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A New Approach to Solve Nonlinear Wave Equations*

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Abstract *From the nonlinear sine-Gordon equation, new transformations are obtained in this paper, which are applied to propose a new approach to construct exact periodic solutions to nonlinear wave equations. It is shown that more new periodic solutions can be obtained by this new approach, and more shock wave solutions or solitary wave solutions can be got under their limit conditions.*

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1 Generalized Transformations from sine-Gordon Equation

The sine-Gordon equation reads

$$u_{tt} - c_0^2 u_{xx} + f_0^2 \sin u = 0\,, \tag{1}$$

which can be solved in the following frame

$$u = u(\xi), \qquad \xi = x - ct\,, \tag{2}$$

where c is a wave velocity. Then equation (1) becomes

$$(c^2 - c_0^2)\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + f_0^2 \sin u = 0\,. \tag{3}$$

Integrating this equation, we get

$$\Big(\frac{\mathrm{d}\omega}{\mathrm{d}\xi}\Big)^2 + \frac{f_0^2}{c^2 - c_0^2}\sin^2\omega = \frac{H}{2}\,, \tag{4}$$

where H is integration constant, $\omega = u/2$. There are two cases to be considered.

Case 1 $c^2 > c_0^2$

Set $\lambda_0^2 = f_0^2/(c^2 - c_0^2)$ and $H = 2\lambda_0^2 m^2$, equation (4) can be rewritten as

$$\frac{\mathrm{d}\omega}{\mathrm{d}\xi} = \lambda_0\sqrt{m^2 - \sin^2\omega}\,. \tag{5}$$

Equation (5) is the first generalized transformation which we get from the nonlinear sine-Gordon equation.

Case 2 $c^2 < c_0^2$

Similarly, we can get

$$\frac{\mathrm{d}\omega}{\mathrm{d}\xi} = \lambda_1\sqrt{m'^2 - \cos^2\omega}\,, \tag{6}$$

where $\lambda_1^2 = -\lambda_0^2$ and $m'^2 = 1 - m^2$. This is the second generalized transformation we get from the nonlinear sine-Gordon equation.

In Ref. [1], based on the sine-Gordon equation, Yan got a transformation

$$\frac{\mathrm{d}\omega}{\mathrm{d}\xi} = \sin\omega\,. \tag{7}$$

We can see that the transformation (7) is just a special case of transformation (6) when '+' is taken in Eq. (6) and $\lambda_1 = 1$, $m'^2 = 1$.

2 A New Approach to Solve Nonlinear Equations

In the following, we will introduce another method based on the transformations given in the former section. Consider a given nonlinear wave equation

$$N(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0\,. \tag{8}$$

We seek its wave solutions in the frame of Eq. (2), then equation (8) can be rewritten as

$$N\Big(u, \frac{\mathrm{d}u}{\mathrm{d}\xi}, \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2}, \ldots\Big) = 0\,, \tag{9}$$

and $u(\xi)$ can be expressed as a finite series of $\sin\omega$ or $\cos\omega$, i.e. the ansatz

$$u = u_1(\xi) = \sum_{j=0}^{n} a_j \cos^j\omega\,, \tag{10a}$$

$$u = u_2(\xi) = \sum_{j=0}^{n} b_j \sin^j\omega\,, \tag{10b}$$

where ω satisfies transformation (5) or (6), then

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}\xi^2} = -\lambda_0^2 \cos\omega\sin\omega\,, \quad \text{or} \tag{11a}$$

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}\xi^2} = \lambda_1^2 \cos\omega\sin\omega\,. \tag{11b}$$

The highest degree of Eq. (10) is

$$\mathrm{O}(u(\xi)) = n\,, \tag{12}$$

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then the highest degree of $\mathrm{d}u/\mathrm{d}\xi$ can be taken as

$$\mathrm{O}\Big(\frac{\mathrm{d}u}{\mathrm{d}\xi}\Big) = n+1\,, \tag{13}$$

and

$$\mathrm{O}\Big(u\frac{\mathrm{d}u}{\mathrm{d}\xi}\Big) = 2n+1, \qquad \mathrm{O}\Big(\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}\Big) = n+2\,,$$
$$\mathrm{O}\Big(\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}\Big) = n+3\,. \tag{14}$$

Thus we can select n in Eq. (10) to balance the highest order of derivative term and nonlinear term in Eq. (9). Then substituting Eq. (10) into Eq. (9), determining the expansion coefficients and other undetermined constants, combining the results from the transformation (5) or (6), one can get exact solutions to the given nonlinear equations.

We know that when $m' \to 1$, the transformation (6) degenerates as the transformation (7), so the solutions got from the above expansion may cover the results obtained by sine-cosine method given by Ref. [1].

3 Applications

In this paper, we will demonstrate the above approach on two examples: KdV equation and mKdV equation.

3.1 *mKdV Equation*

The mKdV equation reads

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0\,. \tag{15}$$

Substituting Eq. (2) into Eq. (15) yields

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi} + \alpha u^2\frac{\mathrm{d}u}{\mathrm{d}\xi} + \beta\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} = 0\,. \tag{16}$$

Integrating this equation yields

$$-cu + \frac{\alpha}{3}u^3 + \beta\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = C_0\,, \tag{17}$$

where C_0 is integration constant.

Considering Eqs. (12), (13), and (14) to balance the highest order of derivative term and nonlinear term in Eq. (17), we can get

$$n = 1\,, \tag{18}$$

so the ansatz solution of Eq. (15) in terms of $\cos\omega$ is

$$u = a_0 + a_1\cos\omega\,, \tag{19}$$

and ω satisfies transformation (5), we know that

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = -a_1\sin\omega\frac{\mathrm{d}\omega}{\mathrm{d}\xi}\,, \tag{20}$$

$$\begin{aligned}\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} &= -a_1\sin\omega\frac{\mathrm{d}^2\omega}{\mathrm{d}\xi^2} - a_1\cos\omega\Big(\frac{\mathrm{d}\omega}{\mathrm{d}\xi}\Big)^2 \\ &= \lambda_0^2(2-m^2)a_1\cos\omega - 2\lambda_0^2 a_1\cos^3\omega\,,\end{aligned} \tag{21}$$

$$u^3 = a_0^3 + 3a_0^2a_1\cos\omega + 3a_0a_1^2\cos^2\omega + a_1^3\cos^3\omega\,. \tag{22}$$

So substituting Eq. (19) into Eq. (17) yields

$$\begin{aligned}&[-ca_0 + \alpha a_0^3/3 - C_0] + [-ca_1 + \alpha a_0^2 a_1 \\ &\quad + \beta\lambda_0^2(2-m^2)a_1]\cos\omega + \alpha a_0 a_1^2\cos^2\omega \\ &\quad + [\alpha a_1^3/3 - 2\beta\lambda_0^2 a_1]\cos^3\omega = 0\,,\end{aligned} \tag{23}$$

from Eq. (23) setting the coefficients of $(\cos\omega)^0$, $\cos\omega$, $\cos^2\omega$ and $\cos^3\omega$ to be zero, we can get the algebraic equations about a_0, a_1, C_0, and c

$$-ca_0 + \alpha a_0^3/3 - C_0 = 0\,, \tag{24a}$$
$$-ca_1 + \alpha a_0^2 a_1 + \beta\lambda_0^2(2-m^2)a_1 = 0\,, \tag{24b}$$
$$\alpha a_0 a_1^2 = 0\,, \tag{24c}$$
$$\alpha a_1^3/3 - 2\beta\lambda_0^2 a_1 = 0\,. \tag{24d}$$

Solving Eqs. (24a), (24b), (24c), and (24d) yields the following solutions

$$C_0 = 0, \qquad a_0 = 0\,,$$
$$a_1 = \pm\sqrt{\frac{6\lambda_0^2\beta}{\alpha}}\,, \qquad c = (2-m^2)\lambda_0^2\beta\,. \tag{25}$$

Similarly, the ansatz solution of Eq. (15) in terms of $\sin\omega$ is

$$u = b_0 + b_1\sin\omega\,, \tag{26}$$

where ω satisfies transformation (5). The corresponding solution is

$$C_0 = 0, \qquad b_0 = 0\,,$$
$$b_1 = \pm\sqrt{-\frac{6\lambda_0^2\beta}{\alpha}}\,, \qquad c = -\lambda_0^2(1+m^2)\beta\,. \tag{27}$$

Actually, setting $\sin\omega = m\sin\varphi$, equation (5) is rewritten as

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \lambda_0\sqrt{1-m^2\sin^2\varphi}\,. \tag{28}$$

Notice that

$$\begin{aligned}\theta(\tau) &= \int_0^{\Psi}\frac{1}{\sqrt{1-m^2\sin^2\psi}}\mathrm{d}\psi \\ &= \int_0^{\tau\equiv\sin\psi}\frac{1}{\sqrt{(1-y^2)(1-m^2y^2)}}\mathrm{d}y\end{aligned} \tag{29}$$

is called the first kind Legendre elliptic integral, where m is a parameter which is known as the modulus. The inverse function $\tau \equiv \sin\varphi$ is called the Jacobi elliptic sine function which is represented by

$$\tau = \sin\psi = \mathrm{sn}\,\theta\,. \tag{30}$$

So from Eq. (28) we know that the transformation (5) admits the following solution

$$\sin\omega = m\,\mathrm{sn}\,(\lambda_0\xi, m)\,. \tag{31}$$

Similarly, $\sqrt{1-\tau^2}$ and $\sqrt{1-m^2\tau^2}$ are defined as the Jacobi elliptic cosine function and the third kind Jacobi elliptic function, respectively. They are expressed as

$$\sqrt{1-\tau^2} = \mathrm{cn}\,\theta, \qquad \sqrt{1-m^2\tau^2} = \mathrm{dn}\,\theta\,, \tag{32}$$

respectively. There are relations between $\mathrm{sn}\,\theta$, $\mathrm{cn}\,\theta$, and $\mathrm{dn}\,\theta$

$$\mathrm{sn}^2\theta + \mathrm{cn}^2\theta = 1, \quad 1 - m^2\mathrm{sn}^2\theta = \mathrm{dn}^2\theta\,. \tag{33}$$

Detailed explanations about Jacobi elliptic functions can be found in Refs. [2] and [3].

Then we get

$$\cos\omega = \mathrm{dn}(\lambda_0\xi, m)\,. \tag{34}$$

Similarly, the transformation (6) admits the following solution

$$\cos\omega = m'\mathrm{sn}(\lambda_1\xi, m')\,, \tag{35}$$

and then we get

$$\sin\omega = \mathrm{dn}(\lambda_1\xi, m')\,, \tag{36}$$

where m' is co-modulus.

Thus the periodic solutions of Eq. (15) are

$$u_1 = a_1\cos\omega = \pm\sqrt{\frac{6\lambda_0^2\beta}{\alpha}}\,\mathrm{dn}\lambda_0(x - ct)\,, \tag{37}$$

and

$$u_2 = b_1\sin\omega = \pm\sqrt{-\frac{6\lambda_0^2\beta}{\alpha}}\,m\,\mathrm{sn}\lambda_0(x - ct)\,. \tag{38}$$

And from Eqs. (29) and (33) we know that when $m \to 0$, $\mathrm{sn}u$, $\mathrm{cn}u$, and $\mathrm{dn}u$ degenerate as $\sin u, \cos u$ and 1, respectively; while when $m \to 1$, $\mathrm{sn}u, \mathrm{cn}u$ and $\mathrm{dn}u$ degenerate as $\tanh u$, $\mathrm{sech} u$ and $\mathrm{sech} u$, respectively. So the solutions (37) and (38) degenerate as another two solutions

$$u_3 = \pm\sqrt{\frac{6\lambda_0^2\beta}{\alpha}}\,\mathrm{sech}\lambda_0(x - ct)\,, \tag{39}$$

and

$$u_4 = \pm\sqrt{-\frac{6\lambda_0^2\beta}{\alpha}}\,\tanh\lambda_0(x - ct)\,, \tag{40}$$

which are shock wave solution and solitary wave solution, respectively.

3.2 *KdV Equations*

The KdV equation reads

$$u_t + uu_x + \beta u_{xxx} = 0\,. \tag{41}$$

We solve it in the frame (2), so the system (41) can be rewritten as

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi} + u\frac{\mathrm{d}u}{\mathrm{d}\xi} + \beta\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} = 0\,. \tag{42}$$

Integrating this equation yields

$$-cu + u^2/2 + \beta\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = A\,, \tag{43}$$

where A is integration constant, here set to be zero.

We suppose the ansatz solution to Eq. (43) is Eq. (10), where ω satisfies the transformation (5). Substituting Eq. (10) into Eq. (43) to balance the nonlinear term and the highest degree differential term gives $n = 2$.

So the ansatz solution to Eq. (41) is

$$u_1 = a_0 + a_1\cos\omega + a_2\cos^2\omega\,, \tag{44a}$$

$$u_2 = b_0 + b_1\sin\omega + b_2\sin^2\omega\,. \tag{44b}$$

Substituting the ansatz solution (44a) into Eq. (43) results in

$$\begin{aligned}[-ca_0 &+ a_0^2/2 + 2\beta\lambda_0^2(m^2 - 1)a_2] + [-ca_1 + a_0a_1 \\ &+ \beta\lambda_0^2(2 - m^2)a_1]\cos\omega + [-ca_2 + (a_1^2 + 2a_0a_2)/2 \\ &+ 4\beta\lambda_0^2(2 - m^2)a_2]\cos^2\omega + [a_1a_2 - 2\beta\lambda_0^2a_1]\cos^3\omega \\ &+ [a_2^2/2 - 6\beta\lambda_0^2a_2]\cos^4\omega = 0\,. \end{aligned}\tag{45}$$

Setting the coefficients of $(\cos\omega)^0$, $\cos\omega$, $\cos^2\omega$, $\cos^3\omega$ and $\cos^4\omega$ to be zero, we can get the algebraic equations about expansion coefficients and c

$$-ca_0 + a_0^2/2 + 2\beta\lambda_0^2(m^2 - 1)a_2 = 0\,, \tag{46a}$$

$$-ca_1 + a_0a_1 + \beta\lambda_0^2(2 - m^2)a_1 = 0\,, \tag{46b}$$

$$-ca_2 + (a_1^2 + 2a_0a_2)/2 + 4\beta\lambda_0^2(2 - m^2)a_2 = 0\,, \tag{46c}$$

$$a_1a_2 - 2\beta\lambda_0^2a_1 = 0\,, \tag{46d}$$

$$a_2^2/2 - 6\beta\lambda_0^2a_2 = 0\,, \tag{46e}$$

from which we can get

$$\begin{aligned} &a_0 = c - 4\beta\lambda_0^2(2 - m^2)\,, \\ &a_1 = 0, \qquad a_2 = 12\beta\lambda_0^2\,, \\ &c^2 = 16\beta^2\lambda_0^4(1 - m^2 + m^4)\,. \end{aligned}\tag{47}$$

Similarly, substituting ansatz solution (44b) into Eq. (43) yields

$$\begin{aligned} &b_0 = c + 4\beta\lambda_0^2(1 + m^2)\,, \\ &b_1 = 0, \qquad b_2 = -12\beta\lambda_0^2\,, \\ &c^2 = 16\beta^2\lambda_0^4(1 - m^2 + m^4)\,. \end{aligned}\tag{48}$$

Thus we can get the solutions to Eq. (41)

$$\begin{aligned} u_1 &= a_0 + a_2\cos^2\omega \\ &= c - 4\beta\lambda_0^2(2 - m^2) + 12\beta\lambda_0^2\mathrm{dn}^2\lambda_0(x - ct)\,, \end{aligned}\tag{49}$$

$$\begin{aligned} u_2 &= b_0 + b_2\sin^2\omega \\ &= c + 4\beta\lambda_0^2(1 + m^2) - 12\beta\lambda_0^2m^2\mathrm{sn}^2\lambda_0(x - ct)\,. \end{aligned}\tag{50}$$

When $m \to 1$, the limit solutions are obtained,

$$u_3 = c - 4\beta\lambda_0^2 + 12\beta\lambda_0^2\mathrm{sech}^2\lambda_0(x - ct)\,, \tag{51}$$

$$u_4 = c + 8\beta\lambda_0^2 - 12\beta\lambda_0^2\tanh^2\lambda_0(x - ct)\,. \tag{52}$$

4 Conclusion

There are many methods proposed to solve nonlinear equations, such as the sine-cosine method,[1] the homogeneous balance method,[4−6] the hyperbolic tangent expansion method,[7−9] the Jacobi elliptic function expansion method,[10,11] the nonlinear transformation method,[12,13] the trial function method[14,15] and others.[16−18] In this letter, a new approach based on the new transformation from nonlinear sine-Gordon equation is proposed to construct the exact solutions to nonlinear equations. And it is shown that the periodic wave solutions obtained by this method can degenerate to generalized solitary wave solutions, so many new shock wave or solitary wave solutions can also be obtained. Actually, this method can be applied to more nonlinear equations, such as $(2m+1)$-order KdV equations, Kawahara equation, modified Kawahara equation, Benjamin–Bona–Mahony equation, and so on.

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