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Jacobi 椭圆函数展开法及其在求解非线性波动方程中的应用*

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给出了 Jacobi 椭圆函数展开法, 且应用该方法获得了几种非线性波方程的准确周期解. 该方法包含了双曲函数展开法, 应用该方法得到的周期解包含了冲击波解和孤波解.

关键词: Jacobi 椭圆函数, 非线性方程, 周期解, 孤波解
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1 引 言

寻找非线性波方程的准确解在非线性的问题中占有重要地位. 最近, 出现了许多求非线性波方程准确解的新方法, 如齐次平衡法[1—3,16,19]、双曲正切函数展开法[4—6]、试探函数法[7,8]、非线性变换法[9,10]和 sine-cosine 方法[11]. 文献[15,17,18,20]也探讨了非线性方程的解. 但是, 这些方法只能求得非线性波方程的冲击波解和孤波解, 不能求得非线性方程的周期解. Porubov 等[12—14]虽然求得了一些非线性波方程的准确周期解, 但是应用的是 Weierstrass 椭圆函数. 本文提出的 Jacobi 椭圆函数展开法不但包含了双曲正切函数展开法, 而且用该方法求得的周期解也包含了冲击波解和孤波解.

2 Jacobi 椭圆函数展开法

考虑非线性波方程

$$N\left(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 u}{\partial x^2},\cdots\right)=0, \tag{1}$$

寻求它的行波解为

$$u=u(\xi),\quad \xi=k(x-ct), \tag{2}$$

其中 k 和 c 分别为波数和波速.

本文的方法是将 $u(\xi)$ 展开为下列 Jacobi 椭圆正弦函数 $\mathrm{sn}\xi$ 的级数:

$$u(\xi)=\sum_{j=0}^{n}a_j\mathrm{sn}^j\xi. \tag{3}$$

它的最高阶数为

$$O(u(\xi))=n. \tag{4}$$

因为

$$\frac{\mathrm{d}u}{\mathrm{d}\xi}=\sum_{j=0}^{n}ja_j\mathrm{sn}^{j-1}\xi\mathrm{cn}\xi\mathrm{dn}\xi, \tag{5}$$

其中 $\mathrm{cn}\xi$ 和 $\mathrm{dn}\xi$ 分别为 Jacobi 椭圆余弦函数和第三种 Jacobi 椭圆函数, 且

$$\mathrm{cn}^2\xi=1-\mathrm{sn}^2\xi,\ \mathrm{dn}^2\xi=1-m^2\mathrm{sn}^2\xi, \tag{6}$$

$m(0<m<1)$ 为模数, 且

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{sn}\xi=\mathrm{cn}\xi\mathrm{dn}\xi,\ \frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{cn}\xi=-\mathrm{sn}\xi\mathrm{dn}\xi,$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{dn}\xi=-m^2\mathrm{sn}\xi\mathrm{cn}\xi. \tag{7}$$

由(5)式, 可以认为 $\mathrm{d}u/\mathrm{d}\xi$ 的最高阶数为

$$O\left(\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)=n+1. \tag{8}$$

类似地, 有

$$O\left(u\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)=2n+1,\ O\left(\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}\right)=n+2,$$

$$O\left(\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}\right)=n+3. \tag{9}$$

在(3)式中选择 n, 使得非线性波方程(1)中的非线性项和最高阶导数项平衡. 应该指出的是, 因为 m

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$\to 1$ 时,$\mathrm{sn}\,\xi \to \tanh \xi$,(3) 式就退化为

$$u(\xi) = \sum_{j=0}^{n} a_j \tanh^j \xi. \tag{10}$$

所以本文的方法包含了双曲正切函数展开法.

3　方法的应用

下面举例说明Jacobi 椭圆正弦函数展开法的应用.

3.1　KdV方程

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0. \tag{11}$$

把(2)式代入上式,求得

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi} + u\frac{\mathrm{d}u}{\mathrm{d}\xi} + \beta k^2 \frac{\mathrm{d}^3 u}{\mathrm{d}\xi^3} = 0. \tag{12}$$

由(3)式,可知

$$O\left(u\frac{\mathrm{d}u}{\mathrm{d}\xi}\right) = 2n+1,\ O\left(\frac{\mathrm{d}^3 u}{\mathrm{d}\xi^3}\right) = n+3. \tag{13}$$

两者平衡,有

$$n = 2. \tag{14}$$

故设方程(12)的解为

$$u(\xi) = a_0 + a_1 \mathrm{sn}\,\xi + a_2 \mathrm{sn}^2 \xi. \tag{15}$$

注意到

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = (a_1 + 2a_2 \mathrm{sn}\,\xi)\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi, \tag{16}$$

$$u\frac{\mathrm{d}u}{\mathrm{d}\xi} = [a_0 a_1 + (a_1^2 + 2a_0 a_2)\mathrm{sn}\,\xi + 3a_1 a_2 \mathrm{sn}^2 \xi + 2a_2^2 \mathrm{sn}^3 \xi]\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi, \tag{17}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} = 2a_2 - (1+m^2)a_1 \mathrm{sn}\,\xi - 4(1+m^2) \cdot a_2 \mathrm{sn}^2 \xi + 2m^2 a_1 \mathrm{sn}^3 \xi + 6m^2 a_2 \mathrm{sn}^4 \xi, \tag{18}$$

$$\frac{\mathrm{d}^3 u}{\mathrm{d}\xi^3} = [-(1+m^2)a_1 - 8(1+m^2)a_2 \mathrm{sn}\,\xi + 6m^2 a_1 \mathrm{sn}^2 \xi + 24m^2 a_2 \mathrm{sn}^3 \xi]\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi, \tag{19}$$

把(15)式代入方程(12),有

$$\langle -[c - a_0 + (1+m^2)\beta k^2]a_1 + \{a_1^2 - 2[c - a_0 + 4(1+m^2)\beta k^2]a_2\}\mathrm{sn}\,\xi\rangle \mathrm{cn}\,\xi\,\mathrm{dn}\,\xi + 3a_1(a_2 + 2m^2\beta k^2)\mathrm{sn}^2\xi\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi + 2a_2 \cdot (a_2 + 12m^2\beta k^2)\mathrm{sn}^3\xi\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi = 0. \tag{20}$$

由此定得

$$a_1 = 0,\ a_0 = c + 4(1+m^2)\beta k^2,\ a_2 = -12m^2\beta k^2. \tag{21}$$

代入(15)式,最后求得

$$u = c + 4(1+m^2)\beta k^2 - 12m^2\beta k^2 \mathrm{sn}^2\xi = c + 4(1-2m^2)\beta k^2 + 12m^2\beta k^2 \mathrm{cn}^2 \xi. \tag{22}$$

这就是 KdV 方程(11)的准确周期解,也就是 KdV 方程的椭圆余弦波解.

取 $m=1$,则(22)式化为

$$u = c - 4\beta k^2 + 12\beta k^2 \mathrm{sech}^2 \xi. \tag{23}$$

这就是 KdV 方程(11)的孤波解. 特别地,取 $c = 4\beta k^2$,则(23)式化为

$$u = 3c\,\mathrm{sech}^2 \sqrt{\frac{c}{4\beta}}(x - ct). \tag{24}$$

3.2　Boussinesq 方程

$$\frac{\partial^2 u}{\partial t^2} - c_0^2\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial^4 u}{\partial x^4} - \beta\frac{\partial^2 u^2}{\partial x^2} = 0. \tag{25}$$

把(2)式代入上式,求得

$$(c^2 - c_0^2)\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} - \alpha k^2\frac{\mathrm{d}^4 u}{\mathrm{d}\xi^4} - \beta\frac{\mathrm{d}^2 u^2}{\mathrm{d}\xi^2} = 0. \tag{26}$$

由(2)式,可知

$$O\left(\frac{\mathrm{d}^2 u^2}{\mathrm{d}\xi^2}\right) = 2n+2,\ O\left(\frac{\mathrm{d}^4 u}{\mathrm{d}\xi^4}\right) = n+4. \tag{27}$$

两者平衡,有

$$n = 2. \tag{28}$$

因而方程(26)仍有形式如(15)式的解. 注意到

$$\frac{\mathrm{d}u^2}{\mathrm{d}\xi} = [2a_0 a_1 + (2a_1^2 + 4a_0 a_2)\mathrm{sn}\,\xi + 6a_1 a_2 \mathrm{sn}^2\xi + 4a_2^2 \mathrm{sn}^3 \xi]\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi, \tag{29}$$

$$\frac{\mathrm{d}^4 u}{\mathrm{d}\xi^4} = -8(1+m^2)a_2 + [(1+m^2)^2 + 12m^2]a_1 \mathrm{sn}\,\xi + 8[2(1+m^2)^2 + 9m^2]a_2 \mathrm{sn}^2\xi - 20m^2(1+m^2) \cdot a_1 \mathrm{sn}^3\xi - 120m^2(1+m^2)a_2 \mathrm{sn}^4\xi + 24m^4 a_1 \mathrm{sn}^5 \xi + 120m^4 a_2 \mathrm{sn}^6 \xi, \tag{30}$$

$$\frac{\mathrm{d}^2 u^2}{\mathrm{d}\xi^2} = 2(a_1^2 + 2a_0 a_2) - 2[(1+m^2)a_0 - 6a_2]a_1 \mathrm{sn}\,\xi - 4[(1+m^2)a_1^2 + 2(1+m^2)a_0 a_2 - 3a_2^2]\mathrm{sn}^2\xi + 2[2m^2 a_0 - 9(1+m^2)a_2]a_1 \mathrm{sn}^3\xi + 2[3m^2 a_1^2 + 6m^2 a_0 a_2 - 8(1+m^2)a_2^2]\mathrm{sn}^4\xi + 24m^2 a_1 a_2 \mathrm{sn}^5\xi + 20m^2 a_2^2 \mathrm{sn}^6 \xi, \tag{31}$$

把(15)式代入方程(26),有

$$2[(c^2-c_0^2)a_2+4(1+m^2)\alpha k^2a_2-\beta(a_1^2+2a_0a_2)]-\{(1+m^2)(c^2-c_0^2)+\alpha k^2[(1+m^2)^2+12m^2]+2\beta[(1+m^2)a_0-6a_2]\}a_1\mathrm{sn}\xi-2\{2(1+m^2)(c^2-c_0^2)a_2+4\alpha k^2[2(1+m^2)^2+9m^2]a_2-2\beta[(1+m^2)a_1^2+2(1+m^2)a_0a_2-3a_2^2]\}a_2\mathrm{sn}^2\xi+2\{m^2(c^2-c_0^2)+10m^2(1+m^2)\alpha k^2-\beta[2m^2a_0-9(1+m^2)a_2]\}a_1\mathrm{sn}^3\xi+2\{3m^2(c^2-c_0^2)a_2+60m^2(1+m^2)\alpha k^2a_2-\beta[3m^2a_1^2+6m^2a_0a_2-8(1+m^2)a_2]\}\mathrm{sn}^4\xi-24m^2(m^2\alpha k^2+\beta a_2)a_1\mathrm{sn}^5\xi-20m^2(6m^2\alpha k^2+\beta a_2)a_2\mathrm{sn}^6\xi=0. \tag{32}$$

由此定得

$$a_1=0,\quad a_2=-\frac{6}{\beta}m^2\alpha k^2,$$

$$a_0=\frac{c^2-c_0^2}{2\beta}+\frac{2}{\beta}(1+m^2)\alpha k^2. \tag{33}$$

代入(15)式,最后求得

$$u=\frac{c^2-c_0^2}{2\beta}+\frac{2}{\beta}(1+m^2)\alpha k^2-\frac{6}{\beta}m^2\alpha k^2\mathrm{sn}^2\xi=\frac{c^2-c_0^2}{2\beta}-\frac{2}{\beta}(2m^2-1)\alpha k^2+\frac{6}{\beta}m^2\alpha k^2\mathrm{cn}^2\xi. \tag{34}$$

这就是 Boussinesq 方程(25)的准确周期解,也就是 Boussinesq 方程的椭圆余弦波解.

取 $m=1$,则(34)式化为

$$u=\frac{c^2-c_0^2}{2\beta}-\frac{2\alpha k^2}{\beta}+\frac{6\alpha k^2}{\beta}\mathrm{sech}^2\xi. \tag{35}$$

这就是 Boussinesq 方程的孤波解. 特别地,取 $c^2-c_0^2=4\alpha k^2$,则(35)式化为

$$u=\frac{3(c^2-c_0^2)}{2\beta}\mathrm{sech}^2\sqrt{\frac{c^2-c_0^2}{4\alpha}}(x-ct). \tag{36}$$

3.3 mKdV方程

$$\frac{\partial u}{\partial t}+\alpha u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3u}{\partial x^3}=0. \tag{37}$$

把(2)式代入上式,求得

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi}+\alpha u^2\frac{\mathrm{d}u}{\mathrm{d}\xi}+\beta k^2\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}=0. \tag{38}$$

由(3)式,可知

$$O\left(u^2\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)=3n+1,\quad O\left(\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}\right)=n+3. \tag{39}$$

两者平衡,有

$$n=1. \tag{40}$$

因而 mKdV 方程(37)有下列形式的周期解:

$$u=a_0+a_1\mathrm{sn}\xi. \tag{41}$$

注意到

$$\frac{\mathrm{d}u}{\mathrm{d}\xi}=a_1\mathrm{cn}\xi\mathrm{dn}\xi, \tag{42}$$

$$u^2\frac{\mathrm{d}u}{\mathrm{d}\xi}=(a_0^2a_1+2a_0a_1^2\mathrm{sn}\xi+a_1^3\mathrm{sn}^2\xi)\mathrm{cn}\xi\mathrm{dn}\xi, \tag{43}$$

$$\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}=[-(1+m^2)a_1+6m^2a_1\mathrm{sn}^2\xi]\mathrm{cn}\xi\mathrm{dn}\xi, \tag{44}$$

把(41)式代入方程(38),有

$$[-c+\alpha a_0^2-\beta(1+m^2)k^2]a_1\mathrm{cn}\xi\mathrm{dn}\xi+2\alpha a_0a_1^2\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi+(\alpha a_1^2+6\beta m^2k^2)a_1\mathrm{sn}^2\xi\mathrm{cn}\xi\mathrm{dn}\xi=0. \tag{45}$$

由此定得

$$a_0=0,\quad a_1=\pm\sqrt{-\frac{6\beta}{\alpha}}mk,\quad c=-\beta(1+m^2)k^2. \tag{46}$$

代入(41)式,求得

$$u=\pm\sqrt{-\frac{6\beta}{\alpha}}mk\mathrm{sn}\xi=\pm\sqrt{\frac{6c}{\alpha(1+m^2)}}\cdot m\mathrm{sn}\sqrt{-\frac{c}{\beta(1+m^2)}}(x-ct). \tag{47}$$

这就是 mKdV 方程(37)的准确周期解. 它要求 $c>0$, $\alpha>0$, $\beta<0$ 或 $c<0$, $\alpha<0$, $\beta>0$.

取 $m=1$,则(47)式化为

$$u=\pm\sqrt{-\frac{6\beta}{\alpha}}k\tanh\xi=\pm\sqrt{\frac{3c}{\alpha}}\tanh\sqrt{-\frac{c}{2\beta}}(x-ct). \tag{48}$$

这就是 mKdV 方程(37)的冲击波解.

3.4 非线性 Klein-Gordon 方程

现在讨论下列两类方程:

$$\frac{\partial^2u}{\partial t^2}-c_0^2\frac{\partial^2u}{\partial x^2}+\alpha u-\beta u^2=0 \tag{49}$$

和

$$\frac{\partial^2u}{\partial t^2}-c_0^2\frac{\partial^2u}{\partial x^2}+\alpha u-\beta u^3=0. \tag{50}$$

在作(2)式的变换后,它们分别化为

$$k^2(c^2-c_0^2)\frac{d^2u}{d\xi^2}+\alpha u-\beta u^2=0 \qquad (51)$$

和

$$k^2(c^2-c_0^2)\frac{d^2u}{d\xi^2}+\alpha u-\beta u^3=0. \qquad (52)$$

以(3)式的解代入方程(51)和(52),分别使其中的非线性项和最高阶导数项平衡,显然分别有

$$n=2 \qquad (53)$$

和

$$n=1. \qquad (54)$$

因而方程(51)和(52)分别有形式如(15)和(41)式的解. 把(15)式代入(51)式,有

$$\begin{aligned}&\{\alpha a_0-\beta a_0^2+2k^2(c^2-c_0^2)a_2\}+[\alpha-(1+m^2)\\&\cdot k^2(c^2-c_0^2)-2\beta a_0]a_1\mathrm{sn}\xi+[\alpha a_2\\&-2\beta a_0a_2-\beta a_1^2-4(1+m^2)k^2(c^2-c_0^2)a_2]\\&\cdot\mathrm{sn}^2\xi+2[k^2(c^2-c_0^2)m^2-\beta a_2]a_1\\&\cdot\mathrm{sn}^3\xi+[6k^2(c^2-c_0^2)m^2\\&-\beta a_2]a_2\mathrm{sn}^4\xi=0.\end{aligned} \qquad (55)$$

由此定得

$$a_1=0,\ a_2=\frac{6}{\beta}m^2k^2(c^2-c_0^2),$$
$$a_0=\frac{\alpha}{2\beta}-\frac{2(1+m^2)}{\beta}k^2(c^2-c_0^2),$$
$$k^4(c^2-c_0^2)^2=\frac{\alpha^2}{16[(1+m^2)^2-3m^2]}. \qquad (56)$$

代入(15)式,求得

$$\begin{aligned}u&=\frac{\alpha}{2\beta}-\frac{2(1+m^2)}{\beta}k^2(c^2-c_0^2)\\&\quad+\frac{6}{\beta}m^2k^2(c^2-c_0^2)\mathrm{sn}^2\xi\\&=\frac{\alpha}{2\beta}-\frac{2(1-2m^2)}{\beta}k^2(c^2-c_0^2)\\&\quad-\frac{6}{\beta}m^2k^2(c^2-c_0^2)\mathrm{cn}^2\xi.\end{aligned} \qquad (57)$$

这就是方程(51)的椭圆余弦波解.

取 $m=1$,则(57)式化为

$$\begin{aligned}u&=\frac{\alpha}{2\beta}-\frac{2}{\beta}k^2(c^2-c_0^2)\\&\quad-\frac{6}{\beta}k^2(c^2-c_0^2)\mathrm{sech}^2\xi.\end{aligned} \qquad (58)$$

这就是方程(51)的孤波解. 特别地,取 $k^2(c^2-c_0^2)=\alpha/4$,则(58)式化为

$$u=\frac{3\alpha}{2\beta}\mathrm{sech}^2\sqrt{\frac{1}{4}\left|\frac{\alpha}{c^2-c_0^2}\right|}(x-ct). \qquad (59)$$

把(41)式代入(52)式,有

$$\begin{aligned}&(\alpha-\beta a_0^2)a_0+[\alpha-3\beta a_0^2-(1+m^2)k^2\\&\cdot(c^2-c_0^2)]a_1\mathrm{sn}\xi-3\beta a_0a_1^2\mathrm{sn}^2\xi\\&-[\beta a_1^2+2m^2k^2(c^2-c_0^2)]a_1\mathrm{sn}^3\xi=0.\end{aligned} \qquad (60)$$

由此定得

$$a_0=0,\ a_1=\pm\sqrt{\frac{2m^2k^2(c^2-c_0^2)}{\beta}},$$
$$k^2=\frac{\alpha}{(1+m^2)(c^2-c_0^2)}. \qquad (61)$$

代入(41)式,求得

$$\begin{aligned}u&=\pm\sqrt{\frac{2m^2k^2(c^2-c_0^2)}{\beta}}\mathrm{sn}\xi\\&=\pm\sqrt{\frac{2m^2\alpha}{\beta(1+m^2)}}\mathrm{sn}\sqrt{\frac{\alpha}{(c^2-c_0^2)(1+m^2)}}(x-ct).\end{aligned} \qquad (62)$$

这就是方程(42)的准确周期解. 它要求 $\alpha>0$, $\beta>0$, $c^2>c_0^2$ 或 $\alpha<0$, $\beta<0$, $c^2<c_0^2$.

取 $m=1$,则(62)式化为

$$\begin{aligned}u&=\pm\sqrt{\frac{2k^2(c^2-c_0^2)}{\beta}}\tanh\xi\\&=\pm\sqrt{\frac{\alpha}{\beta}}\tanh\sqrt{\frac{\alpha}{2(c^2-c_0^2)}}(x-ct).\end{aligned} \qquad (63)$$

这就是方程(52)的冲击波解.

3.5 Variant Boussinesq 方程组

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\alpha\frac{\partial^3u}{\partial t\partial x^2}=0, \qquad (64a)$$

$$\frac{\partial v}{\partial t}+\frac{\partial(uv)}{\partial x}+\beta\frac{\partial^3u}{\partial x^3}=0. \qquad (64b)$$

令

$$u=u(\xi),\ v=v(\xi),\ \xi=k(x-ct), \qquad (65)$$

则方程组(64)化为

$$-c\frac{du}{d\xi}+u\frac{du}{d\xi}+\frac{dv}{d\xi}-\alpha k^2c\frac{d^3u}{d\xi^3}=0,$$
$$-c\frac{dv}{d\xi}+\frac{d(uv)}{d\xi}+\beta k^2\frac{d^3u}{d\xi^3}=0. \qquad (66)$$

显然可设

$$u(\xi)=a_0+a_1\mathrm{sn}\xi+a_2\mathrm{sn}^2\xi,$$
$$v(\xi)=b_0+b_1\mathrm{sn}\xi+b_2\mathrm{sn}^2\xi. \qquad (67)$$

把(67)式代入方程组(66),有

$$\begin{aligned}&[-ca_1+a_0a_1+b_1+\alpha k^2c(1+m^2)a_1]\mathrm{cn}\xi\mathrm{dn}\xi\\&+[-2ca_2+(a_1^2+2a_0a_2)+2b_2+8\alpha k^2c(1+m^2)\\&\cdot a_2]\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi+3(a_2-2\alpha k^2cm^2)a_1\mathrm{sn}^2\xi\mathrm{cn}\xi\mathrm{dn}\xi\\&+2(a_2-12\alpha k^2cm^2)a_2\mathrm{sn}^3\xi\mathrm{cn}\xi\mathrm{dn}\xi=0,\end{aligned}$$

$$[-cb_1+a_0b_1+a_1b_0-\beta k^2(1+m^2)a_1]\mathrm{cn}\xi\mathrm{dn}\xi$$
$$+[-2cb_2+2(a_0b_2+a_1b_1+a_2b_0)-8\beta k^2$$
$$\cdot(1+m^2)a_2]\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi+3[(a_1b_2+a_2b_1)$$
$$+2m^2\beta k^2a_1]\mathrm{sn}^2\xi\mathrm{cn}\xi\mathrm{dn}\xi+4(b_2+6m^2\beta k^2)$$
$$\cdot a_2\mathrm{sn}^3\xi\mathrm{cn}\xi\mathrm{dn}\xi=0. \qquad (68)$$

由此定得

$$a_1=0,\ b_1=0,\ a_2=12cm^2\alpha k^2,\ b_2=-6m^2\beta k^2,$$
$$\begin{aligned}a_0&=c-\frac{b_2}{a_2}-4(1+m^2)\alpha k^2c\\&=c+\frac{\beta}{2\alpha c}-4(1+m^2)\alpha k^2c,\\b_0&=\frac{b_2}{a_2}(c-a_0)+4(1+m^2)\beta k^2\\&=-\frac{\beta^2}{4c^2\alpha^2}+2(1+m^2)\beta k^2.\end{aligned} \qquad (69)$$

代入(67)式,求得

$$\begin{aligned}u&=c+\frac{\beta}{2\alpha c}-4(1+m^2)\alpha k^2c+12cm^2\alpha k^2\mathrm{sn}^2\xi\\&=c+\frac{\beta}{2\alpha c}-4(1-2m^2)\alpha k^2c-12cm^2\alpha k^2\mathrm{cn}^2\xi,\\v&=-\frac{\beta^2}{4c^2\alpha^2}+2(1+m^2)\beta k^2-6m^2\beta k^2\mathrm{sn}^2\xi\\&=-\frac{\beta^2}{4c^2\alpha^2}+2(1-2m^2)\beta k^2\\&\quad+6m^2\beta k^2\mathrm{cn}^2\xi.\end{aligned} \qquad (70)$$

这就是 Variant Boussinesq 方程组(64)的周期解,也就是椭圆余弦波解.

取 $m=1$,则(70)式化为

$$\begin{aligned}u&=c+\frac{\beta}{2\alpha c}+4\alpha k^2c-12c\alpha k^2\mathrm{sech}^2\xi,\\v&=-\frac{\beta^2}{4c^2\alpha^2}-2\beta k^2+6\beta k^2\mathrm{sech}^2\xi.\end{aligned} \qquad (71)$$

这就是 Variant Boussinesq 方程组(64)的孤波解.

4 结 论

本文提出的 Jacobi 椭圆函数展开法可以求得几种非线性波方程的准确周期解.该方法包含了双曲正切函数展开法,并且周期解包含了冲击波解和孤波解.

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EXPANSION METHOD ABOUT THE JACOBI ELLIPTIC FUNCTION AND ITS APPLICATIONS TO NONLINEAR WAVE EQUATIONS *

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ABSTRACT

A Jacobi elliptic function expansion method is proposed to construct the exact periodic solutions of nonlinear wave equations. This new method contains the hyperbolic tangent expansion method, and the periodic solutions obtained by this method, include the shock wave solutions and the solitary wave solutions.

Keywords: Jacobi elliptic function, nonlinear equation, periodic solution, solitary wave solution
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