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Multi-order exact solutions of the complex Ginzburg–Landau equation

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Abstract

The multi-order exact solutions of the one-dimensional complex Ginzburg–Landau equation are obtained by the use of the wave packet theory. In these solutions, the zeroth-order exact solution is a plane wave; the first-order exact solutions are shock waves between the amplitude and the shift of phase, spiral waves between the shift of phase and distance or between the shift of phase and the amplitude; the second-order exact solutions are two periodic waves with certain conditions and their limits – the shock waves and solitary waves can also be found. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Complex Ginzburg–Landau equation; Multi-order exact solutions; Spiral wave solution

1. Introduction

The one-dimensional complex Ginzburg–Landau equation including the non-steady, nonlinear, dispersive and diffusive terms, as well as the linear growth or damping term, is usually written as

$$i\frac{\partial u}{\partial t}+(\alpha_1+i\alpha_2)\frac{\partial^2 u}{\partial x^2}+(\beta_1+i\beta_2)|u|^2u-i\gamma u=0 \tag{1}$$

where $i=\sqrt{-1}$ is a pure imaginary number; α_1, α_2, β_1, β_2 and γ are all real constants. $(\beta_1+i\beta_2)|u|^2u$ represents the nonlinear effect, $\alpha_1\partial^2u/\partial x^2$ and $i\alpha_2\partial^2u/\partial x^2$ represent the dispersive and diffusive effects, respectively, $i\gamma u$ denotes the linear growth or damping.

When $\alpha_2=\beta_2=\gamma=0$, the complex Ginzburg–Landau Eq. (1) degenerated into the following nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t}+\alpha_1\frac{\partial^2 u}{\partial x^2}+\beta_1|u|^2u=0 \tag{2}$$

The complex Ginzburg–Landau Eq. (1) plays an important role in many branches of physics, such as the fluid dynamics, nonlinear optics, chemical and biological dynamics, etc. [1–4]. A lot of studies have shown that the complex Ginzburg–Landau Eq. (1) possesses a rich variety of solutions involving the plane waves, shock waves, solitary waves, spiral waves, as well as the hole, periodic and quasi-periodic solutions, etc. [5–12].

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In this Letter, the multi-order exact solution of the complex Ginzburg–Landau Eq. (1) is obtained by use of the wave packet theory and a relatively simple method.

2. Envelope solutions

We assume that the envelope solutions for Eq. (1) are of the following form

$$u = a(\xi)\mathrm{e}^{i[kx-\omega t+\theta(\xi)]} \tag{3}$$

where k and ω are the wave number and angular frequency, respectively, and

$$\xi = \epsilon\left(x - c_g t\right), \quad c_g = \text{const}. \tag{4}$$

$\epsilon \ll 1$ is small parameter. Both the amplitude $a(\xi)$ and the shift of phase $\theta(\xi)$ are real functions.

Substituting (3) into (1) and equating the real and imaginary parts to zero separately, we have

$$\begin{aligned} & a\left(\omega - \alpha_1 k^2 + \beta_1 a^2\right) \\ &\quad + \epsilon\left[\left(c_g - 2\alpha_1 k\right)a\frac{d\theta}{d\xi} - 2\alpha_2 k\frac{da}{d\xi}\right] \\ &\quad + \epsilon^2\left\{\alpha_1\left[\frac{d^2a}{d\xi^2} - a\left(\frac{d\theta}{d\xi}\right)^2\right]\right. \\ &\quad \left. - \alpha_2\left[2\frac{da}{d\xi}\frac{d\theta}{d\xi} + a\frac{d^2\theta}{d\xi^2}\right]\right\} = 0 \end{aligned} \tag{5a}$$

$$\begin{aligned} & -a\left(\gamma + \alpha_2 k^2 - \beta_2 a^2\right) \\ &\quad + \epsilon\left[-2\alpha_2 ka\frac{d\theta}{d\xi} - \left(c_g - 2\alpha_1 k\right)\frac{da}{d\xi}\right] \\ &\quad + \epsilon^2\left\{\alpha_2\left[\frac{d^2a}{d\xi^2} - a\left(\frac{d\theta}{d\xi}\right)^2\right]\right. \\ &\quad \left. + \alpha_1\left[2\frac{da}{d\xi}\frac{d\theta}{d\xi} + a\frac{d^2\theta}{d\xi^2}\right]\right\} = 0 \end{aligned} \tag{5b}$$

Eqs. (5) constitute the equations of $a(\xi)$ and $\theta(\xi)$ in the envelope solution (3) of the complex Ginzburg–Landau Eq. (1). In the following sections, we look for the exact solutions in each order of Eqs. (5).

3. Zero-order exact solutions

Setting $\epsilon = 0$ which corresponds to that the variation of a and θ with ξ are disregarded in Eqs. (5), we obtain the following plane wave solutions [5,6]

$$\omega - \alpha_1 k^2 + \beta_1 a^2 = 0 \tag{6a}$$

$$\gamma + \alpha_2 k^2 - \beta_2 a^2 = 0 \tag{6b}$$

from which the dispersion relation and amplitude can be determined.

4. First-order exact solutions

Neglecting the terms with ϵ^2 in Eqs. (5) yields

$$\begin{aligned} & 2\alpha_2 k\left(\epsilon\frac{da}{d\xi}\right) - \left(c_g - 2\alpha_1 k\right)\left(\epsilon a\frac{d\theta}{d\xi}\right) \\ &\quad = a\left(\omega - \alpha_1 k^2 + \beta_1 a^2\right) \end{aligned} \tag{7a}$$

$$\begin{aligned} & \left(c_g - 2\alpha_1 k\right)\left(\epsilon\frac{da}{d\xi}\right) + 2\alpha_2 k\left(\epsilon a\frac{d\theta}{d\xi}\right) \\ &\quad = -a\left(\gamma + \alpha_2 k^2 - \beta_2 a^2\right) \end{aligned} \tag{7b}$$

Eqs. (7) constitute the systems of first-order equations in $a(\xi)$ and $\theta(\xi)$, we obtain the following relations

$$\begin{aligned} \delta\left(\epsilon\frac{da}{d\xi}\right) &= \delta_2 a\left(\omega - \alpha_1 k^2 + \beta_1 a^2\right) \\ &\quad - \delta_1 a\left(\gamma + \alpha_2 k^2 - \beta_2 a^2\right) \end{aligned} \tag{8a}$$

$$\begin{aligned} \delta\left(\epsilon a\frac{d\theta}{d\xi}\right) &= -\delta_1 a\left(\omega - \alpha_1 k^2 + \beta_1 a^2\right) \\ &\quad - \delta_2 a\left(\gamma + \alpha_2 k^2 - \beta_2 a^2\right) \end{aligned} \tag{8b}$$

easily by the elimination, with

$$\delta_1 \equiv c_g - 2\alpha_1 k, \quad \delta_2 \equiv 2\alpha_2 k, \quad \delta \equiv \delta_1^2 + \delta_2^2 \tag{9}$$

Eq. (8a) can be rewritten as

$$\frac{da}{a\left(p + qa^2\right)} = \frac{d\xi}{\delta\epsilon} \tag{10}$$

where

$$p \equiv (\omega\delta_2 - \gamma\delta_1) - (\alpha_1\delta_2 + \alpha_2\delta_1)k^2 \tag{11a}$$

$$q \equiv \beta_1\delta_2 + \beta_2\delta_1 \tag{11b}$$

Eq. (10) can be integrated to give

$$a^2 = -\frac{p}{2q}\mathrm{e}^{\frac{p}{\delta\epsilon}(\xi-\xi_0)}\operatorname{sech}\frac{p}{\delta\epsilon}(\xi-\xi_0),$$

$$a^2 < -\frac{p}{q} \tag{12a}$$

$$a^2 = -\frac{p}{2q}\mathrm{e}^{\frac{p}{\delta\epsilon}(\xi-\xi_0)}\operatorname{csch}\frac{p}{\delta\epsilon}(\xi-\xi_0),$$

$$a^2 > -\frac{p}{q} \tag{12b}$$

where ξ_0 is an integration constant. Eqs. (12) represent the shock wave solutions for the amplitude of the complex Ginzburg–Landau Eq. (1) and can be illustrated in Fig. 1 and Fig. 2, from which the following results can be obtained: under the condition of $a^2 < -p/q$, when $\xi \to \xi_0$, then $a^2(\xi) \to -p/(2q)$; when $\xi \to +\infty$ (or $\xi \to -\infty$), then $a^2 \to -p/q$.

Eq. (8b) can rewritten as

$$\delta\epsilon\frac{d\theta}{d\xi} = r + sa^2 \tag{13}$$

with

$$r \equiv -(\omega\delta_1 + \gamma\delta_2) + (\alpha_1\delta_1 - \alpha_2\delta_2)k^2 \tag{14a}$$

$$s \equiv -(\beta_1\delta_1 - \beta_2\delta_2) \tag{14b}$$

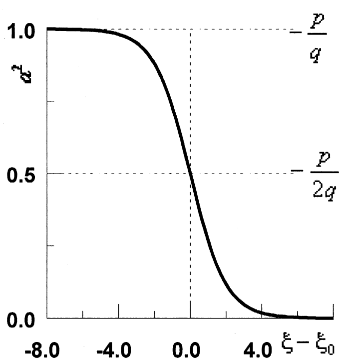

Fig. 1. Shock wave solution for a^2 ($p<0$) under the condition of $a^2 < -p/q$.

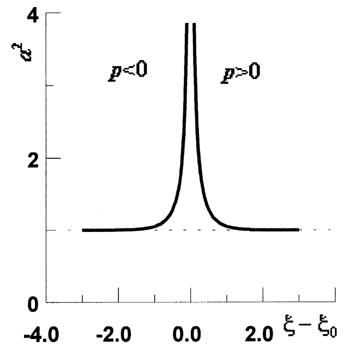

Fig. 2. Shock wave solution for a^2 under the condition of $a^2 < -p/q$.

Substitution of Eq. (12) into Eq. (13) can be integrated as

$$\theta = \frac{1}{\delta\epsilon}\left(r - \frac{p}{q}s\right)(\xi-\xi_0) - \frac{s}{2q}\ln\left[\mathrm{e}^{\frac{2p}{\delta\epsilon}(\xi-\xi_0)} + 1\right],$$

$$a^2 < -\frac{p}{q} \tag{15a}$$

$$\theta = \frac{1}{\delta\epsilon}\left(r - \frac{p}{q}s\right)(\xi-\xi_0) - \frac{s}{2q}\ln\left[\mathrm{e}^{\frac{2p}{\delta\epsilon}(\xi-\xi_0)} - 1\right],$$

$$a^2 > -\frac{p}{q} \tag{15b}$$

which is the solution of the shift of phase of the complex Ginzburg–Landau Eq. (1), which shows that there is a spiral wave relation between the shift of phase and distance, furthermore, when $\xi \to +\infty$, it is an Archimede spiral.

(8a) divided by (8b) yields

$$\frac{da}{ad\theta} = \frac{p + qa^2}{r + sa^2} \tag{16}$$

which is a differential equation of a and θ in the polar coordinates (a, θ) and can be integrated to give

$$|qa^2|^{\frac{r}{2p}}|p + qa^2|^{\frac{1}{2}(\frac{s}{q}-\frac{r}{p})} = \mathrm{e}^{\theta-\theta_0} \tag{17}$$

where θ_0 is an integration constant, when $\frac{s}{q} - \frac{r}{p} = 0$, (17) can be reduced to

$$a = |q|^{-\frac{1}{2}}\mathrm{e}^{\frac{p}{r}(\theta-\theta_0)} \tag{18}$$

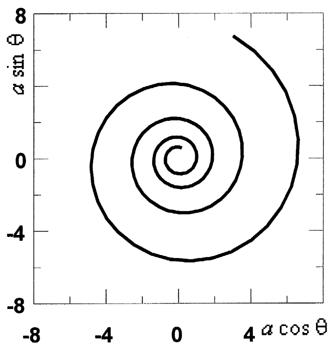

Fig. 3. Spiral wave solution between a and θ.

which represents a spiral and can be illustrated in Fig. 3. Thus (17) denotes the spiral wave solutions of the complex Ginzburg–Landau Eq. (1).

5. Second-order exact solutions

By means of (9), Eqs. (5) can be rewritten as

$$\epsilon^2\left\{\alpha_1\left[\frac{d^2a}{d\xi^2}-a\left(\frac{d\theta}{d\xi}\right)^2\right]-\alpha_2\left[2\frac{da}{d\xi}\frac{d\theta}{d\xi}+a\frac{d^2\theta}{d\xi^2}\right]\right\}$$
$$+\epsilon\left[\delta_1 a\frac{d\theta}{d\xi}-\delta_2\frac{da}{d\xi}\right]$$
$$+a\left(\omega-\alpha_1k^2+\beta_1a^2\right)=0 \tag{19a}$$

$$\epsilon^2\left\{\alpha_2\left[\frac{d^2a}{d\xi^2}-a\left(\frac{d\theta}{d\xi}\right)^2\right]+\alpha_1\left[2\frac{da}{d\xi}\frac{d\theta}{d\xi}+a\frac{d^2\theta}{d\xi^2}\right]\right\}$$
$$+\epsilon\left[-\delta_2 a\frac{d\theta}{d\xi}-\delta_1\frac{da}{d\xi}\right]$$
$$-a\left(\gamma+\alpha_2k^2-\beta_2a^2\right)=0 \tag{19b}$$

Eliminating $2\frac{da}{d\xi}\frac{d\theta}{d\xi}+a\frac{d^2\theta}{d\xi^2}$ and $\frac{d^2a}{d\xi^2}-a(\frac{d\theta}{d\xi})^2$ in Eqs. (19) yields

$$\alpha\epsilon^2\left[\frac{d^2a}{d\xi^2}-a\left(\frac{d\theta}{d\xi}\right)^2\right]-p_2\epsilon\frac{da}{d\xi}+p_1\epsilon a\frac{d\theta}{d\xi}$$
$$+a\left(q_1+q_2a^2\right)=0 \tag{20a}$$

$$\alpha\epsilon^2\left[2\frac{da}{d\xi}\frac{d\theta}{d\xi}+a\frac{d^2\theta}{d\xi^2}\right]-p_1\epsilon\frac{da}{d\xi}-p_2\epsilon a\frac{d\theta}{d\xi}$$
$$+a\left(r_1+r_2a^2\right)=0 \tag{20b}$$

with

$$\alpha=\alpha_1^2+\alpha_2^2,\ p_1=\alpha_1\delta_1-\alpha_2\delta_2=\alpha_1c_g-2\alpha k \tag{21a}$$

$$p_2=\alpha_1\delta_2+\alpha_2\delta_1=\alpha_2c_g,$$
$$q_1=\alpha_1\omega-\alpha_2\gamma-\alpha k^2 \tag{21b}$$

$$q_2=\alpha_1\beta_1+\alpha_2\beta_2,\quad r_1=-(\alpha_1\gamma+\alpha_2\omega),$$
$$r_2=\alpha_1\beta_2-\alpha_2\beta_1 \tag{21c}$$

A similar way as [12], we set

$$b=\frac{d\theta}{d\xi},\quad y=a^2,\quad z=a^2b \tag{22}$$

then Eqs. (20) are reduced to

$$2y\frac{d^2y}{d\xi^2}-\left(\frac{dy}{d\xi}\right)^2-a_1y\frac{dy}{d\xi}$$
$$+a_2yz-4z^2-a_3y^2+a_4y^3=0 \tag{23a}$$

$$\frac{dz}{d\xi}-b_1\frac{dy}{d\xi}-b_2z+b_3y+b_4y^2=0 \tag{23b}$$

with

$$a_1=\frac{2p_2}{\epsilon\alpha},\quad a_2=\frac{4p_1}{\epsilon\alpha},\quad a_3=-\frac{4q_1}{\epsilon^2\alpha},$$
$$a_4=\frac{4q_2}{\epsilon^2\alpha}, \tag{24a}$$

$$b_1=\frac{p_1}{2\epsilon\alpha}=\frac{a_2}{8},\quad b_2=\frac{p_2}{\epsilon\alpha}=\frac{a_1}{2},b_3=\frac{r_1}{\epsilon^2\alpha},$$
$$b_4=\frac{r_2}{\epsilon^2\alpha} \tag{24b}$$

It is difficult to solve Eq. (23a) and Eq. (23b), so we state only two cases, in which Eq. (23a) and Eq. (23b) can be solved.

Case one: the nonlinear Schrödinger equation,

$$\alpha_2=\beta_2=\gamma=0,\quad \alpha=\alpha_1^2,$$
$$p_1=\alpha_1\left(c_g-2\alpha_1k\right),\quad p_2=0,$$
$$q_1=\alpha_1\left(\omega-\alpha_1k^2\right),\quad q_2=\alpha_1\beta_1,$$
$$r_1=0,\quad r_2=0.$$

Thus,

$$a_1 = 0, \quad a_2 = \frac{4}{\alpha_1 \epsilon}\left(c_g - 2\alpha_1 k\right),$$

$$a_3 = -\frac{4}{\alpha_1 \epsilon^2}\left(\omega - \alpha_1 k^2\right), \quad a_4 = \frac{4\beta_1}{\alpha_1 \epsilon^2},$$

$$b_1 = \frac{1}{2\epsilon}\left(c_g - 2\alpha_1 k\right), \quad b_2 = 0,$$

$$b_3 = 0, \quad b_4 = 0.$$

Usually taking $c_g = 2\alpha_1 k$, then $a_2 = 0$, $b_1 = 0$. Thus, Eqs. (23) become

$$2y\frac{d^2 y}{d\xi^2} - \left(\frac{dy}{d\xi}\right)^2 - 4z^2 + \frac{4}{\alpha_1 \epsilon^2}\left(\omega - \alpha_1 k^2\right)y^2 + \frac{4\beta_1}{\alpha_1 \epsilon^2}y^3 = 0 \tag{25a}$$

$$\frac{dz}{d\xi} = 0 \tag{25b}$$

We obtain from Eq. (25b)

$$z \equiv a^2 b = C \tag{26}$$

C is an integration constant. Often setting $b \equiv d\theta/d\xi = 0$ in the nonlinear Schrödinger equation, then $C = 0$. Such that Eq. (25a) is reduced to

$$2y\frac{d^2 y}{d\xi^2} - \left(\frac{dy}{d\xi}\right)^2 + \frac{4}{\alpha_1 \epsilon^2}\left(\omega - \alpha_1 k^2\right)y^2 + \frac{4\beta_1}{\alpha_1 \epsilon^2}y^3 = 0 \tag{27}$$

In term of (22), Eq. (27) can be rewritten as

$$\alpha_1 \epsilon^2 \frac{d^2 a}{d\xi^2} + \left(\omega - \alpha_1 k^2\right)a + \beta_1 a^3 = 0 \tag{28}$$

From (28) the envelope solitary solutions of the nonlinear Schrödinger equation can be easily found.

Case two: $c_g = 0$, $r_1 = 0$ and $r_2 = 0$. Since $\xi = \epsilon x$, we get from (21) and (24)

$$p_1 = -2\alpha k, \quad p_2 = 0, \quad a_1 = 0, \quad b_2 = 0 \tag{29a}$$

$$b_3 = 0, \quad b_4 = 0, \quad \alpha_1 r + \alpha_2 \omega = 0,$$

$$\alpha_1 \beta_2 - \alpha_2 \beta_1 = 0 \tag{29b}$$

Such that, Eqs. (23) can be reduced to

$$2y\frac{d^2 y}{d\xi^2} - \left(\frac{dy}{d\xi}\right)^2 + a_2 yz - 4z^2 - a_3 y^2 + a_4 y^3 = 0 \tag{30a}$$

$$\frac{dz}{d\xi} - b_1 \frac{dy}{d\xi} = 0 \tag{30b}$$

it is explicit that the Eqs. (30) are different from the results given in Ref. [12].

From Eq. (30b) we have

$$z = b_1 y \tag{31}$$

Substituting (31) into Eq. (30a) yields

$$2y\frac{d^2 y}{d\xi^2} - \left(\frac{dy}{d\xi}\right)^2 + \left(a_2 b_1 - 4b_1^2 - a_3\right)y^2 + a_4 y^3 = 0 \tag{32}$$

which is the Painlevé-type equation. Via the ansatz

$$\frac{dy}{d\xi} = v(y) \tag{33}$$

Eq. (32) reduces to

$$2yv\frac{dv}{dy} - v^2 + \left(a_2 b_1 - 4b_1^2 - a_3\right)y^2 + a_4 y^3 = 0 \tag{34}$$

Setting

$$w = v^2 \tag{35}$$

Eq. (34) is reduced to the following linear equation in term of $w(y)$:

$$\frac{dw}{dy} - \frac{1}{y}w = -\left(a_2 b_1 - 4b_1^2 - a_3\right)y - a_4 y^2 \tag{36}$$

from which the following solution can be obtained

$$w = D_1 y - \left(a_2 b_1 - 4b_1^2 - a_3\right)y^2 - \frac{a_4}{2}y^3 \tag{37}$$

where D_1 is an arbitrary constant. Applying (33) and (35), (37) becomes

$$\left(\frac{dy}{d\xi}\right)^2 = -\frac{a_4}{2}y\left[y^2 + \frac{2}{a_4}\left(a_2 b_1 - 4b_1^2 - a_3\right)y + D\right] \tag{38}$$

which is a standard elliptic equation and $D = -\frac{2}{a_4}D_1$.

Supposed that $[\frac{2}{a_4}(a_2 b_1 - 4b_1^2 - a_3)]^2 - 4D > 0$ and $y^2 + \frac{2}{a_4}(a_2 b_1 - 4b_1^2 - a_3)y + D = 0$ has two real roots A and B $(B > A)$, When $a_4 < 0$, the solution of (38) is given by

$$y = A\mathrm{sn}^2\left(\sqrt{-\frac{a_4 B}{8}}\,\xi, m\right) \quad \left(m \equiv \sqrt{\frac{A}{B}}\right) \tag{39}$$

where sn() denotes the Jacobi elliptic sine function and m is the modulus.

Similarly, when $[\frac{2}{a_4}(a_2 b_1 - 4b_1^2 - a_3)]^2 - 4D > 0$ and $y^2 + \frac{2}{a_4}(a_2 b_1 - 4b_1^2 - a_3)y + D = 0$ has two real roots B and $B - A$ $(B > A)$, then when $a_4 > 0$, then solution of (38) is given by

$$y = B\mathrm{dn}^2\left(\sqrt{\frac{a_4 B}{8}}\,\xi, m\right) \quad \left(m \equiv \sqrt{\frac{A}{B}}\right) \tag{40}$$

where dn() denotes the Jacobi elliptic function of the third kind and m is modulus.

Notice that $y = a^2$, we obtain from (39) and (40), respectively

$$a \equiv \sqrt{A}\,\mathrm{sn}\left(\sqrt{-\frac{a_4 B}{8}}\,\xi, m\right) \quad \left(m \equiv \sqrt{\frac{A}{B}}\right) \tag{41}$$

and

$$a \equiv \sqrt{B}\,\mathrm{dn}\left(\sqrt{\frac{a_4 B}{8}}\,\xi, m\right) \quad \left(m \equiv \sqrt{\frac{A}{B}}\right) \tag{42}$$

(41) and (42) are the periodic solutions in amplitude of the complex Ginzburg–Landau equation under the condition (29).

If $m \to 1$, (41) and (42) then degenerate into

$$a \equiv \sqrt{A}\,\tanh\left(\sqrt{-\frac{a_4 B}{8}}\,\xi\right) \tag{43}$$

and

$$a \equiv \sqrt{B}\,\mathrm{sech}\left(\sqrt{\frac{a_4 B}{8}}\,\xi\right) \tag{44}$$

respectively. (43) and (44) are the shock and solitary wave solutions in amplitude of the complex Ginzburg–Landau equation.

6. Conclusion

Applying the wave packet theory the multi-order exact solution of the complex Ginzburg–Landau equation are obtained. The multi-order exact solutions including the familiar plane wave, shock and solitary waves, periodic and spiral solutions have relatively simple form and clear picture.

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