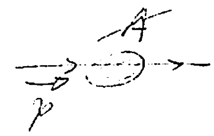
2. Wave function renormalization

1) The self-energy of the electron $\Sigma(p) = I + II + III + \cdots + VI$, which in general can be expressed as

$$A = \{ m a_1 + m a_2 \gamma_5 + a_3 i\gamma p + a_4 i\gamma p \gamma_5 \} (2\pi)^4 i \quad ;$$



(Actually a_2 should be vanished unless time-reversal T is violated.)

The complete propagator for the electron becomes
(dressed)

$$D(p) = \frac{1}{i\not{p}+m} + \frac{1}{i\not{p}+m} A \frac{1}{(2\pi)^4 i} \frac{1}{i\not{p}+m} + \cdots \quad ; \quad (\not{p} \equiv \gamma \cdot p)$$

$$= (i\not{p}+m)^{-1} \left\{ 1 - \frac{1}{(2\pi)^4 i} A \frac{1}{i\not{p}+m} \right\}^{-1}$$

$$= \left\{ \left(1 - \frac{1}{(2\pi)^4 i} A \frac{1}{i\not{p}+m}\right)(i\not{p}+m) \right\}^{-1}$$

$$= \frac{1}{i\not{p}+m - \frac{A}{(2\pi)^4 i}}$$

$$\equiv \frac{1}{i\gamma p B + m C}$$

Since

$$i\not{p} + m - \frac{A}{(2\pi)^4 i} = i\gamma p + m - \{ m a_1 + m a_2 \gamma_5 + a_3 i\gamma p + a_4 i\gamma p \gamma_5 \}$$

$$= i\gamma p \left[1 - a_3 - a_4 \gamma^5 \right] + m \left[1 - a_1 - a_2 \gamma^5 \right]$$

So

$$B = 1 - a_3 - a_4 \gamma_5$$
$$C = 1 - a_1 - a_2 \gamma_5$$

2) Consider now $\dfrac{1}{i\gamma p B + mC}$

where both B and C may contain γ_5. Define $\bar{B}$ and $\bar{C}$ as obtained from B and C by changing the sign of γ_5. Then

$$(i\gamma p B + mC)(-i\gamma p B + m\bar{C})$$

$$= \gamma p B(\gamma p B) - imC(\gamma p B) + i(\gamma p B)\bar{C}m + m^2 C\bar{C};$$

$$= \gamma p \gamma p \bar{B}B + mi\gamma p(B\bar{C} - \bar{C}B) + m^2 C\bar{C};$$

$$= p^2 B\bar{B} + m^2 C\bar{C}$$

$$\left(B\gamma p = \gamma p \bar{B} \right)$$
$$\left(C(\gamma p) = \gamma p \bar{C} \right)$$
$$\left[\begin{array}{l} B\bar{C} = \bar{C}B \\ \bar{B}B = B\bar{B} \\ \gamma p \gamma p = p^2 \end{array} \right]$$

Hence

$$\frac{1}{i\gamma p B + mC} = \frac{-i\gamma p B + m\bar{C}}{B\bar{B}p^2 + m^2 C\bar{C}},$$

with
$$\bar{B} = 1 - a_3 + a_4\gamma_5$$
$$\bar{C} = 1 - a_1 + a_2\gamma_5$$

$$\left(\mathcal{L} = -\bar{\psi}(\not{\partial} + m)\psi \right)$$

3) Without radiative correction, the Dirac eq is

$(i\gamma p + m)\, u(p) = 0$. The propagator is

$$\frac{1}{i\gamma p + m} = \frac{-i\gamma p + m}{p^2 + m^2}$$

And on the pole we have a residue for the two point function

$\bar{u} \quad \times \xleftarrow{p} \times \quad u$ $\qquad \left[\bar{u}(p)(-i\gamma p + m)u(p) \right] = 2m\left\{ \bar{u}(p)\, u(p) \right\}$

Now, we have the following. The denominator is

$$p^2 B\bar{B} + m^2 C\bar{C}$$

$$= p^2(1 - a_3 - a_4\gamma_5)(1 - a_3 + a_4\gamma_5) + (1 - a_1 - a_2\gamma_5)(1 - a_1 + a_2\gamma_5)m^2$$

$$\approx p^2(1 - 2a_3) + m^2 - 2a_1 m^2$$

(Note $a_1, a_2, a_3, a_4 \sim O(g^2)$. Therefore a_1^2, a_1a_2 etc are of the order $O(g^4)$)

Using the on-shell subtraction, we shall expand the denominator around the physical mass of the electron, hence

$$p^2 B\bar{B} + m^2 C\bar{C}$$
$$\simeq p^2(1-2a_3) + m^2 - 2m^2 a_1$$
$$= p^2 + m^2 - [2a_3 p^2 + 2m^2 a_1].$$

In on-shell subtraction scheme, we should expand $[2a_3 p^2 + 2m^2 a_1]$ around $p^2 = -m^2$, i.e.

$$[2a_3 p^2 + 2m^2 a_1] = [2\underline{a_3}(-m^2) + 2m^2 \underline{a_1}] + \frac{d}{dp^2}\bigg|_{p^2=-m^2} [2a_3 p^2 + 2m^2 a_1]\cdot(p^2+m^2) + \cdots$$
$$= [-2m^2\underline{a_3} + 2m^2\underline{a_1}] + \left\{2\underline{a_3} + 2(-m^2)\frac{da_3}{dp^2}\bigg|_{-m^2} + 2m^2\frac{da_1}{dp^2}\bigg|_{-m^2}\right\}\cdot(p^2+m^2) + \cdots$$
$$= -2m^2\underline{a_3} + 2m^2\underline{a_1} + \left(2\underline{a_3} - 2m^2\frac{da_3}{dp^2}\bigg|_{-m^2} + 2m^2\frac{da_1}{dp^2}\bigg|_{-m^2}\right)(p^2+m^2) + \cdots$$

where $\underline{a_3} \equiv a_3(p^2=-m^2)$, i.e. a_3 evaluated at $p^2=-m^2$.

$$\frac{da_3}{dp^2}\bigg|_{-m^2} \equiv \frac{da_3(p^2)}{dp^2}\bigg|_{p^2=-m^2}, \quad \text{etc.}$$

Thus

$$p^2 B\bar{B} + m^2 C\bar{C}$$
$$= (p^2+m^2)\left[1 - 2\underline{a_3} + 2m^2\frac{da_3}{dp^2}\bigg|_{-m^2} - 2m^2\frac{da_1}{dp^2}\bigg|_{-m^2}\right] + 2m^2\underline{a_3} - 2m^2\underline{a_1}$$
$$= \left[1 - 2\underline{a_3} + 2m^2\frac{da_3}{dp^2}\bigg|_{-m^2} - 2m^2\frac{da_1}{dp^2}\bigg|_{-m^2}\right]\left\{p^2 + [m(1+\underline{a_3}-\underline{a_1})]^2\right\}$$

Therefore, the dressed propagator is

$$\frac{1}{\left\{1 - 2\underline{a_3} + 2m^2\frac{da_3}{dp^2}\Big|_{-m^2} - 2m^2\frac{da_1}{dp^2}\Big|_{-m^2}\right\}}\cdot\frac{-i\gamma p(1-a_3-a_4\gamma_5) + m(1-a_1+a_2\gamma_5)}{p^2 + \{m(1+\underline{a_3}-\underline{a_1})\}^2}$$

Requiring the pole at $p^2 = -m^2$, the renormalization of the electron mass is

$$m(\underline{a_3} - \underline{a_1}) = -\delta m \quad , \quad \text{for} \quad \left(m_{bare} = m_{ren} + \delta m \right)$$

so

$$\delta m = m(\underline{a_1} - \underline{a_3}) \quad \Rightarrow \quad m_{bare} = m_{ren}(1 + \underline{a_1} - \underline{a_3})$$

If we define $m_{bare}^2 = m_{ren}^2 + \delta m^2$, then $\delta m^2 = 2m^2(\underline{a_1} - \underline{a_3})$

4) Consider now the new Dirac eq for the wavefunction $W(p)$, at $p^2 = -m_p^2$,

$$\left\{ i\gamma p(1 - \underline{a_3} - \underline{a_4}\gamma_5) + m(1 - \underline{a_1} - \underline{a_2}\gamma_5) \right\} W(p) = 0$$

let $u(p)$ be a solution of the Dirac eq.

$$\left[i\gamma p + m(1 + \underline{a_3} - \underline{a_1}) \right] u(p) = 0 \quad \Rightarrow \quad i\gamma p\, u(p) = -m(1 + \underline{a_3} - \underline{a_1})\, u(p)$$

(Note In the on-shell subtraction $m_{bare}(1 + \underline{a_3} - \underline{a_1}) = (m_{ren} + \delta m) + m(\underline{a_3} - \underline{a_1})$
$= m_{ren}$
$= m_{physical}$)

and assume $W(p) = (1 + \alpha\gamma^5)u(p)$, then require $\left(\alpha \sim O(g^2) \right)$ [Note]

$$\left\{ i\gamma p(1 - \underline{a_3} - \underline{a_4}\gamma_5) + m(1 - \underline{a_1} - \underline{a_2}\gamma_5) \right\} (1 + \alpha\gamma^5)\, u(p)$$

$$= (1 - \underline{a_3} + \underline{a_4}\gamma_5)(1 - \alpha\gamma^5)\, i\gamma p\, U(p) + m(1 - \underline{a_1} - \underline{a_2}\gamma_5)(1 + \alpha\gamma^5)\, U(p)$$

$$= \left\{ -(1 - \underline{a_3} + \underline{a_4}\gamma^5)(1 - \alpha\gamma^5)(1 + \underline{a_3} - \underline{a_1}) + (1 - \underline{a_1} - \underline{a_2}\gamma^5)(1 + \alpha\gamma^5) \right\} m\, U(p)$$

$$= \{ \underline{-1} + \underline{a_3} - \underline{a_4}\gamma^5 + \alpha\gamma^5 - \underline{a_3}\alpha\gamma^5 + \underline{a_4}\alpha - \underline{a_3} + \underline{a_1} + \alpha\underline{a_3}\gamma^5 - \alpha\underline{a_1}\gamma^5$$
$$\underline{+1} - \underline{a_1} - \underline{a_2}\gamma^5 + \alpha\gamma^5 - \underline{a_1}\gamma^5\alpha - \underline{a_2}\alpha \; \} \, m\, U(p)$$

$$\approx \left\{ 2\alpha\gamma^5 - \underline{a_4}\gamma^5 - \underline{a_2}\gamma^5 + (\underline{a_3} - \underline{a_3}) - (\underline{a_1} - \underline{a_1}) \right\} m\, u(p)$$

$$\approx \left\{ 2\alpha - \underline{a_4} - \underline{a_2} \right\} \gamma^5 m\, u(p) \; ;$$

$$= 0$$

Therefore

$$\alpha = \frac{1}{2}(\underline{a_2} + \underline{a_4}),$$

i.e.

$$W(p) = (1+\alpha\gamma^5)\,u(p)$$
$$= \left[1 + \frac{1}{2}(\underline{a_2}+\underline{a_4})\gamma^5\right] u(p) \quad ;$$

$$\overline{W(p)} = \overline{u}(p)(1-\overline{\alpha}\gamma^5)$$

From the dressed propagator,

$$\frac{1}{\left[1 - 2\underline{a_3} + 2m^2\underline{a_3'} - 2m^2\underline{a_1'}\right]} \; \frac{-i\gamma p(1-\underline{a_3}-\underline{a_4}\gamma_5) + m(1-\underline{a_1}+\underline{a_2}\gamma^5)}{p^2 + [m(1+\underline{a_3}-\underline{a_1})]^2},$$

with

$$\underline{a_3'} \equiv \left.\frac{da_3}{dp^2}\right|_{p^2=-m^2}, \quad \text{etc} \;;$$

we can get its residue at the pole, $p^2 = -[m(1+\underline{a_3}-\underline{a_1})]^2$, as

$$\text{(I)} \equiv \overline{W(p)}\,\frac{1}{\left[1-2\underline{a_3}+2m^2\underline{a_3'}-2m^2\underline{a_1'}\right]}\left\{-i\gamma p(1-\underline{a_3}-\underline{a_4}\gamma_5)+m(1-\underline{a_1}+\underline{a_2}\gamma^5)\right\}W(p)$$

$$= \overline{u}(p)(1-\overline{\alpha}\gamma^5)\cdot\left[1+2\underline{a_3}-2m^2\underline{a_3'}+2m^2\underline{a_1'}\right]\left\{-i\gamma p(1-\underline{a_3}-\underline{a_4}\gamma_5)+m(1-\underline{a_1}+\underline{a_2}\gamma_5)\right\}$$
$$\cdot(1+\alpha\gamma^5)\,U(p)$$

Use

$$\text{(1)} \equiv -i\gamma p(1-a_3-a_4\gamma 5)(1+\alpha\gamma^5)U(p)$$
$$= -(1-a_3+a_4\gamma^5)(1-\alpha\gamma^5)\,i\gamma p\,U(p)$$
$$= -(1-a_3+a_4\gamma^5-\alpha\gamma^5)\left[-m(1+a_3-a_1)\right]U(p);$$
$$= m(1+a_4\gamma^5-\alpha\gamma^5-a_1)\,U(p)$$

(From now on, we will suppress the underscore. All the quantities are evaluated at the pole, $p^2 = -[m(1+\underline{a_3}-\underline{a_1})]^2$

Note. $a_3(-m^2) \simeq a_3(-m^2(1+\underline{a_3}-a_1)^2)$ up to one-loop.)

$$\text{(2)} \equiv m(1-a_1+a_2\gamma^5)(1+\alpha\gamma^5)U(p)$$
$$= m(1-a_1+a_2\gamma^5+\alpha\gamma^5)\,U(p)$$

$$\text{(1)}+\text{(2)} = m\left[2-2a_1+a_2\gamma^5+a_4\gamma^5\right]U(p)$$

$$\text{(3)} = (1-\overline{\alpha}\gamma^5)(1+2a_3-2m^2a_3'+2m^2a_1')$$
$$= 1+2a_3-2m^2a_3'+2m^2a_1'-\overline{\alpha}\gamma^5$$

Therefore

$$(\mathrm{I}) = \bar{u}(p)\left\{ 2 - 2a_1 + a_2\gamma^5 + a_4\gamma^5 + 4a_3 - 4m^2 a_3' + 4m^2 a_1' - 2\bar{\alpha}\gamma^5 \right\} m\, u(p)$$

$$= 2m\,\bar{u}(p)\left\{ 1 - a_1 + 2a_3 - 2m^2 a_3' + 2m^2 a_1' + (a_2 + a_4 - 2\bar{\alpha})\gamma^5 \right\} u(p)$$

Because, we require the residue of the dressed propagator should have the form $\bar{u}(p)u(p)$, without terms like $\bar{u}(p)\gamma^5 u(p)$, therefore, we <u>demand</u>

$$\bar{\alpha} = \tfrac{1}{2}(a_2 + a_4) = \alpha$$

or, $a_4^* = a_4$. (Notice that a_2 must be zero unless the time-reversal T is violated.)

so that the γ^5 term vanishes.

Because $m_{ren} = m_{bare}(1 - a_1 + a_3)$, so we can rewrite (I) as

$$2\left[m(1+a_3-a_1)\right]\left[\bar{u}(p)u(p)\right]\cdot\left[1 + a_3 - 2m^2 a_3' + 2m^2 a_1'\right];$$

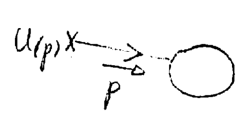
where $u(p) = (1-\alpha\gamma^5)W(p)$; $\bar{u}(p) = \bar{W}(p)(1+\alpha\gamma^5)$ for $\bar{\alpha} = \alpha$.

5) From the above result, we found that the residue of the dressed propagator at pole, $p^2 = -[m(1+a_3-a_1)]^2$, is

$$2\left[m(1+a_3-a_1)\right]\left[\bar{W}(p)W(p)\right]\left[1 + a_3 - 2m^2 a_3' + 2m^2 a_1'\right]; \quad (\because \bar{u}u = \bar{w}w)$$

To make the residue to be 1, ie $\bar{W}(p)W(p)$, we have to renormalize our current source by a factor

$$\frac{1}{\left\{1 + \frac{1}{2}a_3 - m^2 a_3' + m^2 a_1'\right\}}\;\frac{1}{2m(1+a_3-a_1)}\;W(p)$$



$$\equiv \frac{1}{\left\{1 + \frac{1}{2}a_3 - m^2 a_3' + m^2 a_1'\right\}}\cdot\frac{\left[1 + \frac{1}{2}(a_2 + a_4)\gamma^5\right]}{2m(1+a_3-a_1)}\;U(p)$$

Note $W(p) = (1+\alpha\gamma^5)U(p)$. Since the current source will emit physical sources, therefore, we should evaluate $\alpha = \frac{1}{2}(a_2+a_4)$ at pole, i.e. $p^2 = -m^2(1+a_3-a_1)^2$

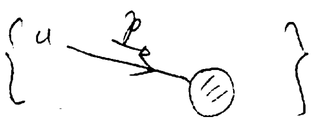
b) To obtain the S-matrix from a Green's function, we have to truncate the external leg, i.e. attaching this source to a propagator, multiplying $p^2+[m(1+\underline{a}_3-\underline{a}_1)]^2$, and going to on-shell.

Here we shall suppress all the underscore, i.e. $\underline{a}_3 = a_3$.

$$\text{(II)} \equiv \lim_{p^2=-[m(1+a_3-a_1)]^2} \left(p^2+[m(1+a_3-a_1)]^2\right) \cdot$$

$$\cdot \left\{ \frac{1}{[1-2a_3+2m^2a_3'-2m^2a_1']} \; \frac{-i\gamma p(1-a_3-a_4\gamma^5)+m(1-a_1+a_2\gamma^5)}{p^2+[m(1+a_3-a_1)]^2} \right\}$$

$$\cdot \left\{ \frac{1}{[1+\frac{1}{2}a_3-m^2a_3'+m^2a_1']} \; \frac{[1+\frac{1}{2}(a_2+a_4)\gamma^5]}{2m(1+a_3-a_1)} \; u(p) \right\}$$

$\cdot \{$  $\}$

Use

$$-i\gamma p(1-a_3-a_4\gamma^5)(1+\tfrac{1}{2}\gamma\gamma^5)u(p)$$
$$=-i\gamma p(1-a_3-a_4\gamma^5)\,w(p)$$
$$= m(1-a_1-a_2\gamma^5)\,w(p),$$

then

$$\text{(II)} = \frac{1}{[1-\frac{3}{2}a_3+m^2a_3'-m^2a_1']} \; \frac{2m(1-a_1)(1+\frac{1}{2}(a_2+a_4)\gamma^5)\,u(p)}{2m(1+a_3-a_1)}$$

$$= \left(1+\tfrac{3}{2}a_3-m^2a_3'+m^2a_1'\right)\left[1-a_1+\tfrac{1}{2}(a_2+a_4)\gamma^5-a_3+a_1\right]$$

$$= 1+\tfrac{1}{2}\underline{a_3}-m^2\underline{a_3'}+m^2\underline{a_1'}+\tfrac{1}{2}\underline{a_2}\gamma^5+\tfrac{1}{2}\underline{a_4}\gamma^5$$

$$\equiv 1+W_V+W_A\gamma^5,$$

with

$$W_V = \tfrac{1}{2}\underline{a_3}-m^2\underline{a_3'}+m^2\underline{a_1'}$$
$$W_A = \tfrac{1}{2}(\underline{a_2}+\underline{a_4})$$

7) Recall that

$$\left\{ i\gamma p(1-a_3-a_4\gamma^5)+m(1-a_1-a_2\gamma^5) \right\} (1+\alpha\gamma^5)u=0, \quad \text{--- } ①$$

with

$$\left[i\gamma p + m(1+a_3-a_1) \right] u = 0. \quad \text{--- } ②$$

Since ②* gives

$$u^* \left[i(p_1\gamma^1+p_2\gamma^2+p_3\gamma^3+ip_0\gamma^4)+m(1+a_3-a_1) \right]^* = 0$$

$$u^* \left[-i(p_1\gamma^1+p_2\gamma^2+p_3\gamma^3-ip_0\gamma^4)+m(1+a_3^*-a_1^*) \right] = 0$$

multiplied by γ^4 from right hand side, get

$$u^*\gamma^4 \left[-i(-p_1\gamma^1-p_2\gamma^2-p_3\gamma^3-ip_0\gamma^4)+m(1+a_3^*-a_1^*) \right] = 0,$$

ie

$$\bar{u} \left[i\gamma \cdot p + m(1+a_3^*-a_1^*) \right] = 0 \quad \text{--- } ③$$

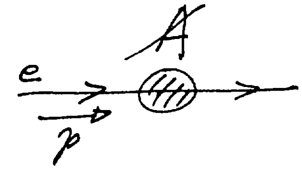
$$u^*(1+\alpha^*\gamma^5)\left\{ -i(p_1\gamma^1+p_2\gamma^2+p_3\gamma^3-ip_0\gamma^4)(1-a_3^*-a_4^*\gamma^5)+m(1-a_1^*-a_2^*\gamma^5) \right\}\gamma^4=0$$

$$\bar{u}(1-\alpha^*\gamma^5)\left\{ i\gamma\cdot p(1-a_3^*+a_4^*\gamma^5)+m(1-a_1^*+a_2^*\gamma^5) \right\}=0 \quad \text{--- } ④$$

Note We see that the difference between ② and ③, and, ① and ④ is at the sign of the γ^5.

8) Summary:

Let the selfenergy of the electron be expressed as



$$A = (2\pi)^4 i \left\{ m a_1 + m a_2 \gamma^5 + a_3 i\gamma p + a_4 i \gamma p \gamma^5 \right\},$$

then the full propagator of electron is

$$\frac{1}{\left[1 - 2\underline{a_3} + 2m^2 \underline{a_3'} - 2m^2 \underline{a_1'}\right]} \quad \frac{-i\gamma p(1 - a_3 - a_4\gamma^5) + m(1 - a_1 + a_2\gamma^5)}{p^2 + \left[m(1 + \underline{a_3} - \underline{a_1})\right]^2}$$

The mass counterterm is

$$\delta m^2 = 2m^2(\underline{a_1} - \underline{a_3}), \quad \text{for} \quad m_{bare}^2 = m_{ren}^2 + \delta m^2$$

in the on-shell subtraction scheme.

$$\left(\delta m = m(\underline{a_1} - \underline{a_3})\right)$$

Define

$$W_V = \frac{1}{2}\underline{a_3} - m^2 \underline{a_3'} + m^2 \underline{a_1'}$$
$$W_A = \frac{1}{2}(\underline{a_2} + \underline{a_4}),$$

then the wavefunction renormalization are

$(1 + W_V + W_A\gamma^5)\, u$	for	incoming particle
$\bar{u}\,(1 + W_V - W_A\gamma^5)$	for	outgoing particle
$(1 + W_V + W_A\gamma^5)\, v$	for	outgoing antiparticle
$-\bar{v}\,(1 + W_V - W_A\gamma^5)$	for	incoming antiparticle

Note: ① a_2 is vanished unless the time-reversal T is violated.

② $\underline{a_3} \equiv a_3(p^2 = -m_{ren}^2)$, $\quad m_{ren}^2 = m_{bare}^2(1 + \underline{a_3} - \underline{a_1})^2$

$\simeq a_3(p^2 = -m^2)$, upto one loop for $m \equiv m_{bare}$

$$\underline{a_3'} \equiv \left.\frac{d a_3}{d p^2}\right|_{p^2 = -m_{ren}^2}$$