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Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations

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Abstract

A Jacobi elliptic function expansion method, which is more general than the hyperbolic tangent function expansion method, is proposed to construct the exact periodic solutions of nonlinear wave equations. It is shown that the periodic solutions obtained by this method include some shock wave solutions and solitary wave solutions. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

It plays an important role to find the exact solutions of nonlinear wave equations in the nonlinear problems. Recently, a number of methods have been proposed, such as the homogeneous balance method [1–3], the hyperbolic tangent function expansion method [4–6], the trial function method [7,8], the nonlinear transformation method [9,10] and sine–cosine method [11]. However, these methods can only obtain the shock and solitary wave solutions and cannot obtain the periodic solutions of nonlinear wave equations. Although Porubov et al. [12–14] have obtained some exact periodic solutions to some nonlinear wave equations, they use the Weierstrass elliptic function and involve complicated deducing. In this Letter, the Jacobi elliptic function expansion method, which is more general than the hyperbolic tangent function expansion method, is proposed and applied to some nonlinear wave equations. It is shown that the periodic solutions obtained by this method include some shock wave solutions and solitary wave solutions.

2. Jacobi elliptic function expansion method

Consider a given nonlinear wave equation

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$$N\left(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 u}{\partial x^2},\dots\right)=0, \tag{1}$$

we seek its wave solutions of the following form:

$$u=u(\xi),\quad \xi=k(x-ct), \tag{2}$$

where k and c are the wave number and wave speed, respectively.

By the Jacobi elliptic function expansion method, $u(\xi)$ can be expressed as a finite series of Jacobi elliptic function, $\operatorname{sn}\xi$, i.e., the ansatz

$$u(\xi)=\sum_{j=0}^{n} a_j \operatorname{sn}^j \xi \tag{3}$$

is made and its highest degree is

$$O\big(u(\xi)\big)=n. \tag{4}$$

Notice that

$$\frac{du}{d\xi}=\sum_{j=0}^{n} j a_j \operatorname{sn}^{j-1}\xi \operatorname{cn}\xi \operatorname{dn}\xi, \tag{5}$$

where $\operatorname{cn}\xi$ and $\operatorname{dn}\xi$ are the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind, respectively. And

$$\operatorname{cn}^2\xi=1-\operatorname{sn}^2\xi,\qquad \operatorname{dn}^2\xi=1-m^2\operatorname{sn}^2\xi \tag{6}$$

with the modulus m $(0<m<1)$. Since

$$\frac{d}{d\xi}\operatorname{sn}\xi=\operatorname{cn}\xi\operatorname{dn}\xi,\qquad \frac{d}{d\xi}\operatorname{cn}\xi=-\operatorname{sn}\xi\operatorname{dn}\xi,\qquad \frac{d}{d\xi}\operatorname{dn}\xi=-m^2\operatorname{sn}\xi\operatorname{cn}\xi, \tag{7}$$

the highest degree of $d^p u/d\xi^p$ is taken as

$$O\left(\frac{d^p u}{d\xi^p}\right)=n+p,\quad p=1,2,3,\dots, \tag{8}$$

and

$$O\left(u^q\frac{d^p u}{d\xi^p}\right)=(q+1)n+p,\quad q=0,1,2,\dots,\quad p=1,2,3,\dots. \tag{9}$$

Thus we can select n in (3) to balance the derivative term of the highest order and the nonlinear term in (1).

We know that when $m\to 1$, $\operatorname{sn}\xi\to\tanh\xi$, thus (3) is degenerated as the following form:

$$u(\xi)=\sum_{j=0}^{n} a_j \tanh^j \xi. \tag{10}$$

So, the Jacobi elliptic function expansion method is more general than the hyperbolic tangent function expansion method.

3. Applications

We illustrate the applications of the Jacobi elliptic sine function expansion method to some nonlinear wave equations.

3.1. Applications to single equation

3.1.1. KdV equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0. \tag{11}$$

Substituting (2) into (11), we have

$$-c\frac{du}{d\xi} + u\frac{du}{d\xi} + \beta k^2\frac{d^3 u}{d\xi^3} = 0. \tag{12}$$

Thus we can deduce from (3) that

$$O\left(u\frac{du}{d\xi}\right) = 2n+1, \qquad O\left(\frac{d^3 u}{d\xi^3}\right) = n+3, \tag{13}$$

thus

$$n = 2. \tag{14}$$

So the KdV equation (11) may have the following form travelling wave solution:

$$u(\xi) = a_0 + a_1 \operatorname{sn}\xi + a_2 \operatorname{sn}^2\xi \tag{15}$$

and

$$\frac{du}{d\xi} = (a_1 + 2a_2 \operatorname{sn}\xi)\operatorname{cn}\xi \operatorname{dn}\xi, \tag{16}$$

$$u\frac{du}{d\xi} = \left[a_0a_1 + \left(a_1^2 + 2a_0a_2\right)\operatorname{sn}\xi + 3a_1a_2\operatorname{sn}^2\xi + 2a_2^2\operatorname{sn}^3\xi\right]\operatorname{cn}\xi\operatorname{dn}\xi, \tag{17}$$

$$\frac{d^2u}{d\xi^2} = 2a_2 - \left(1+m^2\right)a_1\operatorname{sn}\xi - 4\left(1+m^2\right)a_2\operatorname{sn}^2\xi + 2m^2a_1\operatorname{sn}^3\xi + 6m^2a_2\operatorname{sn}^4\xi, \tag{18}$$

$$\frac{d^3u}{d\xi^3} = \left[-\left(1+m^2\right)a_1 - 8\left(1+m^2\right)a_2\operatorname{sn}\xi + 6m^2a_1\operatorname{sn}^2\xi + 24m^2a_2\operatorname{sn}^3\xi\right]\operatorname{cn}\xi\operatorname{dn}\xi. \tag{19}$$

Substituting (15) into (12), we have

$$\begin{aligned}&-\left[c - a_0 + \left(1+m^2\right)\beta k^2\right]a_1\operatorname{cn}\xi\operatorname{dn}\xi + \left\{a_1^2 - 2\left[c - a_0 + 4\left(1+m^2\right)\beta k^2\right]a_2\right\}\operatorname{sn}\xi\operatorname{cn}\xi\operatorname{dn}\xi\\&\quad + 3a_1\left(a_2 + 2m^2\beta k^2\right)\operatorname{sn}^2\xi\operatorname{cn}\xi\operatorname{dn}\xi + 2a_2\left(a_2 + 12m^2\beta k^2\right)\operatorname{sn}^3\xi\operatorname{cn}\xi\operatorname{dn}\xi = 0.\end{aligned} \tag{20}$$

Thus we can determine the coefficients

$$a_1 = 0, \qquad a_2 = -12m^2\beta k^2, \qquad a_0 = c + 4\left(1+m^2\right)\beta k^2. \tag{21}$$

Substituting (21) into (15), a final solution is given,

$$u = c + 4\left(1+m^2\right)\beta k^2 - 12m^2\beta k^2\operatorname{sn}^2\xi = c + 4\left(1-2m^2\right)\beta k^2 + 12m^2\beta k^2\operatorname{cn}^2\xi, \tag{22}$$

which is the exact periodic solution of KdV equation (11). Usually, it is known as the cnoidal wave solution of KdV equation.

Taking $m = 1$, then (22) is reduced to

$$u = c - 4\beta k^2 + 12\beta k^2\operatorname{sech}^2\xi, \tag{23}$$

which is the solitary wave solution of KdV equation. Especially, when $c = 4\beta k^2$, (23) becomes

$$u = 3c\,\mathrm{sech}^2\sqrt{\frac{c}{4\beta}}(x - ct). \tag{24}$$

Similarly, this method can be applied to other single equation, such as:

3.1.2. Boussinesq equation

$$\frac{\partial^2 u}{\partial t^2} - c_0^2\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial^4 u}{\partial x^4} - \beta\frac{\partial^2 u^2}{\partial x^2} = 0. \tag{25}$$

Its ansatz solution is (15). Substituting (2) and (15) into (25) yields

$$\begin{aligned}
&2\big[\big(c^2 - c_0^2\big)a_2 + 4\big(1+m^2\big)\alpha k^2 a_2 - \beta\big(a_1^2 + 2a_0a_2\big)\big] - \Big\{\big(1+m^2\big)\big(c^2 - c_0^2\big) + \alpha k^2\big[\big(1+m^2\big)^2 + 12m^2\big]\\
&\quad + 2\beta\big[\big(1+m^2\big)a_0 - 6a_2\big]\Big\}a_1\,\mathrm{sn}\,\xi - 2\Big\{2\big(1+m^2\big)\big(c^2 - c_0^2\big)a_2 + 4\alpha k^2\big[2\big(1+m^2\big)^2 + 9m^2\big]a_2\\
&\quad - 2\beta\big[\big(1+m^2\big)a_1^2 + 2\big(1+m^2\big)a_0a_2 - 3a_2^2\big]\Big\}a_2\,\mathrm{sn}^2\,\xi + 2\Big\{m^2\big(c^2 - c_0^2\big) + 10m^2\big(1+m^2\big)\alpha k^2\\
&\quad - \beta\big[2m^2a_0 - 9\big(1+m^2\big)a_2\big]\Big\}a_1\,\mathrm{sn}^3\,\xi + 2\Big\{3m^2\big(c^2 - c_0^2\big)a_2 + 60m^2\big(1+m^2\big)\alpha k^2 a_2\\
&\quad - \beta\big[3m^2a_1^2 + 6m^2a_0a_2 - 8\big(1+m^2\big)a_2\big]\Big\}\,\mathrm{sn}^4\,\xi - 24m^2\big(m^2\alpha k^2 + \beta a_2\big)a_1\,\mathrm{sn}^5\,\xi\\
&\quad - 20m^2\big(6m^2\alpha k^2 + \beta a_2\big)a_2\,\mathrm{sn}^6\,\xi = 0,
\end{aligned} \tag{26}$$

from which it is determined that

$$a_1 = 0, \qquad a_2 = -\frac{6}{\beta}m^2\alpha k^2, \qquad a_0 = \frac{c^2 - c_0^2}{2\beta} + \frac{2}{\beta}\big(1+m^2\big)\alpha k^2. \tag{27}$$

Thus the periodic solution of (25) is

$$u = \frac{c^2 - c_0^2}{2\beta} + \frac{2}{\beta}\big(1+m^2\big)\alpha k^2 - \frac{6}{\beta}m^2\alpha k^2\,\mathrm{sn}^2\,\xi = \frac{c^2 - c_0^2}{2\beta} - \frac{2}{\beta}\big(2m^2 - 1\big)\alpha k^2 + \frac{6}{\beta}m^2\alpha k^2\,\mathrm{cn}^2\,\xi. \tag{28}$$

Its corresponding solitary wave solution is

$$u = \frac{c^2 - c_0^2}{2\beta} - \frac{2\alpha k^2}{\beta} + \frac{6\alpha k^2}{\beta}\,\mathrm{sech}^2\,\xi. \tag{29}$$

3.1.3. mKdV equation

$$\frac{\partial u}{\partial t} + \alpha u^2\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0. \tag{30}$$

Its ansatz solution is

$$u = a_0 + a_1\,\mathrm{sn}\,\xi. \tag{31}$$

Substituting (2) and (31) into (25) yields

$$\big[-c + \alpha a_0^2 - \beta\big(1+m^2\big)k^2\big]a_1\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi + 2\alpha a_0a_1^2\,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi + \big(\alpha a_1^2 + 6\beta m^2k^2\big)a_1\,\mathrm{sn}^2\,\xi\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi = 0, \tag{32}$$

from which it is determined that

$$a_0 = 0, \qquad a_1 = \pm\sqrt{-\frac{6\beta}{\alpha}}mk, \qquad c = -\beta\big(1+m^2\big)k^2. \tag{33}$$

Thus the periodic solution of (30) is

$$u=\pm\sqrt{-\frac{6\beta}{\alpha}}mk\,\mathrm{sn}\,\xi=\pm\sqrt{\frac{6c}{\alpha(1+m^2)}}m\,\mathrm{sn}\sqrt{-\frac{c}{\beta(1+m^2)}}(x-ct), \tag{34}$$

which demands that $c>0$, $\alpha>0$, $\beta<0$ or $c<0$, $\alpha<0$, $\beta>0$. And its corresponding shock wave solution is

$$u=\pm\sqrt{-\frac{6\beta}{\alpha}}k\tanh\xi=\pm\sqrt{\frac{3c}{\alpha}}\tanh\sqrt{-\frac{c}{2\beta}}(x-ct). \tag{35}$$

3.1.4. Nonlinear Klein–Gordon equation

We discuss the following two kinds of nonlinear Klein–Gordon equations:

$$\frac{\partial^2 u}{\partial t^2}-c_0^2\frac{\partial^2 u}{\partial x^2}+\alpha u-\beta u^2=0 \tag{36}$$

and

$$\frac{\partial^2 u}{\partial t^2}-c_0^2\frac{\partial^2 u}{\partial x^2}+\alpha u-\beta u^3=0. \tag{37}$$

Their corresponding ansatz solutions are (15) and (31), respectively. Similarly, their exact periodic solutions can be obtained. For (36), it is

$$\begin{aligned}u&=\frac{\alpha}{2\beta}-\frac{2(1+m^2)}{\beta}k^2\left(c^2-c_0^2\right)+\frac{6}{\beta}m^2k^2\left(c^2-c_0^2\right)\mathrm{sn}^2\,\xi\\&=\frac{\alpha}{2\beta}-\frac{2(1-2m^2)}{\beta}k^2\left(c^2-c_0^2\right)-\frac{6}{\beta}m^2k^2\left(c^2-c_0^2\right)\mathrm{cn}^2\,\xi.\end{aligned} \tag{38}$$

Its corresponding solitary wave solution is

$$u=\frac{\alpha}{2\beta}-\frac{2}{\beta}k^2\left(c^2-c_0^2\right)-\frac{6}{\beta}k^2\left(c^2-c_0^2\right)\mathrm{sech}^2\,\xi. \tag{39}$$

For (37), it is

$$u=\pm\sqrt{\frac{2m^2k^2(c^2-c_0^2)}{\beta}}\,\mathrm{sn}\,\xi=\pm\sqrt{\frac{2m^2\alpha}{\beta(1+m^2)}}\,\mathrm{sn}\sqrt{\frac{\alpha}{(c^2-c_0^2)(1+m^2)}}(x-ct), \tag{40}$$

which demands $\alpha>0$, $\beta>0$, $c^2>c_0^2$ or $\alpha<0$, $\beta<0$, $c^2<c_0^2$. Its shock wave solution is

$$u=\pm\sqrt{\frac{2k^2(c^2-c_0^2)}{\beta}}\tanh\xi=\pm\sqrt{\frac{\alpha}{\beta}}\tanh\sqrt{\frac{\alpha}{2(c^2-c_0^2)}}(x-ct). \tag{41}$$

3.2. Applications to coupled equations

The Jacobi elliptic function expansion method can be also applied to coupled equations to obtain their exact periodic solutions. We illustrate this by using the following variant Boussinesq equations:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\alpha\frac{\partial^3 u}{\partial t\partial x^2}=0,\qquad \frac{\partial v}{\partial t}+\frac{\partial(uv)}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0. \tag{42}$$

Setting

$$u=u(\xi),\qquad v=v(\xi),\qquad \xi=k(x-ct), \tag{43}$$

obviously we can make the following ansatz solutions to (42):

$$u(\xi) = a_0 + a_1 \operatorname{sn}\xi + a_2 \operatorname{sn}^2\xi, \qquad v(\xi) = b_0 + b_1 \operatorname{sn}\xi + b_2 \operatorname{sn}^2\xi. \tag{44}$$

Substituting (43) and (44) into (42) yields

$$\begin{aligned} u &= c + \frac{\beta}{2\alpha c} - 4(1+m^2)\alpha k^2 c + 12cm^2\alpha k^2 \operatorname{sn}^2\xi = c + \frac{\beta}{2\alpha c} - 4(1-2m^2)\alpha k^2 c - 12cm^2\alpha k^2 \operatorname{cn}^2\xi, \\ v &= -\frac{\beta^2}{4c^2\alpha^2} + 2(1+m^2)\beta k^2 - 6m^2\beta k^2 \operatorname{sn}^2\xi = -\frac{\beta^2}{4c^2\alpha^2} + 2(1-2m^2)\beta k^2 + 6m^2\beta k^2 \operatorname{cn}^2\xi, \end{aligned} \tag{45}$$

which is the exact periodic solution of (42), i.e., the cnoidal wave solution, their corresponding solitary wave solution is

$$u = c + \frac{\beta}{2\alpha c} + 4\alpha k^2 c - 12c\alpha k^2 \operatorname{sech}^2\xi, \qquad v = -\frac{\beta^2}{4c^2\alpha^2} - 2\beta k^2 + 6\beta k^2 \operatorname{sech}^2\xi. \tag{46}$$

4. Conclusion

In this Letter, the Jacobi elliptic function expansion method is proposed and applied to some nonlinear wave equations. It is shown that this method is more general than the hyperbolic tangent function expansion method. And the periodic wave solutions obtained by the Jacobi elliptic function expansion method contain the shock wave and solitary wave solutions. In the applications, it is shown that the Jacobi elliptic function expansion method can be applied to both single equation and coupled equations. Actually, this method can be applied to obtain solutions to more nonlinear wave equations, as long as the odd- and even-order derivative terms do not coexist in the nonlinear wave equations.

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