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New transformations and new approach to find exact solutions to nonlinear equations

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Abstract

New transformations from the nonlinear sine-Gordon equation are shown in this Letter, based on them a new approach is proposed to construct exact periodic solutions to nonlinear equations. It is shown that more new periodic solutions can be obtained by this new approach and more shock wave solutions or solitary wave solutions can be got under their limit condition.

Keywords: Sine-Gordon equation; Jacobi elliptic function; Nonlinear equation; Periodic wave solution

1. New transformations from sine-Gordon equation

In Ref. [1], based on the sine-Gordon equation, a transformation

$$\frac{d\omega}{d\xi} = \sin\omega, \tag{1}$$

was obtained and applied to solve nonlinear wave equations, many exact solutions have been got since then by that so-called sine–cosine method. We will explain in this Letter that this transformation is just a special case under the limit condition. So we begin with the sine-Gordon equation

$$u_{tt} - c_0^2 u_{xx} + f_0^2 \sin u = 0. \tag{2}$$

To get transformation from Eq. (2), we solve it in the following frame

$$u = u(\xi), \quad \xi = x - ct, \tag{3}$$

where c is wave velocity. Then Eq. (2) becomes

$$\left(c^2 - c_0^2\right)\frac{d^2u}{d\xi^2} + f_0^2 \sin u = 0. \tag{4}$$

Integrating this equation, we get

$$\left(\frac{d\omega}{d\xi}\right)^2 + \frac{f_0^2}{c^2 - c_0^2}\sin^2\omega = \frac{H}{2}, \tag{5}$$

where H is integration constant, $\omega = u/2$. There are two cases to be considered:

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Case 1. $c^2 > c_0^2$

Set $\lambda_0^2 = \frac{f_0^2}{c^2-c_0^2}$ and $H = 2\lambda_0^2 m^2$, Eq. (5) can be rewritten as

$$\frac{d\omega}{d\xi} = \pm\lambda_0\sqrt{m^2 - \sin^2\omega}. \tag{6}$$

Eq. (6) is the first transformation we get from the nonlinear sine-Gordon equation. Then set $\sin\omega = m\sin\varphi$, Eq. (6) reads

$$\frac{d\varphi}{d\xi} = \pm\lambda_0\sqrt{1 - m^2\sin^2\varphi}. \tag{7}$$

Case 2. $c^2 < c_0^2$

Similarly, we can get

$$\frac{d\omega}{d\xi} = \pm\lambda_1\sqrt{m'^2 - \cos^2\omega}, \tag{8}$$

where $\lambda_1^2 = -\lambda_0^2$ and $m'^2 = 1 - m^2$, this is the second transformation we get from the nonlinear sine-Gordon equation. We can see that the transformation (1) is just a special case of transformation (8) when "+" is taken in Eq. (8) and $\lambda_1 = 1$, $m'^2 = 1$.

Actually, from Eq. (7) we know that the transformation (6) admits the following solution

$$\sin\omega = \pm m\,\mathrm{sn}(\lambda_0\xi, m), \tag{9}$$

and then we get

$$\cos\omega = \pm\,\mathrm{dn}(\lambda_0\xi, m), \tag{10}$$

where $\mathrm{sn}(\lambda_0\xi, m)$ and $\mathrm{dn}(\lambda_0\xi, m)$ are Jacobi elliptic sine function and Jacobi elliptic function of the third kind, m and m' are modulus and co-modulus, respectively. Details about Jacobi elliptic functions can be found in Appendix A and references therein.

Similarly, the transformation (8) admits the following solution

$$\cos\omega = \pm m'\,\mathrm{sn}(\lambda_1\xi, m'), \tag{11}$$

and then we get

$$\sin\omega = \pm\,\mathrm{dn}(\lambda_1\xi, m'). \tag{12}$$

2. New approach to find exact solutions to nonlinear equations

Many methods have been proposed to construct exact solutions to nonlinear equations for their important role in understanding the nonlinear problems. Among them there are the sine–cosine method [1], the homogeneous balance method [2–4], the hyperbolic tangent expansion method [5–7], the Jacobi elliptic function expansion method [8,9], the nonlinear transformation method [10,11], the trial function method [12,13] and others [14–16].

In the following, we will introduce another method based on the transformations given in the former section. Consider a given nonlinear wave equation

$$N(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0. \tag{13}$$

We seek its wave solutions in the frame of (3), then Eq. (13) can be rewritten as

$$N\left(u, \frac{du}{d\xi}, \frac{d^2u}{d\xi^2}, \ldots\right) = 0, \tag{14}$$

and $u(\xi)$ can be expressed as a finite series of $\sin\omega$ and $\cos\omega$, i.e., the ansatz

$$u(\xi) = \sum_{j=1}^{n} \cos^{j-1}\omega\,(a_j\cos\omega + b_j\sin\omega) + a_0, \tag{15}$$

where ω satisfies transformations (6) or (8). In this Letter, we only consider the following case:

$$\frac{d\omega}{d\xi} = \sqrt{m^2 - \sin^2\omega}, \tag{16}$$

then

$$\frac{d^2\omega}{d\xi^2} = -\cos\omega\sin\omega. \tag{17}$$

And other forms for the transformations (6) or (8) can be similarly applied to construct exact solutions to nonlinear wave equations.

The highest degree of (15) is

$$O\big(u(\xi)\big) = n, \tag{18}$$

then the highest degree of $\frac{du}{d\xi}$ can be taken as

$$O\left(\frac{du}{d\xi}\right) = n + 1, \tag{19}$$

and

$$O\left(u\,\frac{du}{d\xi}\right)=2n+1,\qquad O\left(\frac{d^2u}{d\xi^2}\right)=n+2,$$

$$O\left(\frac{d^3u}{d\xi^3}\right)=n+3. \tag{20}$$

Thus we can select n in (15) to balance the highest order of derivative term and nonlinear term in (14). Then substitute (15) into (14), determine the expansion coefficients and other undetermined constants, combine the results from the transformation (16), one can got exact solutions to the given nonlinear equations.

We know that when $m' \to 1$, then the transformation (8) degenerates as the transformation (1), so the solutions got from the above expansion contain the results obtained by sine–cosine method given by [1].

3. Applications

In this Letter, we will demonstrate the above approach on two examples: mKdV equation and system of variant Boussinesq equations [2].

3.1. mKdV equation

mKdV equation reads

$$u_t+\alpha u^2u_x+\beta u_{xxx}=0. \tag{21}$$

Substituting (3) into (21) yields

$$-c\frac{du}{d\xi}+\alpha u^2\frac{du}{d\xi}+\beta\frac{d^3u}{d\xi^3}=0. \tag{22}$$

Integrating this equation yields

$$-cu+\frac{\alpha}{3}u^3+\beta\frac{d^2u}{d\xi^2}=C_0, \tag{23}$$

where C_0 is integration constant.

Considering (18), (19) and (20) to balance the highest order of derivative term and nonlinear term in (23), we can get

$$n=1, \tag{24}$$

so the ansatz solution of (21) in term of $\sin\omega$ and $\cos\omega$ is

$$u=a_0+a_1\cos\omega+b_1\sin\omega. \tag{25}$$

We know that

$$\frac{du}{d\xi}=(b_1\cos\omega-a_1\sin\omega)\frac{d\omega}{d\xi}, \tag{26}$$

$$\begin{aligned}\frac{d^2u}{d\xi^2}&=(b_1\cos\omega-a_1\sin\omega)\frac{d^2\omega}{d\xi^2}\\&\quad-(a_1\cos\omega+b_1\sin\omega)\left(\frac{d\omega}{d\xi}\right)^2\\&=-\left(1+m^2\right)b_1\sin\omega-m^2a_1\cos\omega\\&\quad+2a_1\cos\omega\sin^2\omega+2b_1\sin^3\omega,\end{aligned} \tag{27}$$

$$\begin{aligned}u^3&=\left(a_0^3+3a_0a_1^2\right)+3\left(a_0^2+a_1^2\right)b_1\sin\omega\\&\quad+\left(3a_0^2+a_1^2\right)a_1\cos\omega+6a_0a_1b_1\cos\omega\sin\omega\\&\quad+3a_0\left(b_1^2-a_1^2\right)\sin^2\omega\\&\quad+\left(3b_1^2-a_1^2\right)a_1\cos\omega\sin^2\omega\\&\quad+\left(b_1^2-3a_1^2\right)b_1\sin^3\omega.\end{aligned} \tag{28}$$

So substituting (25) into (23) yields

$$\begin{aligned}&\left[-ca_0+\alpha\left(a_0^3+3a_0a_1^2\right)/3-C_0\right]\\&\quad+\left[-cb_1+\alpha\left(a_0^2+a_1^2\right)b_1-\beta\left(1+m^2\right)b_1\right]\sin\omega\\&\quad+\left[-ca_1+\alpha\left(3a_0^2+a_1^2\right)a_1/3-\beta m^2a_1\right]\cos\omega\\&\quad+2\alpha a_0a_1b_1\cos\omega\sin\omega+\alpha a_0\left(b_1^2-a_1^2\right)\sin^2\omega\\&\quad+\left[\alpha\left(3b_1^2-a_1^2\right)a_1/3+2\beta m^2a_1\right]\cos\omega\sin^2\omega\\&\quad+\left[\alpha\left(b_1^2-3a_1^2\right)b_1/3+2\beta m^2b_1\right]\sin^3\omega=0,\end{aligned} \tag{29}$$

from which set the coefficients of $(\cos\omega\sin\omega)^0$, $\sin\omega$, $\cos\omega$, $\cos\omega\sin\omega$, $\sin^2\omega$, $\cos\omega\sin^2\omega$ and $\sin^3\omega$ to be zeros, we can get the algebraic equations about a_0, a_1, b_1, C_0 and c

$$-ca_0+\alpha\left(a_0^3+3a_0a_1^2\right)/3-C_0=0, \tag{30a}$$

$$-cb_1+\alpha\left(a_0^2+a_1^2\right)b_1-\beta\left(1+m^2\right)b_1=0, \tag{30b}$$

$$-ca_1+\alpha\left(3a_0^2+a_1^2\right)a_1/3-\beta m^2a_1=0, \tag{30c}$$

$$2\alpha a_0a_1b_1=0, \tag{30d}$$

$$\alpha a_0\left(b_1^2-a_1^2\right)=0, \tag{30e}$$

$$\alpha\left(3b_1^2-a_1^2\right)a_1/3+2\beta m^2a_1=0, \tag{30f}$$

$$\alpha\left(b_1^2-3a_1^2\right)b_1/3+2\beta m^2b_1=0. \tag{30g}$$

Solving Eqs. (30a)–(30g) yield the following solutions for two cases:

Case 1. $a_1 = 0$

$$C_0 = 0, \qquad a_0 = 0,$$
$$b_1 = \pm\sqrt{-\frac{6\beta}{\alpha}}, \qquad c = -(1+m^2)\beta. \tag{31}$$

Case 2. $b_1 = 0$

$$C_0 = 0, \qquad a_0 = 0,$$
$$a_1 = \pm\sqrt{\frac{6\beta}{\alpha}}, \qquad c = -m^2\beta. \tag{32}$$

Thus the periodic solutions of (21) are

$$u_1 = b_1 \sin\omega = \pm\sqrt{-\frac{6\beta}{\alpha}}\, m\,\mathrm{sn}(x-ct), \tag{33}$$

and

$$u_2 = a_1 \cos\omega = \pm\sqrt{\frac{6\beta}{\alpha}}\,\mathrm{dn}(x-ct). \tag{34}$$

When $m \to 1$, $\mathrm{sn}\,\xi \to \tanh\xi$ and $\mathrm{dn}\,\xi \to \mathrm{sech}\,\xi$, so the solutions (33) and (34) degenerate as another two solutions

$$u_3 = \pm\sqrt{-\frac{6\beta}{\alpha}}\,\tanh(x-ct), \tag{35}$$

and

$$u_4 = \pm\sqrt{\frac{6\beta}{\alpha}}\,\mathrm{sech}(x-ct), \tag{36}$$

which are shock wave solution and solitary wave solution, respectively.

3.2. *System of variant Boussinesq equations*

System of variant Boussinesq equations reads [2]

$$H_t + (Hu)_x + u_{xxx} = 0, \tag{37a}$$
$$u_t + H_x + uu_x = 0. \tag{37b}$$

We solve it in the following frame

$$H = H(\xi), \qquad u = u(\xi), \quad \xi = x - ct. \tag{38}$$

So system (37) can be rewritten as

$$-c\frac{dH}{d\xi} + \frac{d(Hu)}{d\xi} + \frac{d^3u}{d\xi^3} = 0, \tag{39a}$$
$$-c\frac{du}{d\xi} + \frac{dH}{d\xi} + u\frac{du}{d\xi} = 0. \tag{39b}$$

Integrating system (39) yields

$$-cH + Hu + \frac{d^2u}{d\xi^2} = 0, \tag{40a}$$
$$-cu + H + u^2/2 = 0. \tag{40b}$$

where integration constants are set to be zero.

We suppose the ansatz solution to system (40) is

$$H(\xi) = \sum_{j=1}^{n_1} \cos^{j-1}\omega(a_j\cos\omega + b_j\sin\omega) + a_0, \tag{41a}$$
$$u(\xi) = \sum_{l=1}^{n_2} \cos^{l-1}\omega(A_l\cos\omega + B_l\sin\omega) + A_0, \tag{41b}$$

where ω satisfies the transformation (16).

Substituting (41) into (40) to balance the nonlinear term and highest degree differential term gives $n_1 = 2$ and $n_2 = 1$.

So the ansatz solution to system (37) is

$$H(\xi) = a_0 + a_1\cos\omega + b_1\sin\omega + a_2\cos^2\omega + b_2\sin\omega\cos\omega, \tag{42a}$$
$$u(\xi) = A_0 + A_1\cos\omega + B_1\sin\omega. \tag{42b}$$

Substituting ansatz solution (42) into system (40) results in

$$\begin{aligned}
&[-c(a_0+a_2) + a_0A_0 + a_2A_0 + a_1A_1] \\
&\quad + [-ca_1 + (a_1A_0 + a_0A_1 + a_2A_1) - m^2A_1]\cos\omega \\
&\quad + [-cb_1 + (b_1A_0 + b_2A_1 + a_0B_1 + a_2B_1) \\
&\qquad - (1-m^2)B_1]\sin\omega \\
&\quad + [-cb_2 + (b_2A_0 + b_1A_1 + a_1B_1)]\cos\omega\sin\omega \\
&\quad + [ca_2 + (b_1B_1 - a_2A_0 - a_1A_1)]\sin^2\omega \\
&\quad + [(b_2B_1 - a_2A_1) + 2A_1]\cos\omega\sin^2\omega \\
&\quad + [-(b_2A_1 + a_2B_1) + 2B_1]\sin^3\omega = 0,
\end{aligned} \tag{43a}$$

$$\begin{aligned}
&[-cA_0 + (a_0+a_2) + (A_0^2 + A_1^2)/2] \\
&\quad + [-cA_1 + a_1 + A_0A_1]\cos\omega \\
&\quad + [-cB_1 + b_1 + A_0B_1]\sin\omega \\
&\quad + [b_2 + A_1B_1]\cos\omega\sin\omega \\
&\quad + [-a_2 + (B_1^2 - A_1^2)/2]\sin^2\omega = 0.
\end{aligned} \tag{43b}$$

Setting the coefficients of $(\cos\omega\sin\omega)^0$, $\sin\omega$, $\cos\omega$, $\cos\omega\sin\omega$, $\sin^2\omega$, $\cos\omega\sin^2\omega$ and $\sin^3\omega$ to be zeros, we can get the algebraic equations about expansion coefficients and c

$$-c(a_0+a_2)+a_0A_0+a_2A_0+a_1A_1=0, \tag{44a}$$

$$-ca_1+(a_1A_0+a_0A_1+a_2A_1)-m^2A_1=0, \tag{44b}$$

$$-cb_1+(b_1A_0+b_2A_1+a_0B_1+a_2B_1) - \left(1+m^2\right)B_1=0, \tag{44c}$$

$$-cb_2+(b_2A_0+b_1A_1+a_1B_1)=0, \tag{44d}$$

$$ca_2+(b_1B_1-a_2A_0-a_1A_1)=0, \tag{44e}$$

$$(b_2B_1-a_2A_1)+2A_1=0, \tag{44f}$$

$$-(b_2A_1+a_2B_1)+2B_1=0, \tag{44g}$$

$$-cA_0+(a_0+a_2)+\left(A_0^2+A_1^2\right)/2=0, \tag{44h}$$

$$-cA_1+a_1+A_0A_1=0, \tag{44i}$$

$$-cB_1+b_1+A_0B_1=0, \tag{44j}$$

$$b_2+A_1B_1=0, \tag{44k}$$

$$-a_2+\left(B_1^2-A_1^2\right)/2=0, \tag{44l}$$

from which solutions for the three cases can be got.

Case 1. $B_1=a_1=b_1=b_2=0$

$$A_0=c, \qquad A_1=\pm 2i, \qquad a_0=m^2-2, \qquad a_2=2, \tag{45}$$

where $i=\sqrt{-1}$.

Case 2. $A_1=a_1=b_1=b_2=0$

$$A_0=c, \qquad B_1=\pm 2, \qquad a_0=m^2-1, \qquad a_2=2. \tag{46}$$

Case 3. $a_1=b_1=0$

$$A_0=c, \qquad A_1=\pm i, \qquad B_1=-\operatorname{sgn}(\pm i)\cdot\operatorname{sgn}(\pm i), \qquad a_0=m^2-1, \qquad a_2=1, \qquad b_2=\pm i. \tag{47}$$

Then the solutions to system (37) can be got as follows:

$$H_1=a_0+a_2\cos^2\omega=m^2-2+2\,\mathrm{dn}^2\,\xi, \tag{48a}$$

$$u_1=A_0+A_1\cos\omega=c\pm 2i\,\mathrm{dn}\,\xi, \tag{48b}$$

$$H_2=a_0+a_2\cos^2\omega=m^2-1+2\,\mathrm{dn}^2\,\xi, \tag{49a}$$

$$u_2=A_0+B_1\sin\omega=c\pm 2m\,\mathrm{sn}\,\xi, \tag{49b}$$

$$H_3=a_0+a_2\cos^2\omega+b_2\cos\omega\sin\omega = m^2-1+\mathrm{dn}^2\,\xi\pm im\,\mathrm{dn}\,\xi\,\mathrm{sn}\,\xi, \tag{50a}$$

$$u_3=A_0+A_1\cos\omega+B_1\sin\omega = c\pm i\,\mathrm{dn}\,\xi-\operatorname{sgn}(\pm i)\cdot\operatorname{sgn}(\pm i)m\,\mathrm{sn}\,\xi. \tag{50b}$$

When $m\to 1$, $\mathrm{sn}\to\tanh\xi$ and $\mathrm{dn}\,\xi\to\operatorname{sech}\xi$, so under limit condition, the solutions above degenerate as another three solutions

$$H_4=2\operatorname{sech}^2\xi-1, \tag{51a}$$

$$u_4=c\pm 2i\operatorname{sech}\xi, \tag{51b}$$

$$H_5=2\operatorname{sech}^2\xi, \tag{52a}$$

$$u_5=c\pm 2\tanh\xi, \tag{52b}$$

$$H_6=\operatorname{sech}^2\xi\pm i\operatorname{sech}\xi\tanh\xi, \tag{53a}$$

$$u_6=c\pm i\operatorname{sech}\xi-\operatorname{sgn}(\pm i)\cdot\operatorname{sgn}(\pm i)\tanh\xi. \tag{53b}$$

4. Conclusion

In this Letter, new transformations from nonlinear sine-Gordon equation are obtained and based on them a new approach is proposed to construct the exact solutions to nonlinear equations. And it is shown that the periodic wave solutions obtained by this method can degenerate to generalized solitary wave solutions, so other forms of transformations (6) or (8) may be applied to get more new shock wave or solitary wave solutions.

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Appendix A

Notice that

$$u(t)=\int_0^{\phi}\frac{1}{\sqrt{1-m^2\sin^2\varphi}}\,d\varphi = \int_0^{t\equiv\sin\varphi}\frac{1}{\sqrt{(1-x^2)(1-m^2x^2)}}\,dx \tag{A.1}$$

is called the Legendre elliptic integral of the first kind, where m is a parameter which is known as the modulus. The inverse function $t \equiv \sin\varphi$ is called the Jacobi elliptic sine function which is represented by

$$t = \sin\varphi = \operatorname{sn} u. \tag{A.2}$$

Similarly, $\sqrt{1-t^2}$ and $\sqrt{1-m^2t^2}$ are defined as the Jacobi elliptic cosine function and Jacobi elliptic function of the third kind, respectively. They are expressed as

$$\sqrt{1-t^2} = \operatorname{cn} u, \qquad \sqrt{1-m^2t^2} = \operatorname{dn} u, \tag{A.3}$$

respectively.

We see from (A.1) that when $m \to 0$, $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$ degenerate as $\sin u$, $\cos u$ and 1, respectively; while when $m \to 1$, $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$ degenerate as $\tanh u$, $\operatorname{sech} u$ and $\operatorname{sech} u$, respectively. Detailed explanations about Jacobi elliptic functions can be found in Refs. [17,18].

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