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Renormalizing the SM Lagrangian for Precision Tests

1. SM Lagrangian $SU(3)_C \times SU(2)_L \times U(1)_Y$

$$\mathcal{L} = \mathcal{L}\Big(\underbrace{g_3,\ g_2,\ g_1}_{\text{gauge interactions}},\ \underbrace{\lambda,\ v}_{\text{Higgs sector}},\ m_f \Big)$$

m_f: Yukawa interactions

Beyond tree level, we need to
renormalize couplings (those listed above)
and fields (wavefunction renormalization)

2. $SU(3)_c$ coupling α_s

(1) $\alpha_s \equiv \dfrac{g_3^2}{4\pi}$

is usually defined in the $\overline{MS}$-scheme.

$\Rightarrow$ Continuing momentum integrals from 4 to $n \equiv 4-2\varepsilon$ dimensions, and then subtracting off $\left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi\right)$.

<u>Note</u>: To preserve the dimensionless nature of the coupling,

$$g_3 \longrightarrow g_3 \cdot \mu^{\varepsilon} \quad \left(\text{in } n = 4-2\varepsilon \text{ dim.}\right)$$

$\Rightarrow$ A factor $\ln \mu^2$ always comes with $\frac{1}{\varepsilon}$.

$\Rightarrow$ Effective QCD coupling $\alpha_s(\mu)$ with

$$\mu \frac{\partial \alpha_s}{\partial \mu} = 2\beta(\alpha_s) = -\frac{\beta_0}{2\pi}\alpha_s^2 - \frac{\beta_1}{4\pi^2}\alpha_s^3 - \cdots \quad (\beta_2, \beta_3)$$

$$\beta_0 = 11 - \frac{2}{3} n_f$$

n_f : # of quarks with mass less than μ

(2) The new convention is to choose $\mu_0 = M_Z$

$= 91.1876 \pm 0.0021$ GeV

and calculate

$\alpha_s(\mu)$

from $$\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} = \log\left(\frac{\mu^2}{\mu_0^2}\right)$$

with $$\alpha_s(M_Z) = 0.118 \pm 0.003$$

(3) One can also introduce Λ_{QCD} $(\equiv \Lambda)$ to parametrize the μ dependence of $\alpha_s(\mu)$.

The definition of Λ is arbitrary. One way is to define it via

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)} \Big[1 - \underset{(\beta_1, \beta_2)}{\cdots} \Big]$$

Note: $\alpha_s(\mu) \to 0$ as $\mu \to \infty$.

$$\Lambda_{QCD}^{(5)} = 380 \pm 60 \text{ MeV}$$ (with top quark decoupled)

Note. β_0 and β_1 are independent of renormalization scheme.

3. We can trade (g_2, g_1, v)

with (α_{em}, M_W, M_Z)

(1) The tree level relations, denoted by subscript (0),

$$\alpha_{em}^{(0)} = \frac{1}{4\pi} \frac{g_2^2 g_1^2}{g_2^2 + g_1^2}$$

$$M_W^{(0)} = \frac{1}{2} g_2 v$$

$$M_Z^{(0)} = \frac{1}{2} \sqrt{g_2^2 + g_1^2}\, v$$

(2) Conventionly, we use on-shell definition and define

$\alpha_{em}(0) \equiv \alpha(0)$, from $(g-2)_e$, etc.

$M_W \equiv$ on-shell mass (pole mass)

$M_Z \equiv$ on-shell mass (Pole mass)

(3)

Parameter	Measured Value	Precision
$\alpha_{em}^{(0)}$	$[137.03599911(46)]^{-1}$	$4. \times 10^{-9}$
M_Z	$91.1876(21)$	2×10^{-5}
M_W	$80.454\ (59)$	74×10^{-5}

Note. The precision in M_W measurement is poor. Thus, we trade M_W with G_F as input data to fix SM parameters

G_F	$1.16637(1) \times 10^{-5}\ GeV^{-2}$	1×10^{-5}

The Fermi constant G_F is determined from muon lifetime

$$\tau_\mu^{-1} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1 + \frac{3}{5}\frac{m_\mu^2}{M_w^2}\right)\left[F\left(\frac{m_e^2}{m_\mu^2}\right)\right]$$

$$\cdot\left[1 + \left(\frac{25}{8} - \frac{\pi^2}{2}\right)\frac{\alpha}{\pi}\left(1 + \frac{2\alpha}{3\pi}\ln\left(\frac{m_\mu}{m_e}\right)\right)\right]$$

with $\quad F(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x$

4. (α_{em}, G_F, M_Z) scheme

This is the (α_s, M_W, M_Z) on-shell scheme with G_F (instead of M_W) as one of the input data.

$$\left(\text{At tree level, } v^2 = \frac{1}{\sqrt{2}\,G_F} = (246\,\text{GeV})^2 \right)$$

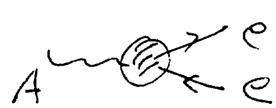
(1) Renormalization :

1. replace

$$e \longrightarrow e_0 = e + \delta e$$

$$M^2_{W,Z} \longrightarrow \underbrace{M^{0\,2}_{W,Z}}_{\text{bare}} = \underbrace{M^2_{W,Z}}_{\text{measurable}} + \underbrace{\delta M^2_{W,Z}}_{\text{Counterterm}}$$

2. fix conterterms by conditions, e.g.

$\delta e = -$ 

at $q^2 = 0$, $q^0 \to 0$

$\delta M_W^2 =$ 

$\Big|_{q^2 = M_W^2}$

$\delta M_Z^2 =$ 

$\Big|_{q^2 = M_Z^2}$

(II) α_{em}

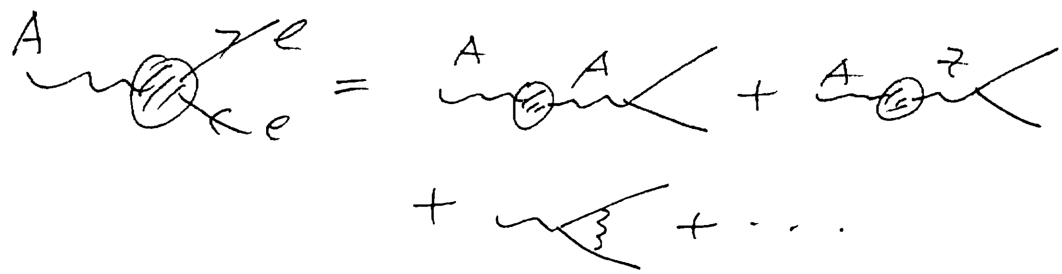
① Denote the vacuum polarization functions as
(2-point functions)

$$\Pi^{ij}_{\mu\nu}(q^2) \equiv -i g_{\mu\nu}\left[A^{ij}(0) + q^2 F^{ij}(q^2)\right] + \begin{pmatrix} q_\mu q_\nu \\ \text{terms} \end{pmatrix}$$

$i, j = A, W^{\pm}, Z$

$$\equiv -i g_{\mu\nu}\left(\Pi_{ij}\right) + \begin{pmatrix} q_\mu q_\nu \\ \text{terms} \end{pmatrix}$$

(don't contribute for gauge bosons coupling to massless fermions)

② 

$$\frac{\delta\alpha}{\alpha} = 2\,\frac{\delta e}{e} = F^{AA}(0) + 2\,\frac{s_W}{c_W}\,\frac{A^{AZ}(0)}{M_Z^2}$$

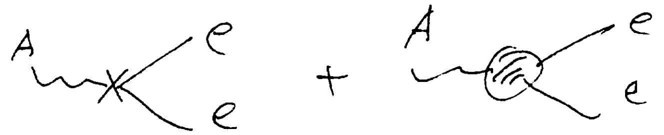
where $s_W^2 \equiv 1 - \dfrac{M_W^2}{M_Z^2}$ (definition of $s_W \equiv \sin\theta_W$)

and $c_W = \dfrac{M_W}{M_Z}$

<u>Note</u> For photon, $\Pi_{AA} = q^2 F^{AA}(q^2)$,

i.e. $A^{AA}(0) = 0$. ($U(1)_{em}$ gauge invariance. Otherwise, photon will gain mass.)

3. When calculating observables at Z-pole, $q^2 = M_Z^2$, one usually encounters



$$\alpha(0)\left\{\left(1+\frac{\delta\alpha}{\alpha}\right)(\cdots) + (-1)\, F^{AA}(M_Z^2) + \cdots \right\}$$

$\Rightarrow$ Rewrite

$$2\,\frac{\delta e}{e} = F^{AA}(0) + \cdots$$

$$= \underbrace{F^{AA}(0) - F^{AA}(M_Z^2)}_{\Delta\alpha,\ \text{finite (but large)}} + F^{AA}(M_Z^2) + \cdots$$

Cancelled

$$\sim Q_f^2 \ln \frac{M_Z^2}{m_f^2} + \cdots$$

$\Rightarrow$ Instead of writing the result as

$$\alpha(0)\,(1 + \Delta\alpha + \cdots)$$

we can write

$$\alpha(M_Z^2)\,(1 + \cdots),$$

i.e. the large correction $\Delta\alpha$ is absorbed into $\alpha(M_Z^2)$

$\Rightarrow$ Resummation

$$\alpha(M_Z^2) = \frac{\alpha(0)}{1-\Delta\alpha}$$

④ What's $\Delta\alpha$?

$$\Delta\alpha \equiv F^{AA}(0) - F^{AA}(M_Z^2)$$

with only light fermions ($m_f < M_Z$) included.

Namely, we have taken $\dfrac{M_Z^2}{m_t^2} \to 0$ limit. (decoupled)

$\Delta\alpha = \sum\limits_{\text{leptons}}$  $+ (\Delta\alpha)^{(5)}_{had}$ (for 5 light quark flavors)

from $e^-e^+ \to$ hadrons data

$$(\Delta\alpha)_{leptons} = \frac{\alpha}{3\pi}\sum_{\ell}\left[\ln\left(\frac{M_Z^2}{m_\ell^2}\right) - \frac{5}{3} + \frac{\alpha}{\pi}(\cdots)\right]$$

$$= 0.0314966 \pm 0.\underbrace{000000}_{6}4$$

$$(\Delta\alpha)^{(5)}_{had} = \frac{-M_Z^2}{4\pi^2\alpha}\,\mathrm{Re}\int_{4m_\pi^2}^{\infty} ds\,\frac{\sigma(e^-e^+ \to \text{hadrons})}{s - M_Z^2 - i\varepsilon}$$

$$= 0.02766 \pm 0.00013$$

$\Rightarrow$ The uncertainty in $(1+\Delta\alpha)$ is dominated by the error in $(\Delta\alpha)^{(5)}_{had}$, which yields about $\dfrac{0.00013}{1.0592} = 0.11\,\%$ accuracy.

(3) M_Z :

(1) The dressed propagator.

$$= \frac{-i g_{\mu\nu}}{q^2 - M_Z^{0\,2} + A^{ZZ}(0) + q^2 F^{ZZ}(q^2)} + (q_\mu q_\nu \text{ term})$$

(2)

$$M_Z^{0\,2} = M_Z^2 + \delta M_Z^2$$

Expand $q^2 F^{ZZ}(q^2)$ around $q^2 = M_Z^2$,

$$q^2 F^{ZZ}(q^2) = M_Z^2 F^{ZZ}(M_Z^2) + \left\{ F^{ZZ}(M_Z^2) + M_Z^2 \left.\frac{d F^{ZZ}(q^2)}{d q^2}\right|_{q^2 = M_Z^2} \right\} \cdot (q^2 - M_Z^2) + \cdots$$

Using on-shell subtraction scheme,

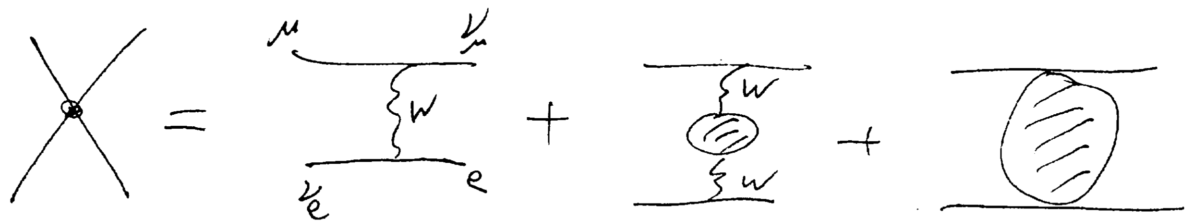
$$\delta M_Z^2 = A^{ZZ}(0) + M_Z^2 F^{ZZ}(M_Z^2)$$

$$\Rightarrow \quad \frac{-i g_{\mu\nu}}{q^2 - M_Z^2} \underbrace{\left\{ 1 - F^{ZZ}(M_Z^2) - M_Z^2 \frac{d F^{ZZ}(M_Z^2)}{d q^2} \right\}}$$

$\left(\text{for } Z^0 \to Z_Z^{1/2} Z = (1 + \tfrac{1}{2} \delta Z_Z) Z \right)$, Wavefunction renormalization $(1 - \delta Z_Z)$

(4) M_W :

Given (α_{em}, G_F, M_Z) and (m_t, m_H),
one can predict M_W. by correlating
G_F to τ_μ (μ-lifetime)



$$G_F = \frac{\pi\alpha(0)}{\sqrt{2}} \frac{1}{M_W^2\left(1-\frac{M_W^2}{M_Z^2}\right)} (1+\Delta r)$$

$$\Delta r = \frac{A^{WW}(0)}{M_W^2} + \cdots$$

$$= \Delta\alpha - \frac{c_W^2}{s_W^2}\Delta\rho + (\Delta r)_{rem}(m_t, m_H)$$

6% 3% 1%

where
$$\Delta\rho \equiv \frac{A_{ZZ}(0)}{M_Z^2} - \frac{A_{WW}(0)}{M_W^2}$$

$$= \frac{3 G_F m_t^2}{8\pi^2\sqrt{2}} + \cdots$$

$$= 0.00992 \left(\frac{m_t}{177.9\,\text{GeV}}\right)^2 + \cdots$$

uncertainty from $\alpha(M_Z)$

$\Rightarrow \Delta r = 0.03434 \pm 0.0017 \pm 0.00014$

Resum $\Delta\alpha$, $\Delta\rho$

$$(1+\Delta r) \longrightarrow \frac{1}{1-\Delta\alpha}\,\frac{1}{1+\frac{c_W^2}{s_W}\Delta\rho} + (\Delta r)_{rem}$$

$$\longrightarrow \frac{1}{1-\Delta\alpha}\,\frac{1}{1+\frac{c_W^2}{s_W^2}\Delta\rho - (\Delta r)_{rem}}$$

$$\Rightarrow \quad G_F = \frac{\pi\alpha}{\sqrt{2}}\,\frac{1}{M_W^2\left(1-\frac{M_W^2}{M_Z^2}\right)}\,\frac{1}{1-\Delta r}$$

with $$(1-\Delta r) = (1-\Delta\alpha)\left(1+\frac{c_W^2}{s_W^2}\Delta\rho\right) - \Delta r_{rem}$$

Therefore, M_W can be predicted from solving

$$\left(1-\frac{M_W^2}{M_Z^2}\right)\frac{M_W^2}{M_Z^2} = \frac{\pi\alpha(0)}{\sqrt{2}\,M_Z^2 G_F\,(1-\Delta r)}$$

$\Rightarrow$ From Z-pole data and m_t measurement, SM predicts

$$M_W = 80.378 \pm 0.023 \text{ GeV}$$

Note: In the above derivation, we have identified

$$\Delta\rho \equiv \frac{A_{ZZ}(0)}{M_Z^2} - \frac{A_{WW}^{(0)}}{M_W^2}$$

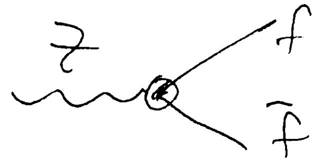
which is the dominant contribution in the ratio

$$R_\nu \equiv \frac{\sigma_{\nu N}^{NC}}{\sigma_{\nu N}^{CC}}$$

with

$$\frac{G_{NC}^{(0)}}{G_{CC}^{(0)}} \equiv \frac{1}{1-\Delta\rho} \quad , \qquad \left(G_{CC}^{(0)} \equiv G_F \right)$$

4. Predict Z-boson observables from SM



g_V^f, g_A^f and $(\cancel{G}_F M_Z^2)$ normalization

(1)

$\Gamma_{Z\to f\bar{f}} \sim (g_V^f)^2 + (g_A^f)^2 \longrightarrow \Gamma_{tot}$, R_{had}, σ_{peak}

$$\left.\begin{aligned} A_{FB}^f &= \frac{3}{4}\,\frac{2 g_V^e g_A^e}{g_V^{e\,2} + g_A^{e\,2}} \cdot \frac{2 g_V^f g_A^f}{g_V^{f\,2} + g_A^{f\,2}} \\ P_\tau &= \frac{2\, g_V^\tau g_A^\tau}{g_V^{\tau\,2} + g_A^{\tau\,2}} \\ A_{LR} &= \frac{2\, g_V^e\, g_A^e}{g_V^{e\,2} + g_A^{e\,2}} \end{aligned}\right\}$$

depend on ratio

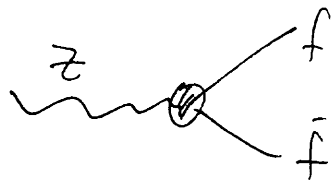
$\frac{g_V^f}{g_A^f} \longleftrightarrow \sin^2\theta_f$

$\Rightarrow$ $\frac{g_V^f}{g_A^f}$ provides precision tests of SM

(2) Effective Couplings from SM: (for $f \neq b$)

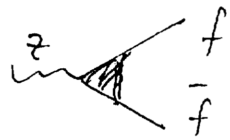
$$g_A^f = I_3^f \sqrt{\rho_f}$$

$$g_V^f = (I_3^f - 2Q_f s_f^2)\sqrt{\rho_f}$$

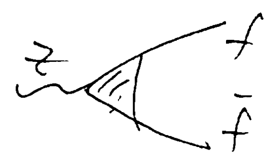


where $s_f^2 \equiv \sin^2\theta_f$

and

$$\rho_f = 1 + (\text{m}, \Delta r) + $$ 

$$= \underbrace{1 + \Delta\rho}_{} + \cdots$$

$$s_f^2 = \underbrace{s_W^2\left(1 + \frac{c_W^2}{s_W^2}\Delta\rho\right)}_{\text{universal}} + \cdots + $$ 

universal — f-dependent (m_t, m_H independent except $f = b$)

Usually,

$$s_f^2 \equiv (1 + \Delta k)\, s_W^2 \equiv k\, s_W^2, \qquad \left(s_W^2 \equiv 1 - \frac{M_W^2}{M_Z^2}\right)$$

So that

$$\Delta k = \frac{c_W^2}{s_W^2}\Delta\rho + \cdots$$

<u>Note</u>: $\sin^2\theta_\ell = 0.2322 \pm 0.0004$ (for f = lepton)

$\sin^2\theta_W = 0.2249 \pm 0.0013$

(3) The partial decay width

$$\Gamma(Z\to f\bar f)=\frac{N_c G_F M_Z^3}{24\pi\sqrt{2}}\,\rho_f\left[\left(1-4|Q_f|s_f^2\right)^2+1\right]$$

with
$$s_f^2\equiv(1+\Delta k)\,s_W^2 \equiv k\, s_W^2$$

$$\Rightarrow \rho_f=\frac{1-\Delta r}{1+\Delta Z}+\cdots \qquad \Sigma_Z(q^2)=\Pi_{ZZ}(q^2)-\frac{\left(\Pi_{ZA}(q^2)\right)^2}{q^2+\Pi_{AA}(q^2)}$$

$$\Delta Z=\mathrm{Re}\,\Sigma_Z'(M_Z^2)$$
$$=-1+(1-\Delta\alpha')(1-\Delta\rho)\left(1+\frac{c^2}{s^2}\Delta\rho\right)+\Delta Z_{rem}$$

$$\Rightarrow \rho_f=\frac{1}{1-\Delta\rho}-\frac{\Delta r_{rem}+\Delta Z_{rem}}{1-\Delta\alpha}+\cdots$$

Also
$$k=1+\frac{c^2}{s^2}\Delta\rho-\frac{c}{s}\,\frac{\Pi_{ZA}(M_Z^2)}{M_Z^2+\Pi_{AA}(M_Z^2)}+\cdots$$
$$=1+\frac{c^2}{s^2}\Delta\rho-\frac{c}{s}\,\frac{1}{1-\Delta\alpha}\left(\frac{\Pi_{ZA}(M_Z^2)}{M_Z^2}\right)_{rem}+\cdots$$

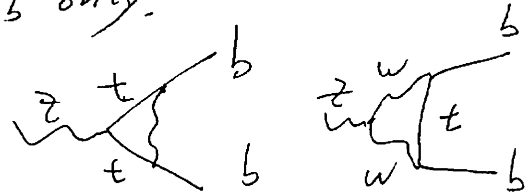
(4) For $Z \to b\bar{b}$

$$\rho_f \to \rho_b = \rho_f (1+\tau)^2 \quad ;$$

$$\kappa \to \kappa_b = \frac{\kappa}{1+\tau}$$

with $\tau = -2\left(\frac{G_F m_t^2}{8\pi^2\sqrt{2}}\right) = -2\left(\frac{m_t}{4\pi v}\right)^2, \quad \left(v^2 = \frac{1}{\sqrt{2}\,G_F}\right)$

new contribution to $Z \to b\bar{b}$ only.



5. New physics and Precision data

(1) In the SM, the physical observables

$(M_W, \; g_V^f, \; g_A^f, \; \ldots)$, or $(\Delta r, \rho_f, \kappa, \cdots)$

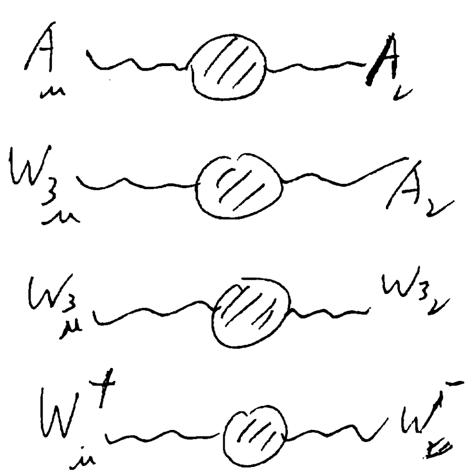
are dominated by $\Delta\alpha$, $\Delta\rho$,

which are defined by the vacuum polarization functions ~~of~~ A_μ, Z_μ, W_μ^+, W_μ^-.

(2) Assume new physics effect only comes in via those 2-point functions

(Oblique corrections).

$\Rightarrow$ There are 4 of them



$\Pi_{QQ}(q^2)$

$\Pi_{3Q}(q^2)$

$\Pi_{33}(q^2)$

$\Pi_{WW}(q^2)$

(3) If the New physics scale Λ_{new} is much larger than M_Z, i.e. $\Lambda_{new} \gg M_Z$, then such effects can be described by 3 parameters (S, T, U) at the 1-loop level.

$$\alpha T = \Delta\rho = \frac{A^{33}(0)}{M_Z^2} - \frac{A^{WW}(0)}{M_W^2}$$

$$\frac{\alpha S}{4 s_W^2} = \frac{1}{s_W} F^{3Q}(M_Z^2) - F^{33}(M_Z^2)$$

$$\frac{\alpha U}{4 s_W^2} = F^{WW}(M_W^2) - F^{33}(M_Z^2)$$

Note: $\Delta\alpha = F^{QQ}(0) - F^{QQ}(M_Z^2)$

$$\Pi^{ij}_{\mu\nu}(q^2) \equiv -i g_{\mu\nu}\left[A^{ij}(0) + q^2 F^{ij}(q^2)\right] + \left(\begin{matrix} q_\mu q_\nu \\ \text{terms}\end{matrix}\right)$$

assumed not to contribute

$\Delta\alpha$	$\Delta\rho = \alpha T$	U	S
heavy object decoupled light: $\sim Q_f^2 \ln \frac{M_Z^2}{m_f^2}$	I-breaking (m_t, m_b) ― dominated by top	I-breaking (small) (in wavefunctions) ― ~ 0 in technicolor	I-conserving ― sensitive to SM Higgs ― heavy degenerate f-multiplet heavy quark doublet (sequential) $\simeq N_C \frac{G_F M_W^2}{12\pi^2\sqrt{2}}$ ― technicolor $(N_{TC}=4)$ $\frac{\alpha}{4 s_W^2}\begin{pmatrix} 0.04 \leftarrow \text{1-doublet} \\ 2.1 \leftarrow \text{(1-generation)} \end{pmatrix}$

<u>Note</u> M_H contributions in SM:

m_t contribution

$$\alpha \cdot \Delta T = \frac{-3}{16\pi C_W^2} \ln \frac{m_H^2}{M_Z^2} \quad + \quad 3\left(\frac{m_t}{4\pi v}\right)^2$$

$$\alpha \cdot \Delta S = \frac{1}{12\pi} \ln \frac{m_H^2}{M_Z^2}$$

$$\alpha \cdot \Delta U \simeq 0$$

(5) In the SM, relations between $(\Delta r, \rho_f, \kappa)$ and (S, T, U)

$$\alpha S \simeq 4s^2\left(c^2(\Delta r_W + \Delta\rho) + (c^2 - s^2)\Delta\kappa\right) \longrightarrow O(\Delta\rho)$$

$$\alpha T \simeq \Delta\rho$$

$$\alpha U = 4s^2\left(s^2(\Delta r_W + 2\Delta\kappa) - c^2\Delta\rho\right) \longrightarrow O(\Delta\rho)$$

where

$$\rho_f \equiv 1 + \Delta\rho$$

$$\kappa \equiv 1 + \Delta\kappa, \quad \text{with} \quad \Delta\kappa \simeq \frac{c^2}{s^2}\Delta\rho + \cdots$$

$$(1 - \Delta r_{oblique}) = (1 - \Delta\alpha)(1 - \Delta r_W)$$

with $\Delta r_W \simeq \frac{-c^2}{s^2}\Delta\rho + \cdots$

$$\Delta\rho = 3\left(\frac{m_t}{4\pi v}\right)^2 + \cdots$$