Radiative corrections to the standard $SU_2 \times U_1$ electro-weak theory.

1. There are 17 free parameters in the Standard Model, with 3 families. Discard the fermion masses, we get 4 free parameters, as

$$(g, g', \mu, \lambda) \qquad \text{or} \qquad (g, \theta, M, m_H).$$

Because of the screening theorem, most of the processes are not sensitive to the unknown physical Higgs mass m_H. One may consider to determine the remaining 3 free parameters, e.g. (g, θ, M), by comparing with the experimental data

1) Veltman chose the 3 free parameters as

$$g, \quad S_\theta, \quad \text{and} \quad M \qquad \left(\begin{array}{l} M = W\text{-boson mass} \\ \theta = \text{weak mixing angle} \\ g = SU_2 \text{ coupling strength} \end{array}\right)$$

Then he invented the ρ-parameter, defined as

$$\rho \equiv \frac{M_p^2}{M_{op}^2 C_\theta^2}, \qquad \left(\begin{array}{l} S_\theta \equiv \sin\theta \\ C_\theta = \cos\theta \end{array}\right)$$

to investigate the heavy Higgs and heavy fermion effects. Where M_p is the physical mass of $W^\pm$-boson, M_{op} is the physical mass of Z-boson (W^0), and C_θ is determined through the low energy experiment ($\overset{(-)}{\nu} e \to \overset{(-)}{\nu} e$).
(The physical mass is defined as the pole of the propagator)

The deviation of ρ from 1 will give us some information about the Higgs sector or the mass splitting of the heavy fermions, etc.

* In Veltman's definition,

the tree relation

$$e = g\, s_\theta$$

will be modified at loop level,

so that $$4\pi\alpha = g^{(1)^2} s_\theta^{(1)^2} (1+\delta_2)$$

2) Marciano chose another set of free parameters as

$$e, \quad M_p \quad \text{and} \quad M_{op}.$$

Where M_p (M_{op}) is the physical mass of $W^\pm$ (Z^0).

e is defined as

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137.03604}.$$

The consequencies of choosing this set of parameters are:

① All of these parameters are physical parameters. Therefore they are gauge invariant (gauge independent).

② The weak mixing angle is defined as

$$C_\theta \equiv \frac{M_p}{M_{op}}.$$

Thus $S_\theta \equiv \sqrt{1 - C_\theta^2} = \sqrt{1 - \frac{M_p^2}{M_{op}^2}}$.

Again S_θ is gauge independent.

③ ρ parameter is 1 by definition

$$\rho \equiv \frac{M_p^2}{M_{op}^2 C_\theta^2} = 1.$$

Note: Here C_θ is defined through $C_\theta \equiv \frac{M_p}{M_{op}}$. This is different from the C_θ shown in Veltman definition. There C_θ is obtained through low energy experiments. We will come back to this point later.

* In Marciano's definition:

electric charge

$$e = g S_\theta .$$

Both e and g are gauge-invariant quantities,

S_θ is S_θ through the definition of

$$C_\theta = \frac{M_p}{M_{op}}$$

④ If one allow both (ρ) and (s_θ) to vary to fit the experimental data, then one should expect $\rho = 1$. The deviation of ρ from 1 indicates the deviation from standard model.

In practice, one can fit the experimental data of the ratios

$$R = \frac{\overset{(-)}{\nu}_\mu N \to \overset{(-)}{\nu}_\mu N}{\overset{(-)}{\nu}_\mu N \to \mu^{\mp} N} \qquad \text{(neutral current and charged current ratio)}$$

$$= \overset{(-)}{f}(\rho, s_\theta)$$

to fix ρ and s_θ simultaneously.

⑤ Marciano called the definition in ③ as one-parameter fit (i.e. fix s_θ only).

And he called the method shown in ④ as two-parameter fit (ie fix ρ and s_θ simultaneously).

3) In this work, we are going to use the parametrization shown in 1), ie choosing the 3 free parameters as

g, s_θ and M.

2. We need three experimental data points to fix the three parameters we chose, ie. g, s_θ and M.

We follow the ones Veltman chose:

(i) Coulomb scattering ($e^-\mu^- \to e^-\mu^-$) to define α.

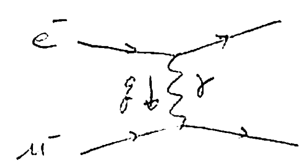
(ii) μ-decay ($\mu^- \to \nu_\mu e^- \bar{\nu}_e$) to define Fermi constant G_F.

(iii) The ratio of $\sigma(\bar{\nu}_\mu e)$ to $\sigma(\nu_\mu e)$ to define the weak mixing angle θ.

Note: All these processes are low energy experiments.

To demostrate how to use the above three processes to fix the three parameters, we work out the results at tree level as follows:

1) Coulomb scattering.



At low energy, there is only one diagram matters.

Its amplitude is

$$\mathcal{M} = \frac{(ie)^2}{q^2}\, \bar{u}_\mu \gamma_\alpha u_\mu \, \bar{u}_e \gamma_\alpha u_e \, .$$

From the precise measurement of differential cross section in $q^2 \to 0$, one can determine

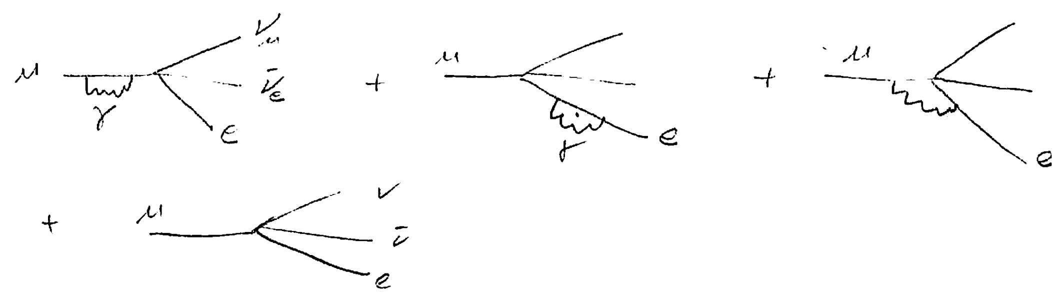
$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137.036}$$

2) μ-decay:

The decay rate of $\mu^- \to \nu_\mu \bar{\nu}_e e^-$ is

$$\Gamma_\mu = \frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1 - \frac{8 m_e^2}{m_\mu^2}\right)\left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right].$$

Here we already include the 1-loop QED corrections, which is defined as the 1-loop four-fermion interactions. The Feynman diagrams are



Take the numerical values

$m_e = 0.511$ MeV

$m_\mu = 105.65916$ MeV

$\alpha = \dfrac{1}{137.036}$

$\tau_\mu = 2.19703 \times 10^{-6}$ sec

$1\ \text{sec}^{-1} = 6.582 \times 10^{-22}$ MeV

we get

$$\frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right) = -0.004204,$$

$$\frac{8 m_e^2}{m_\mu^2} = 0.000187$$

$$\frac{1}{\tau_\mu} = \frac{6.582}{2.197} \times 10^{-16}\ \text{MeV}$$

$$G_F = \begin{cases} 1.164 \times 10^{-8} \times 10^{-3}\ \text{MeV}^{-2} = 1.164 \times 10^{-5}\ \text{GeV}^{-2} & \text{(not including QED)} \\ 1.166 \times 10^{-8} \times 10^{-3}\ \text{MeV}^{-2} = 1.166 \times 10^{-5}\ \text{GeV}^{-2} & \text{(including QED)} \end{cases}$$

We therefore define the fermion constant as

$$G_F = 1.166 \times 10^{-5}\ \text{GeV}^{-2}$$

Note: The 1-loop QED (4-fermion interaction) corrections have been included.

3) $\nu_\mu e$ and $\bar{\nu}_\mu e$ scattering.

At tree level, the total cross sections

$$\sigma(\nu_\mu e) = \frac{8 G_F^2}{\pi} m_e E_\nu (g_L^\nu)^2 \left[(g_L^e)^2 + \frac{1}{3}(g_R^e)^2 \right],$$ and

$$\sigma(\bar{\nu}_\mu e) = \frac{8 G_F^2}{\pi} m_e E_\nu (g_L^\nu)^2 \left[\frac{1}{3}(g_L^e)^2 + (g_R^e)^2 \right].$$

where
$$g_L^f = T_3(f_L) - Q(f) \sin^2\theta ,$$
$$g_R^f = T_3(f_R) - Q(f) \sin^2\theta ,$$

i.e.
$$g_L^e = \frac{-1}{2} + s_\theta^2 .$$
$$g_R^e = s_\theta^2 .$$

Therefore

$$R \equiv \frac{\sigma(\bar{\nu}_\mu e \to \bar{\nu}_\mu e)}{\sigma(\nu_\mu e \to \nu_\mu e)} = \frac{\frac{1}{3}(g_L^e)^2 + (g_R^e)^2}{(g_L^e)^2 + \frac{1}{3}(g_R^e)^2} .$$

Let us express it in terms of another quantity:

The neutral-current part Lagrangian is defined as

$$\mathcal{L}^{NC} = \frac{ig}{c_\theta} Z_\mu \left[g_L^f \bar{f}_L \gamma_\mu f_L + g_R^f \bar{f}_R \gamma_\mu f_R \right]$$
$$= \frac{ig}{2c_\theta} Z_\mu \bar{f} \gamma_\mu \left[(g_L^f + g_R^f) + (g_L^f - g_R^f)\gamma_5 \right] f$$
$$\equiv \frac{ig}{4c_\theta} Z_\mu \bar{f} \gamma_\mu (a + b\gamma_5) f .$$

Hence

$$\left.\begin{aligned} g_L^e + g_R^e &= \frac{a}{2} \\ g_L^e - g_R^e &= \frac{b}{2} \end{aligned}\right\} \Rightarrow \begin{aligned} g_L^e &= \frac{1}{4}(a+b) = \frac{b}{4}\left(1+\frac{a}{b}\right) = \frac{b}{4}(1+\xi) \\ g_R^e &= \frac{1}{4}(a-b) = \frac{b}{4}\left(\frac{a}{b}-1\right) = \frac{b}{4}(\xi-1) \end{aligned}$$

And

$$a = -1 + 4s^2$$
$$b = -1$$

$$\xi \equiv \frac{a}{b}$$

$$R \equiv \frac{\sigma(\bar{\nu}_\mu e)}{\sigma(\nu_\mu e)} = \frac{\frac{1}{3}(1+\xi)^2 + (1-\xi)^2}{(1+\xi)^2 + \frac{1}{3}(1-\xi)^2} \times \frac{3}{3}$$

$$= \frac{1 - \xi + \xi^2}{1 + \xi + \xi^2}, \quad \text{with} \quad \xi = \frac{a}{b} = 1 - 4 S_\theta^2.$$

<u>Note</u>: If we define $\xi = \frac{b}{a}$ instead of $\xi = \frac{a}{b}$, then we will have the same relation as $R = \frac{1-\xi+\xi^2}{1+\xi+\xi^2}$. Therefore we should look for the roots of ξ so that $|S_\theta| \leq 1$.

From the experimental data of R, one can extract the value of ξ, therefore get

$$S_\theta^2 = \frac{1}{4}(1-\xi), \qquad \xi \simeq 0.12 \qquad \left(\text{For } R \simeq \frac{1}{1.27}\right)$$

$$\simeq 0.22$$

3. Once we have the three experimental data points, we need to solve these three equations to fix the three free parameters:

(1) $\alpha = \frac{1}{137.036}$

(2) $G_F = 1.166 \times 10^{-5}\ \text{GeV}^{-2}$

(3) $\frac{1}{4}(1-\xi) = 0.22 \qquad (\xi \equiv 1 - 4 S_\theta^2, \text{ and } \xi \simeq 0.12)$

1) Before we solve these equations, we should recall some relations in the free Lagrangian.

In tree Lagrangian

$$e = g s_\theta \,, \qquad \frac{1}{4}(1-\xi)^2 = s_\theta^2$$

$$G_F = \frac{g^2}{4\sqrt{2}} \frac{1}{M^2} \,.$$

And

$$M_0^2 = \frac{M^2}{c_\theta^2} \,. \qquad \left(c_\theta^2 = 1 - s_\theta^2 = 1 - \frac{1}{4}(1-\xi)^2 \right).$$

Therefore, we have

(1) $$\alpha = \frac{e^2}{4\pi} = \frac{g^2 s_\theta^2}{4\pi} = \frac{1}{137.036}$$

(2) $$\frac{g^2}{4\sqrt{2}} \frac{1}{M^2} = 1.166 \times 10^{-5} \ GeV^{-2}$$

(3) $$\frac{1}{4}(1-\xi) = s_\theta^2 = 0.22 \,.$$

2) From (3), we get $s_\theta^2 = 0.22 = (0.469)^2$

Put in (1), we get

$$g^2 = \frac{1}{137.036} \frac{4\pi}{s_\theta^2} = (0.6456)^2 \,.$$

put in (2), we have

$$M^2 = \frac{g^2}{4\sqrt{2} \cdot (1.166)} \times 10^5 \ GeV^2 = (79.5 \ GeV)^2 \,.$$

Also

$$M_0^2 = \frac{M^2}{c_\theta^2} = (90.0 \ GeV)^2$$

We therefore get the numbers of these three free parameters.

$$g = 0.6456$$

$$s_\theta = 0.469$$

$$M = 79.5 \ GeV$$

Consequently, the mass of Z^0 is obtained through the relation

$$M_0 = \frac{M}{c_\theta} = 90.0 \ (GeV)$$

4. We have established how to fix the three parameters, g, S_θ and M from the three specific experimental data points in the tree level.

However, to test the Standard Model, one should check the radiative corrections also.
The procedures are the same as what we did in tree level, but here we should deal with the "bare parameters" instead of the finite quantities.

1) After doing the loop calculations, one get the infinite quantities δ_1, δ_2 and δ_3, so that

$$(1) \quad \alpha = \frac{g^{(1)^2} S_\theta^{(1)^2}}{4\pi}\left[1+\delta_2\right]$$

$$(2) \quad G_F = \frac{g^{(1)^2}}{4\sqrt{2}M^{(1)^2}}\left[1+\delta_1\right]$$

$$(3) \quad \frac{1}{4}(1-\xi) = S_\theta^{(1)^2}\left[1-\delta_3\right] \equiv \left(S_\theta^{ren}\right)^2 \equiv \left(S_\theta^{physical}\right)^2$$

Where the bare parameters g, S_θ and M are infinite, such that the left-hand side quantities, ie. α, G_F and ξ, are finite.
This is the consequences of a renormalized theory.

Now the logic is somewhat different from previous sections.

We are going to work out the three experimental data points to get the three bare parameters up to one loop.

* We will choose our renormalization scheme (ie choose counterterms) so that

$$\frac{1}{4}(1-\xi) = S_\theta^{(1)^2}(1-\delta_3)$$
$$\equiv \left(S_\theta^{\text{Ren}}\right)^2$$
$$\equiv \left(S_\theta^{\text{physical}}\right)^2$$
$$= \left(S_\theta^{\text{tree}}\right)^2$$

Therefore, $S_\theta^{\text{tree}} = S_\theta^{\text{1-loop renormalized}}$.

i.e. The value of S_θ will not get corrected due to the loop correction.

(Note: This is purely due to the way we chose our renormalization scheme. Obviously, we can choose the other way.)

Thus when we talk about the shift of ρ, we only have to consider the shifts of

M and M_0, not C_θ.

So

$$\delta\rho = \frac{\delta M_p^2}{M_p^2} - \frac{\delta M_{0p}^2}{M_{0p}^2},$$

(The shift quantity is defined as the difference between 1-loop and free results.)

for

$$\rho \equiv \frac{M_p^2}{M_{0p}^2 C_\theta^2}$$

Then the one-loop results of any physical process calculated using these three bare parameters should give a finite answer.

2) Solve the above three equations, one can get the three bare parameters as

$$(3) \qquad S_\theta^{(1)\,2} = \frac{1}{4}(1-\xi)\frac{1}{1-\delta_3} = \frac{1}{4}(1-\xi)(1+\delta_3)$$

$$(1) \qquad g^{(1)\,2} = \frac{4\pi\alpha}{S_\theta^{(1)\,2}}\frac{1}{1+\delta_2} = \frac{4\pi\alpha}{\frac{1}{4}(1-\xi)(1+\delta_3)}(1-\delta_2) = \frac{4\pi\alpha}{\frac{1}{4}(1-\xi)}(1-\delta_2-\delta_3)$$

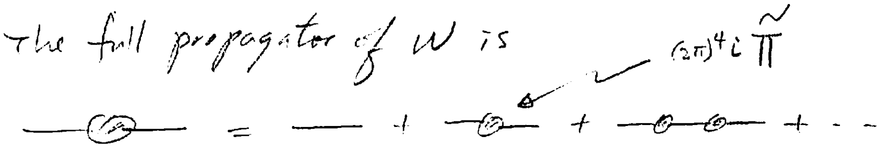
$$(2) \qquad M^{(1)\,2} = \frac{g^{(1)\,2}}{4\sqrt{2}\,G_F}[1+\delta_1] = \frac{4\pi\alpha}{\sqrt{2}(1-\xi)G_F}(1+\delta_1-\delta_2-\delta_3)$$

<u>Note</u>: ① Since we choose the three free parameters as g, S_θ and M, therefore the electric charge $e = \sqrt{4\pi\alpha}$ is <u>not</u> a bare parameter. It is finite.

② The bare parameters $S_\theta^{(1)}$, $g^{(1)}$ and $M^{(1)}$ are superindexed to explicitly express them as infinite quantities.

③ $M^{(1)}$ is <u>not</u> the physical mass of W-boson. The physical mass of W-boson is defined as the pole of W's propagator.

The full propagator of W is



$$= \frac{1}{(2\pi)^4 i}\frac{1}{p^2+M^{(1)2}} + \frac{1}{(2\pi)^4 i}\frac{1}{p^2+M^{(1)2}}(2\pi)^4 i\,\tilde{\Pi}\frac{1}{(2\pi)^4 i}\frac{1}{p^2+M^{(1)2}} + \cdots$$

$$= \frac{1}{(2\pi)^4 i}\frac{1}{p^2+M^{(1)2}}\left\{1 + (2\pi)^4 i\,\tilde{\Pi}\frac{1}{(2\pi)^4 i}\frac{1}{p^2+M^{(1)2}} + \cdots\right\}$$

$$= \left\{\left[1-\tilde{\Pi}\frac{1}{p^2+M^{(1)2}}\right](2\pi)^4 i\,(p^2+M^{(1)})^2\right\}^{-1}$$

$$= \frac{1}{(2\pi)^4 i}\frac{1}{\{p^2+M^{(1)2}-\tilde{\Pi}\}}$$

Therefore the pole of the propagator is at

$$M_p^2 = M^{(1)2} - \tilde{\pi} \,,$$

(Note. The self energy of W is $— \circledcirc — \equiv \Pi \equiv (2\pi)^4 i \, \tilde{\pi}$)

$$= \frac{4\pi\alpha}{\sqrt{2}(1-\xi)G_F}(1+\delta_1-\delta_2-\delta_3) - \tilde{\pi}(-M_p^2) \,.$$

The argument of $\tilde{\pi}$ indicates that one should evaluate $p^2 = -M_p^2$, not $M^{(1)2}$. However, the self-energy of W, ie $\tilde{\pi}$, is proportional to $\frac{g^2}{\pi^2}$, therefore we can treat the right-hand side as the tree quantities. In other words

$$M_p^2 = \frac{4\pi\alpha}{\sqrt{2}(1-\xi)G_F}(1+\delta_1-\delta_2-\delta_3) - \tilde{\pi}(-M^2)$$

where $M^2 = (79.5 \text{ GeV})^2$ is the result from tree calculations. Similarly, g, S_θ and M in $\delta_1, \delta_2, \delta_3$ or $\tilde{\pi}$ should be put to be their tree values when doing the numerical calculations (e.g. mass shift of W).

The same arguments hold for the three eqs.

(1) $$g^{(1)2} = \frac{4\pi\alpha}{\frac{1}{4}(1-\xi)}(1-\delta_2-\delta_3)$$

(2) $$M^{(1)2} = \frac{4\pi\alpha}{\sqrt{2}(1-\xi)G_F}(1+\delta_1-\delta_2-\delta_3)$$

(3) $$S_\theta^{(1)2} = \frac{1}{4}(1-\xi)(1+\delta_3)$$

On the right-hand side, α, G_F and ξ are the data from the experimental measurements. All the factors g, S_θ and M show on the righ-hand side should be put to be their $\underline{\underline{\text{tree}}}$ values.

(4) One can also ask for the mass shift of Z°.
First, one should remember the mass of Z°, M_0, is <u>not</u> a free parameter in this particular prescription. Therefore one should express $M_0 = \frac{M}{C_\theta}$ everywhere in the Feynman rules.

Therefore the mass of Z° is defined as the pole of Z° propagator, which is

$$\frac{1}{(2\pi)^4 i}\,\frac{1}{\left\{ p^2 + \frac{M^{(1)2}}{C_\theta^{(1)2}} - \tilde{\Pi}_0 \right\}} .$$

Therefore Z° mass is

$$M_{op}^2 = \frac{M^{(1)2}}{C_\theta^{(1)2}} - \tilde{\Pi}_0(-M_{op}^2),$$

with

$$\frac{1}{C_\theta^{(1)2}} = \frac{1}{1 - S_\theta^{(1)2}} = \frac{1}{1 - \frac{1}{4}(1-\xi)(1+\delta_3)} = \frac{1}{1-\frac{1}{4}(1-\xi)}\left(1 + \frac{\frac{1}{4}(1-\xi)}{1-\frac{1}{4}(1-\xi)}\delta_3\right).$$

$$\left(\text{Since } \frac{1}{1-a(1+\varepsilon)} = \frac{1}{1-a-a\varepsilon} = \frac{1}{(1-a)(1-\frac{a}{1-a}\varepsilon)} = \frac{1}{1-a}\left(1 + \frac{a}{1-a}\varepsilon\right), \quad \varepsilon \ll 1 \right)$$

Hence

$$M_{op}^2 = \frac{4\pi\alpha}{\sqrt{2}(1-\xi)G_F\left[1-\frac{1}{4}(1-\xi)\right]}\left\{1 + \delta_1 - \delta_2 - \delta_3 + \frac{\frac{1}{4}(1-\xi)}{1-\frac{1}{4}(1-\xi)}\delta_3\right\} - \tilde{\Pi}_0(-M_{op}^2)$$

Notice that on the righ-hand side $M_0^2 = (90.0\ \text{GeV})^2$ is the tree value.
Also

$$\frac{\frac{1}{4}(1-\xi)}{1-\frac{1}{4}(1-\xi)} = \left[\frac{S_\theta^2}{C_\theta^2}\right]_{\text{tree}} .$$

Therefore, one can replace $\{\ \}$ by $\left\{1 + \delta_1 - \delta_2 - \delta_3 + \frac{S_\theta^2}{C_\theta^2}\delta_3\right\}$, as long as one remember that all the parameters on the right-hand side are the quantities obtained from tree calculations.

3) We have got the physical masses of W and Z^0 at one loop. Then we ask what is the physical quantity s_θ^2.
Recall that in eq. (3), $s_\theta^{(1)2}$ is a bare parameter, which is infinite.

Before we go on and answer this question, recall that Marciano defined $C_\theta^M \equiv \frac{M_p}{M_{op}}$. Substituting the results of M_p and M_{op}, one get

$$C_\theta^M = \left[1 - \frac{1}{4}(1-\xi)\right]\left\{1 - \frac{1-\xi}{3+\xi}\delta_3 - \frac{\tilde{\pi}(-M^2)}{M^2} + \frac{\tilde{\pi}_0(-M_0^2)}{M_0^2}\right\}.$$

Now, let us answer the question: What is the physical quantity s_θ^2 at one loop?
Because s_θ^2 is chosen to be a free parameter, therefore the bare parameter $s_\theta^{(1)2}$ is infinite. Then we need to choose a renormalization scheme to absorb the infinite part into its counterterm, ie.

$$\underbrace{(s_\theta^{Ren})^2}_{\text{renormalized quantity (finite)}} = \underbrace{s_\theta^{(1)^2}}_{\text{bare parameter (infinite)}} + \underbrace{\delta s_\theta^{(1)^2}}_{\text{Counterterm (infinite)}}$$

① We may choose the modified minimum subtraction $\overline{MS}$ as our renormalization scheme for s_θ^2, then we get the renormalized quantity $(s_\theta^{Ren})^2$. To identify this renormalized quantity $(s_\theta^{Ren})^2$ as the physical quantity $(s_\theta^{phy})^2$, we have to choose an energy scale. Because we define our s_θ^2 through the low energy experiments $\sigma(\bar{\nu}_\mu e)$ and $\sigma(\nu_\mu e)$, our energy scale μ^2 is just in the limit $q^2 \to 0$.

We therefore define the physical quantity $(s_\theta^{phy})^2$ as the renormalized quantity evaluated at $\mu^2 \to 0$.

② Another way to define $(s_\theta^{phy})^2$ is to choose our renormalized quantity $(s_\theta^{Ren})^2$ as the physical quantity $(s_\theta^{phy})^2$, also

$$
\begin{aligned}
\frac{1}{4}(1-\xi) &= s_\theta^{(1)^2}(1-\delta_3) \\
&\equiv (s_\theta^{Ren})^2 \\
&\equiv (s_\theta^{phy})^2 \\
&= (s_\theta^{tree})^2 .
\end{aligned}
$$

By choosing this particular renormalization scheme, we have the $(s_\theta^{phy})^2$ up to one loop is just the value of $(s_\theta^{tree})^2$ at tree level. Therefore s_θ^2 does <u>not</u> get corrected in this scheme.

Then the shift of ρ from its tree value $\rho = 1$ is just

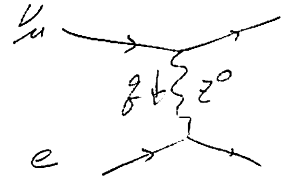
$$\delta\rho = \frac{\delta M_\rho^2}{M_\rho^2} - \frac{\delta M_{o\rho}^2}{M_{o\rho}^2}$$

(δM_ρ^2 is defined as the mass shift of W, ie the difference between 1-loop and tree results.)

for $\rho \equiv \dfrac{M_\rho^2}{M_{o\rho}^2 C_\theta^2}$, because C_θ^2 will never get shifted in this particular renormalization scheme.

5. Find the explicit expressions of δ_1, δ_2 and δ_3 in Standard Model.

1) $R = \frac{\sigma(\bar{\nu}_\mu e)}{\sigma(\nu_\mu e)}$:



(1) At tree level, the corresponding amplitude in the limit $g^2 \to 0$ is

$$\mathcal{M}^{(0)} = \frac{(ig)^2}{16\, C_0^2} \frac{1}{M_0^2} \bar{\nu}_\mu \gamma_\lambda (1+\gamma_5) \nu_\mu \bar{e} \gamma_\lambda (a + b\gamma_5) e,$$

and the ratio R is

$$R = \frac{\sigma(\bar{\nu}_\mu e)}{\sigma(\nu_\mu e)} = \frac{1-\xi+\xi^2}{1+\xi+\xi^2}, \quad \text{with} \quad \xi = \frac{a}{b} = \frac{-1+4S_0^2}{-1}.$$

(2) Up to one loop, the amplitude becomes

$$\mathcal{M}^{(1)} = \frac{(ig^{(1)})^2}{16\, C_0^{(1)2}} \frac{1}{M_0^{(1)2}} \bar{\nu}_\mu \gamma_\lambda (1+\gamma_5) \nu_\mu \bar{e} \gamma_\lambda (A + B\gamma_5) e$$

Obviously, the measured quantity ξ is equal to $\frac{A}{B}$, ie

$$\xi = \frac{A}{B}, \quad \text{with} \quad \begin{aligned} A &= a(1+\delta a) \\ B &= b(1+\delta b) \end{aligned}.$$

Note: ① Since this is low energy experiment, we will evaluate at $g^2 \to 0$.

② Because neutrino should remain massless, therefore neutrino should remain to be left-handed polarized. Thus the neutrino part of the amplitude, $\bar{\nu}_\mu \gamma_\lambda (1+\gamma_5)\nu_\mu$, remains unchanged.

③ The correction due to Z^0 self-energy should be expressed in terms of the $Z\bar{e}e$ coupling, ie

$$\left[\frac{1}{q^2+M_0^{(1)2}-\tilde{\Pi}_0}\right]_{-q^2\to 0} = \frac{1}{M_0^{(1)2}-\tilde{\Pi}_0} = \frac{1}{M_0^{(1)2}}\left(1+\frac{\tilde{\Pi}_0}{M_0^{(1)2}}\right).$$

However, since $\xi = \frac{A}{B} = \frac{a}{b}(1+\delta a-\delta b)$, therefore only the diagrams which give different contributions to δa or δb will contribute ξ. Hence the Z^0-self-energy diagrams and $Z^0\bar{\nu}\nu$ coupling corrections are irrelevent to this quantity ξ.

(3) Procedures :

① Calculate δa and δb from all relevent diagrams. (Only consider those diagrams contribute to δa and δb differently.)

② Get the bare parameter $S_\theta^{(1)2}$ through ξ.

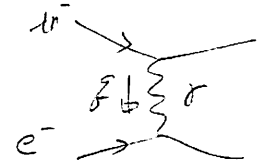
Since
$$\xi = \frac{a}{b}(1+\delta a-\delta b) = (1-4S_\theta^{(1)2})\left[1+\delta a-\delta b\right],$$

so
$$S_\theta^{(1)2} = \frac{1}{4}-\frac{\xi}{4}\left[1+\delta a-\delta b\right] = \frac{1}{4}(1-\xi)\left[1+\frac{\xi}{1-\xi}(\delta a-\delta b)\right].$$

③ Find δ_3 :

Because
$$S_\theta^{(1)2} \equiv \frac{1}{4}(1-\xi)(1+\delta_3),$$
therefore
$$\delta_3 = \frac{\xi}{1-\xi}(\delta a-\delta b).$$

2) Coulomb scattering:



(1) The tree amplitude

$$\mathcal{M}^{(0)} = (i g s_\theta)^2 \frac{1}{q^2} \bar{\mu} \gamma_\lambda \mu \bar{e} \gamma_\lambda e$$

(2) Up to one loop,

$$\mathcal{M}^{(1)} = \left(i g^{(1)} s_\theta^{(1)}\right)^2 \frac{1}{q^2} \bar{\mu} \gamma_\lambda \mu \bar{e} \gamma_\lambda e \cdot (1 + \delta c)$$

Note: ① Only diagrams which develope a pole in $q^2 = 0$ need to be considered since only those are relevant in the low-energy limit.

② There should be no term proportional to γ_5 remains in the total amplitude, so that there is no parity violation at low energy in the scattering of two electrically charged particles.

(3) Procedures:

① Comparing the differential cross sections with experimental data, one get

$$4\pi\alpha = g^{(1)^2} s_\theta^{(1)^2} \cdot (1 + \delta c)$$

② Since we define

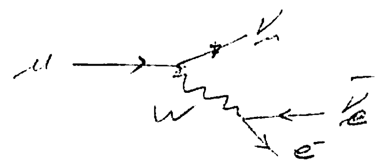
$$4\pi\alpha = g^{(1)^2} s_\theta^{(1)^2} (1 + \delta_2),$$

therefore we can identify δ_2 as δc, ie

$$\delta_2 = \delta c.$$

3) Muon decay

(1) The tree amplitude



$$\mathcal{M}^{(0)} = \frac{ig^2}{8M^2}\,\bar\nu_\mu \gamma_\lambda(1+\gamma_5)\mu\,\bar e \gamma_\lambda(1+\gamma_5)\nu_e .$$

The decay rate at tree level is

$$\Gamma^{(0)} = \frac{m_\mu^5}{192\pi^3}\cdot\left(\frac{g^2}{4\sqrt{2}M^2}\right)^2\cdot\left(1-8\frac{m_e^2}{m_\mu^2}\right)$$

(2) Up to one loop, its decay rate is

$$\Gamma^{(1)} = \Gamma^{(0)}\left\{1+\frac{\alpha}{2\pi}\left(\frac{25}{4}-\pi^2\right)+\delta\Gamma\right\},$$

where the 2nd term comes from parts of the box diagrams, and is identified as the QED corrections on four-fermion direct interaction.

Recall that the "tree value" of G_F has already included the 2nd term contribution.

Therefore to identify

$$G_F \equiv \frac{g^{(1)2}}{4\sqrt{2}M^{(1)2}}\left[1+\delta_1\right]$$

one should subtract the QED contribution off.

Also the amplitude, up to one-loop, is

$$\mathcal{M}^{(1)} = \left(\frac{i\,g^{(1)}}{2\sqrt{2}}\right)^2\frac{1}{M^{(1)2}}(1+\delta_1)\,\bar\nu_\mu \gamma_\lambda(1+\gamma_5)\mu\,\bar e\gamma_\lambda(1+\gamma_5)\nu_e .$$

<u>Note</u>: We should introduce a fictitious photon mass λ to regularize the infrared divergence.

The infrard divergences come from three-vertex diagrams will be cancelled by those from self-energy diagrams. The infrard divergences come from box diagrams are cancelled by the external leg bremstrahlung.

6. Find the physical quatity s_θ^2 up to one loop:
As discussed in section 4, the quantity s_θ^2 is defined (by fixing the renormalization scheme, or choosing counterm) so that

$$\left(s_\theta^{phy}\right)^2_{1\text{-loop}} = \left(s_\theta\right)^2_{tree} = \frac{1}{4}(1-\xi),$$

where ξ is defined as

$$R = \frac{\sigma(\bar{\nu}_\mu e)}{\sigma(\nu_\mu e)} = \frac{1-\xi+\xi^2}{1+\xi+\xi^2}.$$

7. Other parametrization :

Since the mass of Z^0 will be measured to about 100 MeV accuracy, one may think of using M_{op} as one data point instead of using the low energy experiment $R = \frac{\sigma(\bar{\nu}_\mu e)}{\sigma(\nu_\mu e)}$.

In other words, one may still choose the three free parameters as g, S_0 and M, but choose the three data points as

(i) Coulomb scattering $\Rightarrow \alpha$

(ii) μ - decay $\Rightarrow G_F$

(iii) Z^0 mass $\Rightarrow S_\theta^2$.

Then use the resulting bare parameters $g^{(1)}$, $S_\theta^{(1)2}$ and $M^{(1)}$ to calculate the ration

$R = \frac{\sigma(\bar{\nu}_\mu e)}{\sigma(\nu_\mu e)}$ up to one loop, and define

$R = \frac{1-\xi+\xi^2}{1+\xi+\xi^2}$, and $\frac{1}{4}(1-\xi) \equiv (S_\theta^{phy})^2_{1\text{-loop}}$.

Finally define

$$\rho \equiv \frac{M_p^2}{M_{op}^2 \left(C_\theta^{phy}\right)^2_{1\text{-loop}}}.$$