \pm\sqrt{\frac{k}{2\nu}}$, $c = \pm(\frac{1}{2} - r)\sqrt{2k\nu}$, the solution reduces to

$$u = \frac{1}{[1+\mathrm{e}^{\pm\sqrt{k/2\nu}\xi}]} = \frac{1}{2}\left(1 \mp \tanh\frac{1}{2}\sqrt{\frac{k}{2\nu}}\xi\right); \tag{36}$$

2) when $B = a$, $a = \pm r\sqrt{\frac{k}{2\nu}}$, $c = \pm(\frac{r}{2} - 1)\sqrt{2k\nu}$, the constant r comes into the solution

$$u = \frac{r}{[1+\mathrm{e}^{\pm r\sqrt{k/2\nu}\xi}]} = \frac{r}{2}\left(1 \mp \tanh\frac{r}{2}\sqrt{\frac{k}{2\nu}}\xi\right). \tag{37}$$

(D) When the reaction term is (25), then $p = 2m+1$, $d = \frac{1}{m}$; so the trial function of Eq. (21) is with the following form

$$u = \frac{B}{[1+\mathrm{e}^{a\xi}]^{1/m}}, \tag{38}$$

its corresponding solutions are

1) when $B^m = 1$, $a = \pm m\sqrt{\frac{k}{2\nu}}$, $c = \pm(\frac{1}{2} - r)\sqrt{2k\nu}$, the solution is

$$u = \left[\frac{1}{1+\mathrm{e}^{\pm m\sqrt{k/2\nu}\xi}}\right]^{1/m} = \left[\frac{1}{2}\left(1 \mp \tanh\frac{m}{2}\sqrt{\frac{k}{2\nu}}\xi\right)\right]^{1/m}; \tag{39}$$

2) when $B^m = r$, $a = \pm mr\sqrt{\frac{k}{2\nu}}$, $c = \pm(\frac{r}{2} - 1)\sqrt{2k\nu}$, its solution is

$$u = \left[\frac{r}{1+\mathrm{e}^{\pm mr\sqrt{k/2\nu}\xi}}\right]^{1/m} = \left[\frac{r}{2}\left(1 \mp \tanh\frac{mr}{2}\sqrt{\frac{k}{2\nu}}\xi\right)\right]^{1/m}. \tag{40}$$

Similar solution of other reaction-diffusion equations is hopeful.

4 Conclusion

It is clear that the above standard routine derivations are simple and fast in finding analytical solutions for some nonlinear PDE with particular form. The main idea is to start from the Eq. (3) and properly determine d and other constants. These particular exact solutions are useful as a standard for the test of numerical methods of high order accuracy, say from 3rd- to 6th-order of accuracy, rapidly developed in recent years. They also may serve as simple model for some natural phenomena.

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References :

[1] Kuramoto Y. Chemical Oscillations , Waves and Turbulence [M] . Berlin : Springer-Verlag , 1984.

[2] Kuramoto Y. Diffusion-induced chaos in reaction-diffusion systems [J] . Prog Theor Phys Suppl , 1978 ,**64**(suppl) :346 - 350.

[3] Burgers J M. A mathematical model illustrating the theory of turbulence [J] . Adv Appl Mech , 1949 ,**1**(2) :171 - 175.

[4] LIU Shi-da , LIU Shi-kuo. KdV-Burgers modeling of turbulence [J] . Science in China A , 1992 , **35**(5) :576 - 577.

[5] Kawahara T. Formulation of saturated solutions in a nonlinear dispersive system with instability and dissipation [J] . Phys Rev Lett , 1983 ,**51**(5) :381 - 387.

[6] Kwok W, Chow. A class of exact , periodic solutions of nonlinear envelope equations [J] . J Math Phys , 1995 ,**36**(8) :4125 - 4137.

[7] Kudryashov N A. Exact solutions of the generalized Kuramoto-Sivashinsky equation [J] . Phys , Lett A , 1990 ,**147**(5 ,6) :287 - 291.

[8] WANG Ming-liang , ZHOU Yu-bin , LI Zhi-bin. Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics [J] . Phys Lett A , 1996 ,**216**(1) :67 - 75.

[9] KONG De-xing , HU Hai-rong. Geometric approach for finding exact solutions to nonlinear partial differential equations [J] . Phys Lett A , 1998 ,**246**(1 ,2) :105 - 112.

[10] XIONG Shu-lin. A kind of analytical solutions of Burgers-KdV equation [J] . Kexue Tongbao (Chinese Science Bulletin) , 1989 ,**34**(1) :26 - 29. (in Chinese)

[11] LIU Shi-da , LIU Shi-kuo. Heteroclinic orbit on the KdV-Burgers equation and Fisher equation [J] . Commun Theor Phys ,1991 ,**16**(4) :497 - 500.

[12] LIU Shi-da , LIU Shi-kuo. The traveling wave solutions of KdV-Burgers-Kuramoto equation [J] . Progress in Natural Science , 1999 ,**9**(10) :912 - 918. (in Chinese)

[13] Benney D J. Long nonlinear waves in fluid flow [J] . J Math and Phys , 1966 ,**45**(1) :52 - 60.