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New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations

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Abstract

New Jacobi elliptic functions are applied in Jacobi elliptic function expansion method to construct the exact periodic solutions of nonlinear wave equations. It is shown that more new periodic solutions can be obtained by this method and more shock wave solutions or solitary wave solutions can be got at their limit condition. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Much effort has been spent on the construction of exact solutions of nonlinear equations, for their important role in understanding the nonlinear problems. Recently, many methods have been proposed, such as the homogeneous balance method [1–3], the hyperbolic tangent expansion method [4–6], the trial function method [7,8], the nonlinear transformation method [9,10] and sine–cosine method [11]. Many exact solutions have been obtained, however, these methods can only obtain the shock and solitary wave solutions and cannot obtain the periodic solutions of nonlinear wave equations. Although Porubov et al. [12–14] have obtained some exact periodic solutions to some nonlinear wave equations, they use the Weierstrass elliptic function and involve complicated deducing. We [15] have proposed the Jacobi elliptic function expansion method and applied it to some nonlinear wave equations. Many periodic solutions based on the Jacobi elliptic sine function finite expansion were obtained by this method, including some shock wave solutions and solitary wave solutions. Further studies show that new periodic solutions can be got in solving some nonlinear wave equations, if we apply different Jacobi elliptic function expansions. In this Letter, we will show the details about these new Jacobi elliptic function expansion and new periodic solutions.

2. Jacobi elliptic function expansion method

Consider a given nonlinear wave equation

$$N\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}, \dots\right) = 0, \tag{1}$$

we seek its wave solutions

$$u = u(\xi), \quad \xi = k(x - ct), \tag{2}$$

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where k and c are the wavenumber and wave speed, respectively.

In [15], $u(\xi)$ is expressed as a finite series of Jacobi elliptic sine function, $\operatorname{sn}\xi$ (a brief introduction to the definition of Jacobi elliptic functions is given in Appendix A), i.e., the ansatz

$$u(\xi)=\sum_{j=0}^{n} a_j \operatorname{sn}^j \xi \tag{3}$$

is made and its highest degree is

$$O\big(u(\xi)\big)=n. \tag{4}$$

Notice that

$$\frac{du}{d\xi}=\sum_{j=0}^{n} j a_j \operatorname{sn}^{j-1}\xi \operatorname{cn}\xi \operatorname{dn}\xi, \tag{5}$$

where $\operatorname{cn}\xi$ and $\operatorname{dn}\xi$ are the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind, respectively. And

$$\operatorname{cn}^2\xi=1-\operatorname{sn}^2\xi, \qquad \operatorname{dn}^2\xi=1-m^2\operatorname{sn}^2\xi \tag{6}$$

with the modulus m $(0<m<1)$. Since

$$\frac{d}{d\xi}\operatorname{sn}\xi=\operatorname{cn}\xi\operatorname{dn}\xi, \qquad \frac{d}{d\xi}\operatorname{cn}\xi=-\operatorname{sn}\xi\operatorname{dn}\xi,$$
$$\frac{d}{d\xi}\operatorname{dn}\xi=-m^2\operatorname{sn}\xi\operatorname{cn}\xi, \tag{7}$$

the highest degree of $du/d\xi$ is taken as

$$O\left(\frac{du}{d\xi}\right)=n+1 \tag{8}$$

and

$$O\left(u\frac{du}{d\xi}\right)=2n+1, \qquad O\left(\frac{d^2u}{d\xi^2}\right)=n+2,$$
$$O\left(\frac{d^3u}{d\xi^3}\right)=n+3. \tag{9}$$

Thus we can select n in (3) to balance the highest order of derivative term and nonlinear term in (1).

We know that, when $m\to 1$, then $\operatorname{sn}\xi\to\tanh\xi$, thus (3) degenerates as the following form:

$$u(\xi)=\sum_{j=0}^{n} a_j \tanh^j \xi. \tag{10}$$

So shock wave or solitary wave solutions can be obtained by the Jacobi elliptic function expansion method, too.

We can get periodic solutions and solitary solutions to some nonlinear wave equations for selected n. Actually, we can get the same periodic solutions (including solitary wave solutions) based on different Jacobi elliptic functions for some nonlinear wave equations, detailed results have been shown in Ref. [15] that the same solution can be expressed in term of different Jacobi elliptic functions. But we can get different periodic solutions and solitary wave solutions based on different Jacobi elliptic functions for some other nonlinear wave equations. We will show the detailed results for two equations, mKdV equation and nonlinear Klein–Gordon equation.

3. mKdV equation

$$\frac{\partial u}{\partial t}+\alpha u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0. \tag{11}$$

Substituting (2) into (11) yields

$$-c\frac{du}{d\xi}+\alpha u^2\frac{du}{d\xi}+\beta k^2\frac{d^3u}{d\xi^3}=0. \tag{12}$$

3.1. Jacobi elliptic sine function expansion

Considering (4), (8) and (9) to balance the highest order of derivative term and nonlinear term in (11), we can get

$$n=1, \tag{13}$$

so the ansatz solution of (11) in term of $\operatorname{sn}\xi$ is

$$u=a_0+a_1\operatorname{sn}\xi. \tag{14}$$

Substituting (14) into (12) yields

$$\big[-c+\alpha a_0^2-\beta\big(1+m^2\big)k^2\big]a_1\operatorname{cn}\xi\operatorname{dn}\xi$$
$$+2\alpha a_0 a_1^2\operatorname{sn}\xi\operatorname{cn}\xi\operatorname{dn}\xi$$
$$+\big(\alpha a_1^2+6\beta m^2k^2\big)a_1\operatorname{sn}^2\xi\operatorname{cn}\xi\operatorname{dn}\xi=0, \tag{15}$$

from which it is determined that

$$a_0=0, \qquad a_1=\pm\sqrt{-\frac{6\beta}{\alpha}}mk,$$
$$c=-\beta\big(1+m^2\big)k^2. \tag{16}$$

Thus the periodic solution of (11) is

$$u = \pm\sqrt{-\frac{6\beta}{\alpha}}mk\,\operatorname{sn}\xi = \pm\sqrt{\frac{6c}{\alpha(1+m^2)}}m\,\operatorname{sn}\sqrt{-\frac{c}{\beta(1+m^2)}}(x-ct), \tag{17}$$

which demands that $c > 0$, $\alpha > 0$, $\beta < 0$ or $c < 0$, $\alpha < 0$, $\beta > 0$. And its corresponding shock wave solution is

$$u = \pm\sqrt{-\frac{6\beta}{\alpha}}k\tanh\xi = \pm\sqrt{\frac{3c}{\alpha}}\tanh\sqrt{-\frac{c}{2\beta}}(x-ct). \tag{18}$$

3.2. Jacobi elliptic cosine function expansion

Apart from the expansion of Jacobi elliptic sine function expansion, other Jacobi elliptic function expansions can also be applied to construct the periodic solutions of nonlinear wave equations. The ansatz solution in term Jacobi elliptic cosine function expansion can be written as

$$u(\xi) = \sum_{j=0}^{n} b_j \operatorname{cn}^j \xi. \tag{19}$$

To balance the highest order of derivative term and nonlinear term in (11), we can get the ansatz solution of (11) in term of $\operatorname{cn}\xi$:

$$u = b_0 + b_1 \operatorname{cn}\xi. \tag{20}$$

Substituting (20) into (12), we have

$$u = \pm\sqrt{\frac{6c}{\alpha(2m^2-1)}}m\,\operatorname{cn}\sqrt{\frac{c}{\beta(2m^2-1)}}(x-ct). \tag{21}$$

This is another periodic solution of (11). For $m \to 1$, $\operatorname{cn}\xi \to \operatorname{sech}\xi$, thus (21) degenerates as the following form:

$$u = \pm\sqrt{\frac{6c}{\alpha}}\operatorname{sech}\sqrt{\frac{c}{\beta}}(x-ct). \tag{22}$$

This is the solitary solution of (11).

3.3. The third kind of Jacobi elliptic function expansion

The ansatz solution in term of the third kind of Jacobi elliptic function expansion can be written as

$$u(\xi) = \sum_{j=0}^{n} c_j \operatorname{dn}^j \xi. \tag{23}$$

To balance the highest order of derivative term and nonlinear term in (11), we can get the ansatz solution of (11) in term $\operatorname{dn}\xi$:

$$u = c_0 + c_1 \operatorname{dn}\xi. \tag{24}$$

Substituting (24) into (12), we have

$$u = \pm\sqrt{\frac{6c}{\alpha(2-m^2)}}\operatorname{dn}\sqrt{\frac{c}{\beta(2-m^2)}}(x-ct). \tag{25}$$

This is another periodic solution of (11). For $m \to 1$, $\operatorname{dn}\xi \to \operatorname{sech}\xi$, thus (25) degenerates as (22), the solitary solution of (11).

3.4. The Jacobi elliptic function $\operatorname{cs}\xi$ expansion

The ansatz solution in term of Jacobi elliptic function $\operatorname{cs}\xi$ expansion can be written as

$$u(\xi) = \sum_{j=0}^{n} d_j \operatorname{cs}^j \xi, \tag{26}$$

where $\operatorname{cs}\xi = \operatorname{cn}\xi/\operatorname{sn}\xi$. To balance the highest order of derivative term and nonlinear term in (11), we can get the ansatz solution of (11) in term $\operatorname{cs}\xi$:

$$u = d_0 + d_1 \operatorname{cs}\xi. \tag{27}$$

Substituting (27) into (12), we have

$$u = \pm\sqrt{-\frac{6c}{\alpha(2-m^2)}}\operatorname{cs}\sqrt{\frac{c}{\beta(2-m^2)}}(x-ct). \tag{28}$$

This is another periodic solution of (11). For $m \to 1$, $\operatorname{cs}\xi \to \operatorname{csch}\xi$, thus (28) degenerates as

$$u = \pm\sqrt{-\frac{6c}{\alpha}}\operatorname{csch}\sqrt{\frac{c}{\beta}}(x-ct), \tag{29}$$

which is another solitary solution of (11).

4. Nonlinear Klein–Gordon equation

We discuss the following nonlinear Klein–Gordon equation:

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} + \alpha u - \beta u^3 = 0. \tag{30}$$

Substituting (2) into (30) yields

$$k^2\left(c^2 - c_0^2\right)\frac{d^2 u}{d\xi^2} + \alpha u - \beta u^3 = 0. \tag{31}$$

Its corresponding ansatz solution in term of $\operatorname{sn}\xi$ is (14) and exact periodic solution can be obtained as

$$u = \pm\sqrt{\frac{2m^2\alpha}{\beta(1+m^2)}}\operatorname{sn}\sqrt{\frac{\alpha}{(c^2-c_0^2)(1+m^2)}}(x-ct), \tag{32}$$

which demands $\alpha > 0$, $\beta > 0$, $c^2 > c_0^2$ or $\alpha < 0$, $\beta < 0$, $c^2 < c_0^2$. Its shock wave solution is

$$u = \pm\sqrt{\frac{\alpha}{\beta}}\tanh\sqrt{\frac{\alpha}{2(c^2-c_0^2)}}(x-ct). \tag{33}$$

The ansatz solution to (31) in term of $\operatorname{cn}\xi$ is (20) and exact periodic solution is

$$u = \pm\sqrt{\frac{2m^2\alpha}{\beta(2m^2-1)}}\operatorname{cn}\sqrt{\frac{\alpha}{(c^2-c_0^2)(2m^2-1)}} \times (x-ct). \tag{34}$$

Its corresponding solitary wave solution is

$$u = \pm\sqrt{\frac{2\alpha}{\beta}}\operatorname{sech}\sqrt{\frac{\alpha}{(c^2-c_0^2)}}(x-ct). \tag{35}$$

The ansatz solution to (31) in term of $\operatorname{dn}\xi$ is (24) and exact periodic solution is

$$u = \pm\sqrt{\frac{2\alpha}{\beta(2-m^2)}}\operatorname{dn}\sqrt{\frac{\alpha}{(c^2-c_0^2)(2-m^2)}}(x-ct), \tag{36}$$

whose corresponding solitary wave solution is (35).

The ansatz solution to (31) in term of $\operatorname{cs}\xi$ is (27) and exact periodic solution is

$$u = \pm\sqrt{-\frac{2\alpha}{\beta(2-m^2)}}\operatorname{cs}\sqrt{-\frac{\alpha}{(c^2-c_0^2)(2-m^2)}} \times (x-ct), \tag{37}$$

whose corresponding solitary wave solution is

$$u = \pm\sqrt{-\frac{2\alpha}{\beta}}\operatorname{csch}\sqrt{-\frac{\alpha}{(c^2-c_0^2)}}(x-ct). \tag{38}$$

5. Conclusion

In this Letter, the Jacobi elliptic function expansion method based on different Jacobi elliptic functions is applied to some nonlinear wave equations. And it is shown that the periodic wave solutions obtained by the Jacobi elliptic function expansion based on different Jacobi elliptic functions may be different, so many new periodic solutions can be got, so many new shock wave or solitary wave solutions can also be obtained.

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Appendix A

Notice that

$$u(t) = \int_0^{\phi} \frac{1}{\sqrt{1-m^2\sin^2\varphi}}\,d\varphi = \int_0^{t\equiv\sin\varphi} \frac{1}{\sqrt{(1-x^2)(1-m^2x^2)}}\,dx \tag{A.1}$$

is called the Legendre elliptic integral of the first kind, where m is a parameter which is known as the modulus. The inverse function $t \equiv \sin\varphi$ is called the Jacobi elliptic sine function which is represented by

$$t = \operatorname{sn} u. \tag{A.2}$$

Similarly, $\sqrt{1-t^2}$ and $\sqrt{1-m^2t^2}$ are defined as the Jacobi elliptic cosine function and Jacobi elliptic function of the third kind, respectively. They are expressed as

$$\sqrt{1-t^2} = \operatorname{cn} u, \qquad \sqrt{1-m^2t^2} = \operatorname{dn} u, \tag{A.3}$$

respectively.

We see from (A.1) that, when $m \to 0$, $\mathrm{sn}\, u$, $\mathrm{cn}\, u$ and $\mathrm{dn}\, u$ degenerate as $\sin u$, $\cos u$ and 1, respectively; while, when $m \to 1$, $\mathrm{sn}\, u$, $\mathrm{cn}\, u$ and $\mathrm{dn}\, u$ degenerate as $\tanh u$, $\mathrm{sech}\, u$ and $\mathrm{sech}\, u$, respectively. Detailed explanations about Jacobi elliptic functions can be found in Refs. [16,17].

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