Equatorial Envelope Rossby Solitons in a Shear Flow①

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ABSTRACT

A simple shallow–water model on an equatorial β–plane is employed to investigate the nonlinear equatorial Rossby solitons in a mean zonal flow with meridional shear by the asymptotic method of multiple scales. The cubic nonlinear Schrödinger (NLS, for short) equation, satisfied for large amplitude equatorial envelope Rossby solitons in shear basic flow, is derived. The effects of basic flow shear on the nonlinear equatorial Rossby solitons are also analyzed.

Key words: Envelope solitons, NLS

1. Introduction

In the last decades, the theory of equatorial waves has become the conceptual cornerstone for the equatorial atmospheric dynamics and provided a dynamical framework for much of our understanding the slowly evolving, meteorologically significant large–scale phenomena in low latitudes. The equatorial waves have been used for various purposes, especially in explaining some fundamental features of tropical climate. The phenomena that have been explained by using the equatorial wave theory include the Walker circulation (Gill, 1980; Lim and Chang, 1983), the low–frequency Madden–Julian oscillation (e.g., Lau and Peng 1987; Wang and Rui, 1990), and the ENSO (Lau and Shen, 1988). However, in the application of the equatorial wave theories noted above, the impacts of the atmospheric basic state on the structures of equatorial waves need to be well understood and have been the subject of a number of studies (Boyd, 1978; Zhang and Webster, 1989). On the other hand, the nonlinear theory of equatorial waves has also received considerable attention (Domaracki and Loesch, 1977; Ripa, 1982; Boyd, 1980, 1983, 1984, 1985). Boyd (1980) applied the method of multiple scales to the primitive equations to show that long, small amplitude Rossby waves evolved in longitude and time according to the nonlinear Korteweg–de Vries (KdV) or modified KdV (mKdV) equation. Kindle's (1983) numerical experiments showed that general wind stresses readily excite solitons and the strong El Niños normally generate a train of two or three solitary waves. Kindle's results add credibility to the purely analytical theory of Boyd (1980) and raise some questions, too. For example, the powerful 1982 El Niño may generate very strong Rossby solitons whose size raises one obvious question is the obvious uncertainty as to whether the analytical theory of Boyd (1980), which was derived through a small amplitude perturbation expansion, can be legitimately applied to the moderate to large amplitude solitary waves created by such a powerful El Niño. Extending the perturbation theory to the next highest order is the simplest analytical way of resolving this uncertainty (Boyd, 1985). However, for KdV– or mKdV–type Rossby solitons, the long–wave

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approximation ($L_y / L_x \ll 1$) must be required. In fact, in the real atmosphere, especially for large–amplitude Rossby waves, the long–wave approximation sometimes may be incorrect. However, for the envelope Rossby solitons depicted by the NLS equation, the long–wave approximation is not necessarily required. In this case, the envelope Rossby soliton model may be more appropriate than the KdV or mKdV soliton models studied by Boyd (1980). On the other hand, Boyd (1980, 1983, 1985) assumed that the mean state was one of rest, thus ignoring mean currents. That is obviously unrealistic, although he believed that this was justified both by the complexity of the theory even without mean flow and by the fact that weak shear will not qualitatively alter most results. To be sure, a weak mean current will change the latitudinal structure, and add some corrections to the speed and amplitude, furthermore, in this paper, it will be shown that the shear mean flow plays an important role in the formation of equatorial Rossby envelope solitons unless the shear is so strong as to create critical latitudes or instability.

The envelope Rossby solitons in the barotropic shear and uniform flows were first investigated by Benney (1979) and Yamagata (1980), independently. Afterward, Luo (1991) tried to use this envelope Rossby soliton to explain atmospheric blocking highs observed in the atmosphere. In order to tackle the questions arising from the observations and theoretical studies in particular, and to gain a better understanding of tropical atmospheric dynamics in general, it is necessary to investigate further the effects of the varying basic states upon the equatorially trapped nonlinear Rossby waves in the atmosphere. The purpose of this study is to examine the effects of basic zonal flows on the large–amplitude equatorial Rossby waves. In the next section, the problem of nonlinear equatorial Rossby waves in a mean zonal flow with meridional shear is resolved by the asymptotic method of multiple scales and the NLS equation is derived. The final section is a summary and discussion.

2. Derivation of the NLS equation

Charney (1963) pointed out that, in the absence of condensation tropical motions are quasi–horizontal and quasi–non–divergent. To keep the discussion as simple as possible, we consider only a simple model equation, important, however, in describing in an idealized way the dynamics of large–scale flows instead of dealing with the complicated, full set of primitive equations. The governing equation is the quasi–geostrophic potential vorticity equation of shallow–water model on an equatorial β–plane. This gives, in standard notation,

$$\left(\frac{\partial}{\partial t}-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\beta y+\nabla^2\psi-\frac{\beta^2 y^2}{c_0^2}\psi\right)=0\ , \tag{1}$$

where x and y are the local Cartesian coordinates pointing east and north, respectively. β is the northward gradient of the planetary vorticity f, c_0^2 is the square of velocity of pure gravity waves. ψ is the stream function of the two–dimensional motion related to the horizontal velocity components by the definitions

$$u=-\frac{\partial\psi}{\partial y}\ ,\quad v=\frac{\partial\psi}{\partial x} \tag{2}$$

and the Laplacian operator ∇^2 is defined by

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\ . \tag{3}$$

Note that the potential vorticity of Eq.(1) is quite different from midlatitude quasigeostrophic potential vorticity $q=f+\nabla^2\psi-\lambda^2\psi$ (where $f=f_0+\beta y$, $\lambda=f_0/c_0$), which is widely used in midlatitude atmosphere dynamics studies. Eq.(1), which is most readily and systematically achieved by introducing nondimensional coordinates and using formal asymptotic expansion techniques (Gill, 1982; Liu, 1990), filters out the high–frequency inertia–gravity waves, mixed Rossby–gravity waves as well as Kelvin waves. As a result, the analysis of Rossby waves is greatly facilitated. Since we mainly discuss the wave motions near the equator or $y\approx 0$, the appropriate boundary condition is that v vanishes as $y\rightarrow\pm\infty$, or equivalently that

$$\frac{\partial\psi}{\partial x}=0\ , \quad \text{as} \quad y\rightarrow\pm\infty\ , \tag{4}$$

which is an approximation of the homogeneous boundary condition on a sphere (i.e. $v=0$ at the poles) (Lindzen, 1967). In the actual atmospheric situations there is upper limit to $|y|$, the position of the pole and boundary conditions should be different ones. However, approximations in the boundary conditions have little effects on the solutions of lower modes. It is convenient to convert Eq.(1) into non–dimensional form by taking the following scaling rules:

$$t=\left(\frac{1}{\beta L}\right)t_*\ , \quad (x,y)=L(x_*,y_*)\ , \quad \psi=(UL)\psi_* \tag{5}$$

where the non–dimensional variables are marked by an asterisk, $L=\sqrt{c_0/\beta}$ is the equatorial Rossby radius of deformation and U the characteristic velocity scale. Substitution of (5) into (1) yields

$$\frac{\partial}{\partial t}\hat{\nabla}^2\psi+\varepsilon J(\psi,\hat{\nabla}^2\psi)+\frac{\partial\psi}{\partial x}=0\ , \tag{6}$$

where

$$J(a,b)=\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}\ , \quad \varepsilon=\frac{U}{\beta L^2}\ , \quad \hat{\nabla}^2=\nabla^2-y^2\ , \tag{7}$$

J is the Jacobian of (a,b),ε is the equatorial equivalent of the conventional Rossby number. Now that the non–dimensional form of Eq.(1) has been derived, the subscript asterisks can be dropped for simplicity. Here, and in the rest of the paper, expressions are written in non–dimensional form and all symbols stand for dimensionless quantities. It should be pointed out that in Benney (1979), the weak shear was chosen to be a perturbation parameter so that a NLS equation, satisfied for large amplitude Rossby wave in the shear basic flow, can be derived by the asymptotic method. But in Yamagata (1980), the local Rossby number was considered as a perturbation parameter. Here, if $U=10$ m s^{-1}, $c_0^2=10^5$ m^2 s^{-2}, then $\varepsilon\sim O(10^{-2})$ is the small–amplitude parameter and naturally is used as the perturbation parameter. The nonlinear problem posed by (6) may be developed in the nondimensional parameter ε which is a measure of the magnitude of nonlinear products. Attention is focused on systems in which nonlinearity and dispersion are of the same order of magnitude, without a loss of generality, long time and space scales are incorporated in (6) by the derivative transformations

$$\frac{\partial}{\partial t}\rightarrow\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial T_1}+\varepsilon^2\frac{\partial}{\partial T_2}\ , \quad \frac{\partial}{\partial x}\rightarrow\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial X_1}+\varepsilon^2\frac{\partial}{\partial X_2}\ , \tag{8}$$

where long time and space scales are defined as

$$T_1 = \varepsilon t \ , \quad T_2 = \varepsilon^2 t; \quad X_1 = \varepsilon x, \quad X_2 = \varepsilon^2 x \ . \tag{9}$$

Substitution of (8) into (6) yields

$$\begin{aligned}&\left\{\left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T_1}\right) + \varepsilon^2\frac{\partial}{\partial T_2} + \varepsilon\left[\left(\frac{\partial \psi}{\partial x} + \varepsilon\frac{\partial \psi}{\partial X_1} + \varepsilon^2\frac{\partial \psi}{\partial X_2}\right)\frac{\partial}{\partial y}\right.\right.\\&\left.\left. - \frac{\partial \psi}{\partial y}\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2}\right)\right]\right\}\times\left[\hat{\nabla}^2 + 2\varepsilon\frac{\partial^2 \psi}{\partial x \partial X_1}\right.\\&\left. + \varepsilon^2\left(\frac{\partial^2 \psi}{\partial X_1^2} + 2\frac{\partial^2 \psi}{\partial x \partial X_2}\right) + 2\varepsilon^3\frac{\partial^2 \psi}{\partial X_1 \partial X_2} + \varepsilon^4\frac{\partial^2 \psi}{\partial X_2^2}\right]\\&+\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2}\right)\psi = 0 \ .\end{aligned} \tag{10}$$

The basic state upon which a wave perturbation is imposed is a time−independent zonal flow (with a overbar) independent of the x coordinate. In the presence of a small perturbation (denoted by a prime) the total stream function is

$$\begin{aligned}\psi &= -\int^y \bar{u}(s)ds + \psi' \\ &= -\int^y \bar{u}(s)ds + \sum_{m=1}^{\infty} \varepsilon^m \psi_m(x,y,t;X_1,X_2,T_1,T_2) \ .\end{aligned} \tag{11}$$

In the above expression, the perturbation stream function ψ' is assumed to have uniformly valid asymptotic expansions of the power series. The choice of expansions (11) and (8) is based upon the uniform occurrence of $O(\varepsilon)$ nonlinear products in (6) and the desire to match anticipated nonlinear forcing with long time and space scales in the same power of ε. Substitution of (11) into (10) yields the system of equations

$O(\varepsilon)$:

$$\mathscr{L}(\psi_1) = 0 \ , \tag{12}$$

$O(\varepsilon^2)$:

$$\begin{aligned}\mathscr{L}(\psi_2) = &-\left(\frac{\partial}{\partial T_1} + \bar{u}\frac{\partial}{\partial X_1}\right)\hat{\nabla}^2\psi_1 - (1-\bar{u}'')\frac{\partial \psi_1}{\partial X_1}\\&-2\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\frac{\partial^2 \psi_1}{\partial x \partial X_1} - J(\psi_1, \hat{\nabla}^2\psi_1) \ ,\end{aligned} \tag{13}$$

$O(\varepsilon^3)$:

$$\begin{aligned}\mathscr{L}(\psi_3) = &-\left(\frac{\partial}{\partial T_2} + \bar{u}\frac{\partial}{\partial X_2}\right)\hat{\nabla}^2\psi_1 - (1-\bar{u}'')\frac{\partial \psi_1}{\partial X_2} - 2\left(\frac{\partial}{\partial T_1} + \bar{u}\frac{\partial}{\partial X_1}\right)\frac{\partial^2 \psi_1}{\partial x \partial X_1}\\&- J\left(\psi_1, 2\frac{\partial^2 \psi_1}{\partial x \partial X_1}\right) - \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi_1}{\partial X_1^2} + 2\frac{\partial^2 \psi_1}{\partial x \partial X_2} + 2\frac{\partial^2 \psi_2}{\partial x \partial X_1}\right)\\&- \frac{\partial \psi_1}{\partial X_1}\frac{\partial}{\partial y}\hat{\nabla}^2\psi_1 + \frac{\partial \psi_1}{\partial y}\frac{\partial}{\partial X_1}\hat{\nabla}^2\psi_1 - \left(\frac{\partial}{\partial T_1} + \bar{u}\frac{\partial}{\partial X_1}\right)\hat{\nabla}^2\psi_2\\&- (1-\bar{u}'')\frac{\partial \psi_2}{\partial X_1} - J(\psi_1, \hat{\nabla}^2\psi_2) - J(\psi_2, \hat{\nabla}^2\psi_1) \ ,\end{aligned} \tag{14}$$

and so forth, where the operator $\mathscr{L}(\ )$ is defined as

$$\mathscr{L}(\ )=\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\hat{\nabla}^2+(1-\bar{u}'')\frac{\partial}{\partial x}\ . \tag{15}$$

Obviously, ψ_1 satisfies the linear system (12), which by hypothesis has a solution

$$\psi_1=A(X_1,X_2;T_1,T_2)\Phi_1(y)\exp[i(kx-\omega t)]+\text{c.c.} \tag{16}$$

where the wave amplitude A is assumed to be a smooth slowly varying function of the long time and space scales, $\Phi_1(y)$ describes the latitudinal modal structure of the packet, k is the zonal wave number, ω is the wave angular frequency, c.c. is an abbreviation for " complex conjugate" of its preceding term. Substituting (16) into (12) leads to the following equation for Φ_1:

$$\left[L_1+\frac{k(1-\bar{u}'')}{\hat{\omega}}\right]\Phi_1=0\ , \tag{17}$$

where the operator L_1 is defined by

$$L_1=\frac{d^2}{dy^2}-k^2-y^2\ , \tag{18}$$

and $\hat{\omega}$ is the Doppler–shifted frequency, which will be determined as the eigenvalue of Eq.(17), is related to ω by the Doppler relation

$$\hat{\omega}=k\bar{u}-\omega=k(\bar{u}-c)\ . \tag{19}$$

Eq.(17) under the boundary condition $\Phi_1|_{y\to\pm\infty}=0$ poses a standard Sturm–Liouville problem. The effects of zonal flows on linear equatorial trapped waves were treated in detail by many researchers (e.g. Zhang and Webster, 1989). If the effects of zonal flows are neglected (i.e. $\bar{u}=0$), then (17) reduces to an eigen–value problem of the so–called Weber equation, just the same as the Schrödinger equation for a simple harmonic oscillator. The solutions of Eq.(17) exist if and only if the coefficients satisfy the condition

$$-k^2-\frac{k}{\omega}=2n+1\qquad (n=0,1,2,\cdots)\ , \tag{20}$$

where n is the latitudinal mode number. Eq.(20) is the eigenvalues of (17), and the corresponding eigenfunctions are

$$\Phi_1(y)=C_nH_n(y)\exp\left(-\frac{1}{2}y^2\right)\qquad (n=0,1,2,\cdots)\ , \tag{21}$$

where C_n is an arbitrary constant specifying the amplitude of the nth mode and $H_n(y)$ is the nth order Hermite polynomial. The dispersion relationships for the frequency ω and phase speed c of free equatorial Rossby oscillations for given n are easily derived from Eq.(20):

$$\omega=-\frac{k}{k^2+2n+1}\ ,\qquad c=-\frac{1}{k^2+2n+1}\qquad (n=0,1,2,\cdots)\ . \tag{22}$$

Obviously, this is a special case of the Matsuno (1966)'s results. it can be seen that the high–frequency inertia–gravity waves, mixed Rossby–gravity waves as well as Kelvin waves have already been filtered out in Eq.(1). For long Rossby waves for which $k\approx O(\varepsilon^{1/2})$, $\varepsilon\ll 1$ it follows that

$$c(k,n)-c(0,n)\approx O(\varepsilon)\ . \tag{23}$$

In other words, waves whose wavenumbers differ by $O(1)$ will have phase speeds which differ only by a small amount. Eq.(23) can be taken as the mathematical definition of " weak" dispersion. It implies that even if a packet of long Rossby waves is not sharply peaked in wavenumber space about some central packet wavenumber k, the packet nonetheless will spread very slowly, on time scale $T \approx O(\varepsilon^{-1})$, because its constituent waves are all traveling at approximately the same speed. If the amplitude of the packet is also $O(\varepsilon)$, then one can balance the nonlinearity against the dispersion to create the singly–humped soliton. In Boyd (1980), it was shown that the zeroth–order equations describe linear, nondispersive Rossby waves, at the next order, the effects of dispersion and nonlinearity enter simultaneously and the KdV or mKdV equation is obviously derived as a long–wave approximation. With this approximation, high–frequency inertia–gravity waves, mixed Rossby–gravity waves as well as short Rossby waves are filtered out, therefore the KdV or mKdV is a consistent description of ultra–long Rossby waves of small amplitude. However, the assumption of long–wave approximation depends on not only the scale of motions, but also the ratio of the meridional scale to the zonal scale (i.e., $L_y / L_x \ll 1$).

The ε–order solution, therefore, fixes the frequency $\hat{\omega}$, even though A is still undetermined. To this order, the solution is an arbitrary superposition of linear ($\varepsilon \to 0$) Rossby waves, at the next order, the effects of nonlinearity enter. Insertion of (16) into (13) and using (17) yields

$$\begin{aligned}\mathscr{L}(\psi_2) = &-\left[\left(\frac{\partial A}{\partial T_1} + \bar{u}\frac{\partial A}{\partial X_1}\right)L_1\Phi_1 + (1-\bar{u}''-2k\hat{\omega})\Phi_1\frac{\partial A}{\partial X_1}\right]\exp[i(kx-\omega t)\\ &- ik\left(\Phi_1\frac{d}{dy}L_1\Phi_1 - \frac{d\Phi_1}{dy}L_1\Phi_1\right)A^2\exp[2i(kx-\omega t)] + \text{c.c}\ .\\ = &\frac{1-\bar{u}''}{\bar{u}-c}\Phi_1\left(\frac{\partial A}{\partial T_1} + c_1\frac{\partial A}{\partial X_1}\right)\exp[i(kx-\omega t)]\\ &+ ikQ(y)A^2\exp[2i(kx-\omega t)] + \text{c.c.}\ ,\end{aligned} \tag{24}$$

where

$$c_1 = c + \frac{2k^2(\bar{u}-c)^2}{1-\bar{u}''}\ , \qquad Q(y) = \Phi_1^2\frac{d}{dy}\left(\frac{1-\bar{u}''}{\bar{u}-c}\right)\ . \tag{25}$$

Solutions of the homogeneous part of (24) subject to (4) are identical to the $O(\varepsilon)$ solutions described by (17) in their (x,y,t) functional structure. Inspection of the linear inhomogeneous terms of (24) reveals that for ψ_2 there exists a resonant forcing arising from the introduction of multiple scales. That is, the first term on the right–hand side of (24), which has the structure of the free Rossby waves packet, is resonant with the linear operator of the left–hand side of (24). Nonlinear inhomogeneous terms may also be resonant under special conditions. The assumed validity of (11) is assured by demanding that inhomogeneous terms of (24) are orthogonal to the solutions of the homogeneous equation. Orthogonality exists when the following condition is satisfied:

$$\lim_{\substack{\tau\to\infty\\ x\to\infty}}\frac{1}{2\tau x}\int_{-x}^{x}\int_{-\infty}^{+\infty}\int_0^{\tau}(\psi_1)[\text{inhomogeneous terms of (24)}]dxdydt = 0\ . \tag{26}$$

For the purpose of integration, the amplitude coefficients of A are held constant during integration over (x,y,t). The effect of this strategy is to force A to vary in a manner making (26) true, in this way, the presently unknown functional dependence of A will be determined at the next higher order approximations. Therefore, using the orthogonality condition of the left–hand side operator to avoid secular growth, we obtain the following equation

$$\frac{\partial A}{\partial T_1}+c_g\frac{\partial A}{\partial X_1}=0\ , \tag{27}$$

where

$$c_g=c+\frac{I_1}{I}\ ,\qquad I_1=2k^2\int_{-\infty}^{+\infty}\Phi_1^2dy\ ,\qquad I=\int_{-\infty}^{+\infty}\frac{1-\bar{u}''}{(\bar{u}-c)^2}\Phi_1^2dy\ . \tag{28}$$

It is clear from (27) that for the $O(\varepsilon^2)$ problem, the amplitude A propagates at the group velocity. The remaining inhomogeneous terms on the right–hand side of (24) are nonresonant and the following particular solution is yielded:

$$\mathscr{L}(\psi_2)=ikQ(y)A^2\exp[2i(kx-\omega t)]+\text{c.c.}\ , \tag{29}$$

we assume that (29) has the following wave–like solution

$$\psi_2=B(X_1,X_2;T_1,T_2)\Phi_2(y)\exp[2i(kx-\omega t)]+\text{c.c.} \tag{30}$$

Substitution (30) into (29) yields the equation to Φ_2

$$B\left[L_2+\frac{k(1-\bar{u}'')}{\hat{\omega}}\right]\Phi_2=\frac{A^2Q(y)}{2(\bar{u}-c)}\ , \tag{31}$$

and its boundary condition $\Phi_2\big|_{y\to\pm\infty}=0$, where the operator L_2 is defined by

$$L_2=\frac{d^2}{dy^2}-(2k)^2-y^2\ . \tag{32}$$

Obviously, B and A^2 are not two independent variables, B is related to A^2 by Eq.(31). For simplicity, we assumed that

$$B=A^2\ . \tag{33}$$

Substitution (16) and (30) into (14) results in the following solution:

$$\begin{aligned}\mathscr{L}(\psi_3)=&-\Bigg\{\Bigg[\left(\frac{\partial A}{\partial T_2}+\bar{u}\frac{\partial A}{\partial X_2}\right)L_1+(1-\bar{u}'')\frac{\partial A}{\partial X_2}+2ik\left(\frac{\partial^2 A}{\partial T_1\partial X_1}+i\hat{\omega}\frac{\partial A}{\partial X_2}\right)\\&+i(\hat{\omega}+2k\bar{u})\frac{\partial^2 A}{\partial X_1^2}\Bigg]\Phi_1-ikA^*B\Bigg[\left(\Phi_1\frac{d}{dy}+2\frac{d\Phi_1}{dy}\right)L_2\Phi_2\\&-\left(2\Phi_2\frac{d}{dy}+\frac{d\Phi_2}{dy}\right)L_1\Phi_1\Bigg]\Bigg\}\exp[i(kx-\omega t)]+\text{c.c.}+\square\ ,\end{aligned} \tag{34}$$

where A^* denotes the complex conjugate of A, □ stands for those terms which are associated with $\exp[\pm 2i(kx-\omega t)]$ and $\exp[\pm 3i(kx-\omega t)]$. Using (17), (27), (31) and (33), then (34) can be rewritten as follows:

$$\begin{aligned}\mathscr{L}(\psi_3)=&\Bigg\{\frac{1-\bar{u}''}{\bar{u}-c}\left[\frac{\partial A}{\partial T_2}+c_1\frac{\partial A}{\partial X_2}+ik\frac{\bar{u}-c}{1-\bar{u}''}(c+2c_g-3\bar{u})\frac{\partial^2 A}{\partial X_1^2}\right]\Phi_1\\&+ik\left[\frac{\Phi_1}{2}\frac{d}{dy}\left(\frac{Q}{\bar{u}-c}\right)+\frac{Q}{(\bar{u}-c)}\frac{d\Phi_1}{dy}\right]|A|^2A\Bigg\}\\&\times\exp[i(kx-\omega t)]+\text{c.c.}+\square\ .\end{aligned} \tag{35}$$

It is not necessary to solve for ψ_3. Instead solvability conditions associated with (35) will establish equations which determine the evolution of the envelope amplitude and the mean flow. Obviously, the inhomogeneities on the right–hand side of (35) may be evaluated in terms of the lower order solutions and again those inhomogeneities contain terms which are homogeneous solutions of the operator on the left. To avert resonance and hence to keep our expansion (11) uniformly valid in time, the inhomogeneity must be orthogonal to the homogeneous solution. This solvability conditions determine that the free amplitude A must satisfy the following NLS equation:

$$i\left(\frac{\partial A}{\partial T_2}+c_g\frac{\partial A}{\partial X_2}\right)+\alpha\frac{\partial^2 A}{\partial X_1^2}+\delta|A|^2A=0\ , \tag{36}$$

where

$$\alpha=\frac{I_2}{I}\ ,\quad \delta=\frac{I_3}{I}\ ,\quad I_2=k\int_{-\infty}^{+\infty}\frac{(c+2c_g-3\bar{u})}{\bar{u}-c}\Phi_1^2\,dy\ ,$$

$$I_3=k\int_{-\infty}^{+\infty}\frac{1}{(\bar{u}-c)}\left[\frac{\Phi_1^2}{2}\frac{d}{dy}\left(\frac{Q}{\bar{u}-c}\right)+\frac{\Phi_1 Q}{(\bar{u}-c)}\frac{d\Phi_1}{dy}\right]dy\ , \tag{37}$$

where α and δ are the so–called dispersion and Landau coefficients respectively. In the above derivation, we have assumed that any critical level does not exist, i.e. $\bar{u}\neq c$. The set of Eqs.(17), (36) and (37) determines completely the modal structure of a nonlinear equatorial Rossby envelope soliton in a shear basic flow with no critical level.

Note that the dispersion and Landau coefficients α and δ vanish as $k\rightarrow 0$, this indicates that the ultra–long, nondispersive ($c_g=c$, see (22) and (28)) Rossby waves cannot be described by the NLS equation (36). In the limit $k\rightarrow 0$, the proper replacement equation is the KdV or mKdV equation of Boyd (1980). If we introduced the following coordinate transformation defined by Jeffrey and Kawahara (1982):

$$T=T_2\ ,\qquad X=\frac{1}{\varepsilon}(X_2-c_gT_2)=X_1-c_gT_1\ , \tag{38}$$

then Eq.(36) can be transformed into the canonical form

$$i\frac{\partial A}{\partial T}+\alpha\frac{\partial^2 A}{\partial X^2}+\delta|A|^2A=0\ . \tag{39}$$

3. Discussion and conclusions

In this paper, the asymptotic technique is used to investigate nonlinear equatorial Rossby waves in a mean zonal flow with meridional shear by employing a simple shallow–water model on an equatorial β–plane. The nonlinear NLS equation (39), which describes the amplitude evolution of the equatorial Rossby solitons, and also embodies the main characteristics of nonlinear equatorial Rossby solitons in a shear basic flow, was derived. Because the coefficients α and δ are related to the states of basic flow $\bar{u}$, if there is no shear in the basic flow (i.e. $\bar{u}$=constant), then $Q=0$ (see (25)) and $\delta=0$ (see (37)), and the nonlinear terms in the nonlinear NLS equation (39) disappear. This indicates that a necessary condition for the formation of equatorial envelope Rossby solitons is the nonlinear interaction between the equatorial Rossby waves and the shear basic flows.

The NLS is completely integrable (Zakhrov and Shabat, 1972). When the signs of the dispersive and nonlinear terms in the NLS equation (39) are opposite ($\alpha\delta < 0$), the asymptotic solution is simply a wavetrain qualitatively similar to the linear solution. But the most striking difference is that the nonlinearity also acts to widen the wave packet, giving "defocusing" or "superlinear" dispersion in the sense that the wavetrain spreads out more rapidly than in a linear theory. Instability is possible if and only if the Landau constant δ and the dispersive coefficient α in the NLS equation (39) are of same sign ($\alpha\delta < 0$)—which is also the necessary condition for solitary waves. In this case, the asymptotic solution to the NLS equation consists of a wavetrain plus a finite number of envelope solitary waves. For the special case of the single soliton, this gives the two–parameter family of traveling single solitons.

$$A(X,T)=\sqrt{\frac{2\alpha}{\delta}}M\operatorname{sech}M(X-2\alpha\xi T)\exp\{i[\xi X-\alpha(\xi^2-M^2)T]\}\ , \tag{40}$$

where the amplitude–related parameter M, and velocity–related parameter ξ are independent parameters and are determined by the initial condition. Ablowitz et al. (1974) showed that no solitons will form if $\int_{-\infty}^{+\infty}|A(X,0)|dX < 0.904$. This is in marked contrast to the KdV equation where at least one soliton will form no matter how small the initial disturbance, so long it is of the right sign. Substituting (40) and (16) into (11), we obtain the stream function of the equatorial Rossby envelope solitons

$$\psi=-\int^{y}\overline{u}(s)ds+\sqrt{\frac{2\alpha}{\delta}}M\operatorname{sech}M(x-Vt)\Phi_1(y)\exp[i(Kx-\Omega t)]\ , \tag{41}$$

where

$$V=c_g+2\varepsilon\alpha\xi\ ,\qquad K=k+\varepsilon\xi\ ,\qquad \Omega=\omega+\varepsilon\xi c_g+\alpha\varepsilon^2(\xi^2-M^2)\ . \tag{42}$$

From (28) and (37), we also note that the existence of envelope solitons also requiring the following condition must be met:

$$I=\int_{-\infty}^{+\infty}\frac{1-\overline{u}''}{(\overline{u}-c)^2}\Phi_1^2\,dy\neq 0\ . \tag{43}$$

This indicates that the shear is not so strong as to create barotropic instability (Kuo, 1949). In fact, if the instability occurs, the permanent shape of solitons cannot be maintained. It is not difficult to see that the dipole envelope Rossby soliton exhibits an isolated dispersive structure having a shape. We also note that the condition $k\to 0$ is not required in deriving the envelope soliton solution. However, for KdV or mKdV soliton the condition $k\to 0$ must be required because the long wave approximation is used. For this case, the KdV or mKdV soliton exhibits a nondispersive isolated structure of sech^2 shape (Malguzzi and Malanotte–Rizzoli, 1984). Without forcing, the KdV– or mKdV–type soliton cannot properly model the physical mechanism of the equatorial modons breakdown. In the real atmosphere, because the nonlinearity exists, the envelope amplitude of Rossby waves is not usually a constant. Especially for large–amplitude Rossby waves, the long–wave approximation sometimes may be incorrect. For an envelope soliton the long–wave approximation requirement is not necessary. Therefore, the envelope soliton studied here is physically more reasonable for the westward–traveling modons events of the KdV soliton (Boyd, 1985). Moreover, numerical experiments on the Rossby solitons evolution and application of the model proposed in this paper to numerical simulations of some coherent flow structures in the tropical atmosphere will be published elsewhere.

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切变基流中赤道 Rossby 包络孤立波

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利用一个简单的赤道 β 平面浅水模式和多尺度摄动法，从描写赤道 Rossby 波的正压大气位涡度方程中推导出在切变基本维向流中赤道 Rossby 波包演变所满足的非线性 Schrödinger 方程，并得到其单个包络孤立波子波解，并分析基本流切变对非线性赤道 Rossby 孤立子的影响。

关键词: 包络孤立波，非线性 Schrödinger 方程