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The Homoclinic Orbit Solution for Functional Equation*

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Abstract *In this paper, some examples, such as iterated functional systems, scaling equation of wavelet transform, and invariant measure system, are used to show that the homoclinic orbit solutions exist in the functional equations too. And the solitary wave exists in generalized dynamical systems and functional systems.*

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1 Introduction

We have shown that the solitary wave solutions exist in conservative or dissipative partial differential equations, where the solitary wave corresponds to the homoclinic orbit for ordinary differential equations.[1−5] Because we want to know whether homoclinic orbit can exist in functional equations, we will show in this paper that the homoclinic orbit exists in iterated functional systems, scaling equation systems in wavelet transform and invariant measure systems.

2 Iterated Functional System

Early in 1989, Prof. Hao[6] pointed out that the homoclinic orbit exists in iterated functional systems.

The logistic map

$$x_{n+1} = f(x_n) = 4x_n(1 - x_n)\,, \tag{1}$$

as shown in Fig. 1 obviously has two unstable fixed points $x^* = 0$ and $x^* = 3/4$. Starting from $x = 1/2$, twice iteration will make it arrive at the unstable fixed point $x^* = 0$, i.e., it leads to definite number sequence

$$1/2,\ 1,\ 0,\ 0,\ 0,\ \ldots\,. \tag{2}$$

In other words, the forward iteration falls into the unstable fixed point $x^* = 0$. The backward iterations, i.e. from x_{n+1} to x_n, return to the same fixed point $x^* = 0$ too.

Therefore, the iterative sequence from $x = 1/2$ forms as follows:

$$0,\ 0\,, \cdots,\ 1/2,\ 1,\ 0,\ 0\,. \tag{3}$$

This is a homoclinic orbit which approaches $x^* = 0$ as a limit when $n \to \pm\infty$. In Fig. 1, it is denoted by thick lines with arrows. The points 1/2 and 1 are called homoclinic points, which are immersed into the stable set of $x^* = 0$ and the unstable set of $x^* = 0$, and they are homoclinical to $x^* = 0$.

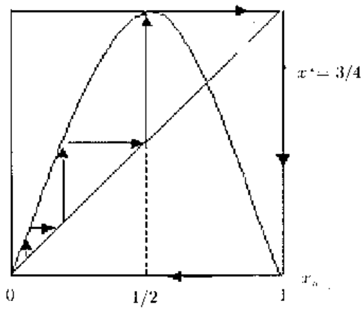

Fig. 1 The homoclinic orbit in map Eq. (1).

3 Scaling Equation for Wavelet Transform

Wavelet transform is a powerful multi-resolution analysis tool.[7,8] The father wavelet or scaling function $\phi(t)$ has one particularly desirable property: it is 0 everywhere except a small closed interval.

For example, Haar father wavelet is

$$\phi(t) = \begin{cases} 1\,, & 0 \le t \le 1\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{4}$$

as shown in Fig. 2. Because $\phi(t)$ approaches to 0 as $t \to \pm\infty$, then $\phi(t)$ is a homoclinic orbit.

It is easily proved that $\phi(t)$ satisfies the following scaling equation (or two-scale relation):[9]

$$\phi(t) = \phi(2t) + \phi(2t-1)\,. \tag{5}$$

From Fig. 2, we see that the left-hand side of Eq. (5) is a square wave, the scale of the first term $\phi(2t)$ on the right-hand side of Eq. (5) is the half as large as $\phi(t)$, and the second term $\phi(2t-1)$ is the result that $\phi(2t)$ shifts rightwards by half unit.

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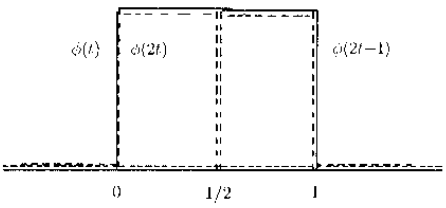

Fig. 2 Haar father wavelet.

In addition, scaling function for tent map

$$\phi(t)=\begin{cases} t\,, & 0\le t<1\,,\\ t-2\,, & 1\le t<2\,,\\ 0\,, & \text{otherwise}\,,\end{cases} \tag{6}$$

is also a homoclinic orbit. It is the solution of the following functional equation

$$\phi(t)=\frac{1}{2}\phi(2t)+\phi(2t-1)+\frac{1}{2}\phi(2t-2)\,. \tag{7}$$

The quadratic Battle Lemarie scaling function

$$\phi(t)=\begin{cases} \frac{1}{2}t^2\,, & 0\le t<1\,,\\ -t^2+3t-\frac{3}{2}\,, & 1\le t<2\,,\\ \frac{1}{2}(t-3)^2\,, & 2\le t<3\,,\\ 0\,, & \text{otherwise}\,,\end{cases} \tag{8}$$

is a continuous and differentiable homoclinic orbit and satisfies the following functional equation

$$\phi(t)=\frac{1}{4}\phi(2t)+\frac{3}{4}\phi(2t-1)+\frac{3}{4}\phi(2t-2)+\frac{1}{4}\phi(2t-3). \tag{9}$$

4 Invariant Measure of One-Dimensional Map

If $s(x)$ is a one-dimensional map in unit interval $[0,1]$, then density evolution equation of the probability density function $f(x)$ for the successive iterated sequence x_0, $s(x_0)$, $s^2(x_0)$, ... can be written as[10,11]

$$Pf(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_{S^{-1}([0,x])}f(u)\mathrm{d}u\,, \tag{10}$$

where P is called Frobenius Perron operator, S^{-1} is an inverse map of s.

For example, the tent map

$$S(x)=\begin{cases} 2x\,, & 0\le x<\frac{1}{2}\,,\\ 2(1-x)\,, & \frac{1}{2}\le x<1\,,\end{cases} \tag{11}$$

whose invariant measure $f(x)$ computed from Eq. (10) is

$$Pf(x)=\frac{1}{2}f\Big(\frac{x}{2}\Big)+\frac{1}{2}f\Big(\frac{x}{2}-1\Big), \tag{12}$$

that is, $f(x)$ satisfies the functional equation

$$f(x)=\frac{1}{2}\Big[f\Big(\frac{x}{2}\Big)+f\Big(\frac{x}{2}-1\Big)\Big], \tag{13}$$

which is similar to Eq. (5). The scale of $f(x)$ is the half of $f(x/2)$. The solution of Eq. (13) is

$$f(x)=\begin{cases} 1\,, & 0\le x<1\,,\\ 0\,, & \text{otherwise}\,,\end{cases} \tag{14}$$

which is a homoclinic orbit too.

In addition, the dyadic transform is

$$S(x)=\begin{cases} 2x\,, & 0\le x<\frac{1}{2}\,,\\ 2x-1\,, & \frac{1}{2}\le x<1\,,\end{cases} \tag{15}$$

whose density function $f(x)$ is the evolution equation

$$Pf(x)=\frac{1}{2}f\Big(\frac{x}{2}\Big)+\frac{1}{2}f\Big(\frac{x}{2}+\frac{1}{2}\Big), \tag{16}$$

i.e., $f(x)$ satisfies functional equation

$$f(x)=\frac{1}{2}f\Big(\frac{x}{2}\Big)+\frac{1}{2}f\Big(\frac{x}{2}+\frac{1}{2}\Big). \tag{17}$$

The solution of Eq. (17) is

$$f(x)=\begin{cases} 1\,, & 0\le x<1\,,\\ 0\,, & \text{otherwise}\,,\end{cases} \tag{18}$$

which is also a homoclinic orbit.

So, it is obvious that there exists a homoclinic orbit among the iterated functional systems, the scaling equation systems in wavelet transform and invariant measure systems. It is a generalized homoclinic orbit.

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