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Spiral Solutions of the Two-Dimensional Complex Ginzburg–Landau Equation*

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Abstract *The multi-order exact solutions of the two-dimensional complex Ginzburg–Landau equation are obtained by making use of the wave-packet theory. In these solutions, the zeroth-order exact solution is a plane wave, the first-order exact solutions are shock waves for the amplitude and spiral waves both between the amplitude and the shift of phase and between the shift of phase and the distance.*

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1 Introduction

The two-dimensional complex Ginzburg–Landau equation including the non-steady, nonlinear, dispersive and diffusive terms, as well as the linear growth (or damping) term is usually written as

$$\mathrm{i}\frac{\partial u}{\partial t} + (\alpha_1 + \mathrm{i}\alpha_2)\nabla^2 u + (\beta_1 + \mathrm{i}\beta_2)|u|^2 u - \mathrm{i}\gamma u = 0\,, \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$ is a pure imaginary number, α_1, α_2, β_1, β_2 and γ are real constants. $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplacian. $(\beta_1 + \mathrm{i}\beta_2)|u|^2 u$ represents the nonlinear effects, $\alpha_1\nabla^2 u$ and $\mathrm{i}\alpha_2\nabla^2 u$ represent the dispersive and diffusive effects, respectively. $\mathrm{i}\gamma u$ denotes the linear growth or damping.

When $\alpha_2 = \beta_2 = \gamma = 0$, the two-dimensional complex Ginzburg–Landau equation (1) degenerates into the following two-dimensional nonlinear Schrödinger equation,

$$\mathrm{i}\frac{\partial u}{\partial t} + \alpha_1\nabla^2 u + \beta_1|u|^2 u = 0\,. \tag{2}$$

A lot of studies have shown that the complex Ginzburg–Landau equation (1) possesses a rich variety of solutions involving the plane waves, shock and solitary waves, the hole, periodic and spiral solutions, etc.[1−8] In particular, the spiral solutions play an important role in many branches of physics, such as the fluid dynamics, nonlinear optics, chemical and biological dynamics, etc.[9−12] However, up to now no one has found analytically the spiral solutions of the complex Ginzburg–Landau equation.

In this paper, the spiral solutions of the two-dimensional complex Ginzburg–Landau equation (1) are obtained by making use of the wave-packet theory and a relatively simple method.

2 Envelope Solution

We assume that the envelope solution of Eq. (1) in polar coordinates (ρ, θ) is of the form

$$u(\rho,\theta,t) = a(R)\,\mathrm{e}^{\mathrm{i}[K\rho + m\theta - \omega t + \phi(R)]}\,, \tag{3}$$

where K is the wave number, m is the arm number of spirals, ω is the angular frequency, and

$$R = \epsilon(\rho - c_g t)\,, \quad c_g = \text{const.}\,, \tag{4}$$

$\epsilon \ll 1$ is a small parameter. Both the amplitude $a(R)$ and the shift of phase $\phi(R)$ are real functions. In fact, the envelope solution (3), in which a are connected only with R, implies that the dense spiral $(K\rho \gg 1)$ with small arm number $(m \le 4)$ is considered.

Substituting Eq. (3) into Eq. (1) and equating the real and imaginary parts to zero, respectively, we have

$$\begin{aligned} a[\omega - \alpha_1 K^2 + \beta_1 a^2] &+ \epsilon\Big[(c_g - 2\alpha_1 K)a\frac{\mathrm{d}\phi}{\mathrm{d}R} \\ &- 2\alpha_2 K\frac{\mathrm{d}a}{\mathrm{d}R}\Big] + \epsilon^2\Big\{\alpha_1\Big[\frac{\mathrm{d}^2 a}{\mathrm{d}R^2} - a\Big(\frac{\mathrm{d}\phi}{\mathrm{d}R}\Big)^2\Big] \\ &- \alpha_2\Big[2\frac{\mathrm{d}a}{\mathrm{d}R}\frac{\mathrm{d}\phi}{\mathrm{d}R} + a\frac{\mathrm{d}^2\phi}{\mathrm{d}R^2}\Big]\Big\} = 0\,, \end{aligned} \tag{5a}$$

$$\begin{aligned} -a[\gamma + \alpha_2 K^2 - \beta_2 a^2] &+ \epsilon\Big[-2\alpha_2 K a\frac{\mathrm{d}\phi}{\mathrm{d}R} \\ &- (c_g - 2\alpha_1 K)\frac{\mathrm{d}a}{\mathrm{d}R}\Big] + \epsilon^2\Big\{\alpha_2\Big[\frac{\mathrm{d}^2 a}{\mathrm{d}R^2} - a\Big(\frac{\mathrm{d}\phi}{\mathrm{d}R}\Big)^2\Big] \\ &+ \alpha_1\Big[2\frac{\mathrm{d}a}{\mathrm{d}R}\frac{\mathrm{d}\phi}{\mathrm{d}R} + a\frac{\mathrm{d}^2\phi}{\mathrm{d}R^2}\Big]\Big\} = 0\,, \end{aligned} \tag{5b}$$

respectively. In the following sections, we look for the exact solution in the first two orders of Eqs (5).

3 Plane Wave Solution

Setting $\epsilon = 0$, which corresponds to that the variations of a and ϕ with R are disregarded in Eqs (5), we obtain

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the following plane waves solutions

$$\omega - \alpha_1 K^2 + \beta_1 a^2 = 0\,, \tag{6a}$$

$$\gamma + \alpha_2 K^2 - \beta_2 a^2 = 0\,, \tag{6b}$$

from which the dispersion relation and amplitude can be determined.

4 Spiral Solutions

Neglecting the terms with ϵ^2 in Eqs (5) yields

$$2\alpha_2 K\Big(\epsilon\frac{\mathrm{d}a}{\mathrm{d}R}\Big) - (c_g - 2\alpha_1 K)\Big(\epsilon a\frac{\mathrm{d}\phi}{\mathrm{d}R}\Big) = a(\omega - \alpha_1 K^2 + \beta_1 a^2)\,, \tag{7a}$$

$$(c_g - 2\alpha_1 K)\Big(\epsilon\frac{\mathrm{d}a}{\mathrm{d}R}\Big) + 2\alpha_2 K\Big(\epsilon a\frac{\mathrm{d}\phi}{\mathrm{d}R}\Big) = -a(\gamma + \alpha_2 K^2 - \beta_2 a^2)\,. \tag{7b}$$

Equations (7) constitute the systems of the first-order equations in $a(R)$ and $\phi(R)$, we can obtain the following relations

$$\delta\Big(\epsilon\frac{\mathrm{d}a}{\mathrm{d}R}\Big) = \delta_2 a(\omega - \alpha_1 K^2 + \beta_1 a^2) - \delta_1 a(\gamma + \alpha_2 K^2 - \beta_2 a^2)\,, \tag{8a}$$

$$\delta\Big(\epsilon a\frac{\mathrm{d}\phi}{\mathrm{d}R}\Big) = -\delta_1 a(\omega - \alpha_1 K^2 + \beta_1 a^2) - \delta_2 a(\gamma + \alpha_2 K^2 - \beta_2 a^2) \tag{8b}$$

easily by the elimination with

$$\delta_1 \equiv c_g - 2\alpha_1 K\,, \quad \delta_2 \equiv 2\alpha_2 K\,, \quad \delta \equiv \delta_1^2 + \delta_2^2\,. \tag{9}$$

Equation (8a) can be rewritten as

$$\frac{\mathrm{d}a}{a(p + qa^2)} = \frac{\mathrm{d}R}{\delta\epsilon}\,, \tag{10}$$

where

$$p \equiv (\omega\delta_2 - \gamma\delta_1) - (\alpha_1\delta_2 + \alpha_2\delta_1)K^2\,, \tag{11a}$$

$$q \equiv \beta_1\delta_2 + \beta_2\delta_1\,. \tag{11b}$$

Equation (10) can be integrated to give

$$a^2 = \begin{cases} -\dfrac{p}{2q}\,\mathrm{e}^{(p/\delta\epsilon)(R-R_0)}\mathrm{sech}\dfrac{p}{\delta\epsilon}(R - R_0), & a^2 < -\dfrac{p}{q}, \\ -\dfrac{p}{2q}\,\mathrm{e}^{(p/\delta\epsilon)(R-R_0)}\mathrm{csch}\dfrac{p}{\delta\epsilon}(R - R_0), & a^2 > -\dfrac{p}{q}, \end{cases} \tag{12}$$

where R_0 is an integration constant. Equation (12) represents the shock wave solutions for the amplitude of the two-dimensional Ginzburg–Landau equation (1), which is illustrated in Figs 1 and 2.

Equation (8b) is rewritten as

$$\delta\epsilon\frac{\mathrm{d}\phi}{\mathrm{d}R} = r + sa^2 \tag{13}$$

with

$$r \equiv -(\omega\delta_1 + \gamma\delta_2) + (\alpha_1\delta_1 - \alpha_2\delta_2)K^2\,, \tag{14a}$$

$$s \equiv -(\beta_1\delta_1 - \beta_2\delta_2)\,. \tag{14b}$$

Substituting Eq. (12) into Eq. (13) and then integrating yield

$$\phi = \begin{cases} \dfrac{1}{\delta\epsilon}\Big(r - \dfrac{p}{q}s\Big)(R - R_0) - \dfrac{s}{2q}\ln[\mathrm{e}^{(2p/\delta\epsilon)(R-R_0)} + 1]\,, & a^2 < -\dfrac{p}{q}\,, \\ \dfrac{1}{\delta\epsilon}\Big(r - \dfrac{p}{q}s\Big)(R - R_0) - \dfrac{s}{2q}\ln[\mathrm{e}^{(2p/\delta\epsilon)(R-R_0)} - 1]\,, & a^2 > -\dfrac{p}{q}\,, \end{cases} \tag{15}$$

which are the solutions for the shift of phase of complex Ginzburg–Landau equation (1). They show that there is a spiral wave relation between ϕ and R, furthermore, when $R \to +\infty$, there is an Archimede spiral between ϕ and R.

Equation (8a) divided by Eq. (8b) yields

$$\frac{\mathrm{d}a}{a\mathrm{d}\phi} = \frac{p + qa^2}{r + sa^2}\,, \tag{16}$$

which is a differential equation of a and ϕ in the polar coordinates (a, ϕ) and can be integrated to give

$$|qa^2|^{r/2p}|p + qa^2|^{(s/q - r/p)/2} = \mathrm{e}^{\phi - \phi_0}\,, \tag{17}$$

where ϕ_0 is an integration constant. When $s/q - r/p = 0$, equation (17) can be reduced to

$$a = |q|^{-1/2}\,\mathrm{e}^{p(\phi - \phi_0)/r}\,, \tag{18}$$

which represents a spiral and is illustrated in Fig. 3. Thus equation (17) denotes the spiral wave solutions of the complex Ginzburg–Landau equation (1).

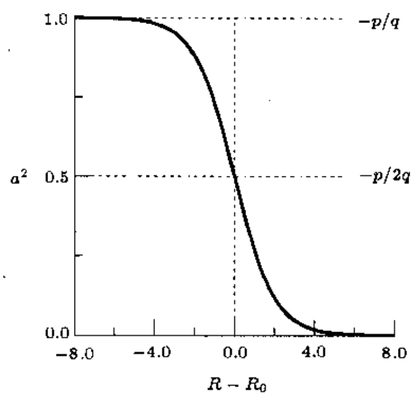

Fig. 1 Shock wave solution for a^2 $(p < 0)$ under the condition of $a^2 < -p/q$.

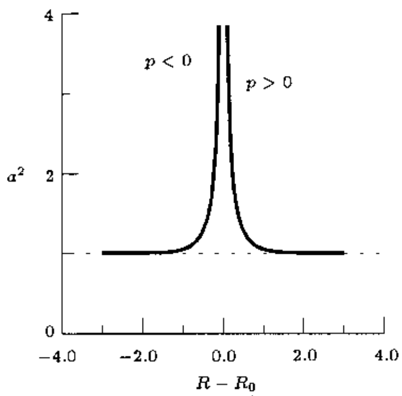

Fig. 2 Shock wave solution for a^2 under the condition of $a^2 > -p/q$.

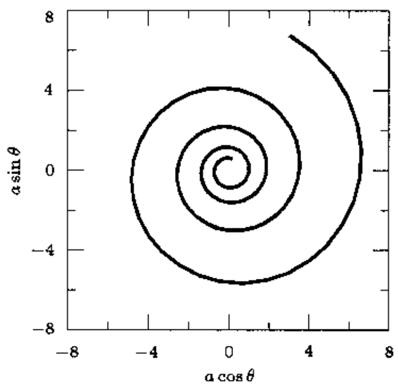

Fig. 3 Spiral wave solution between a and θ.

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