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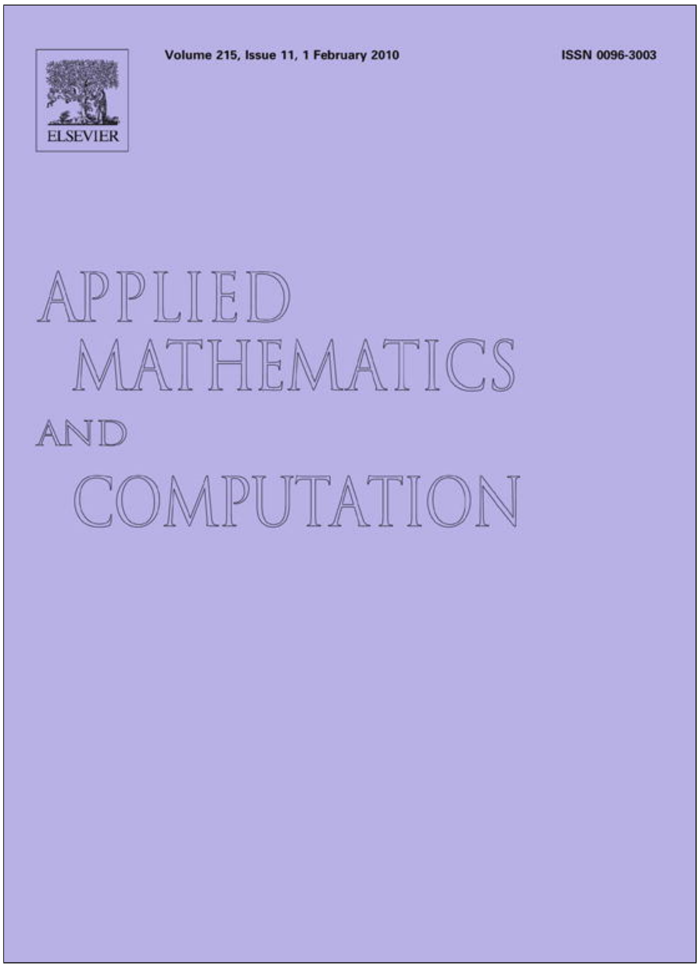

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Novel exact solutions to the short pulse equation

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ABSTRACT

In this paper, the bridge connecting the short pulse equation (SPE for short) with the sine-Gordon equation is applied to construct the novel solutions to the short pulse equation. It is shown that the solutions of the sine-Gordon equation can be used to obtain many different kinds of solutions to the short pulse equation with the aid of symbolic computation and plot representation of Maple.

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1. Introduction

The short pulse equation (SPE for short)

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1}$$

was first introduced by Schäfer and Wayne [1] as a model equation to describe the propagation of ultra-short light pulses in silica optical fibres. Different from the celebrated nonlinear Schrödinger equation (NLSE for short) which is used to model the evolution of slowly varying wave trains, the SPE is proposed to describe the pulse whose spectrum is not narrowly localized around the carrier frequency. It has been proven that as the pulse length shortens, the NLSE approximation describing the optical pulses becomes steadily less accurate, while the SPE provides a better approximation [2].

Contrary to the well studied NLSE, we know a little to SPE. It has been proven that the SPE is an integrable equation possessing a Lax pair [3] of the Wadati–Konno–Ichikawa type [4] and the bi-Hamiltonian structure [5]. Usually, Eq. (1) is difficult to solve, for example, if we solve Eq. (1) in the frame of the following traveling wave transformation:

$$\zeta = x - \omega t, \tag{2}$$

then Eq. (1) can be rewritten as

$$-\omega u_{\zeta\zeta} = u + u u_{\zeta}^2 + \frac{1}{2} u^2 u_{\zeta\zeta}. \tag{3}$$

Due to the coexistence of the terms $u_{\zeta\zeta}$, u and uu_{ζ}^2, Eq. (3) is difficult to be solved by direct integration to derive explicit analytical closed form, so some special transformations have to be introduced. For example, Parkes [6] introduced a new dependent variable z

$$z = \frac{u - v}{|v|}, \tag{4}$$

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and assumed that z is an implicit or explicit function of η, where

$$\eta = x - vt - x_0, \tag{5}$$

v and x_0 are arbitrary constants and $v \neq 0$. Through above transformations, he obtained periodic-hump solution, solitary loop solution and periodic loop solution, 'figure-eight' solution and other type solution to Eq. (1). The solutions found in above mentioned references have been shown to result from a delicate nonlocal balance between dispersion and nonlinearity, and their stable propagation is confirmed by numerical simulations [7]. Most importantly, the transformation between the SPE and the sine-Gordon equation was discovered in Ref. [3] and the derivation of this transformation was considerably simplified in Ref. [8], and later it was used in Ref. [9] to obtain exact loop and pulse solutions of the SPE from the well-known kink and breather solutions of the sine-Gordon equation. The recursion operator found in Ref. [3] was used to study the N-loop soliton solutions to SPE [10].

Since SPE is a current research interest in nonlinear optical fibres theory, in this paper, based on the transformations introduced by Sakovich and Sakovich [3], we will show systematical results for the SPE (1) by using the solutions of the sine-Gordon equation [11,12] derived with the knowledge of elliptic equation and Jacobian elliptic functions [13–17], where many novel solutions will be found.

2. Bridge connecting SPE to sine-Gordon equation

In order to solve the SPE, certain dependent or independent variable transformations must be introduced. Starting from the Eq. (1), we define

$$x = w(y, \tau); \quad t = \tau, \tag{6}$$

then we have

$$\frac{\partial}{\partial x} = \frac{1}{w_y}\frac{\partial}{\partial y}; \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{w_\tau}{w_y}\frac{\partial}{\partial y}. \tag{7}$$

Substituting this transformation into Eq. (1) yields

$$[w_y^3 + w_y u_y^2]u = w_y^2 u_{\tau y} - w_\tau w_y u_{yy} - (w_{\tau y} w_y - w_\tau w_{yy})u_y - \frac{1}{2} w_y u^2 u_{yy} + \frac{1}{2} w_{yy} u^2 u_y. \tag{8}$$

If we set

$$u(x, t) = R(y, \tau), \tag{9}$$

then from Eq. (8) we have

$$[w_y^3 + w_y R_y^2]R = w_y^2 R_{\tau y} - w_\tau w_y R_{yy} - (w_{\tau y} w_y - w_\tau w_{yy})R_y - \frac{1}{2} w_y R^2 R_{yy} + \frac{1}{2} w_{yy} R^2 R_y. \tag{10}$$

It is obvious that the key step to solve the Eq. (10) is to build the bridge between $w(y, \tau)$ and $R(y, \tau)$, in Ref. [3], the relation between $w(y, \tau)$ and $R(y, \tau)$ is

$$\begin{aligned} w_\tau &= -\frac{1}{2}R^2 = -\frac{1}{2}X_\tau^2, \\ w_y &= \cos X, \end{aligned} \tag{11}$$

and from the constraint

$$w_{\tau y} = w_{y\tau}, \tag{12}$$

one can derive the well-known sine-Gordon equation

$$X_{\tau y} = \sin X(y, \tau). \tag{13}$$

Since the solutions to the sine-Gordon equation has been well studied, we can apply the above relations to derive the solutions to SPE easily. In this paper, applying the solutions of the sine-Gordon equation found in Ref. [11] by using the different transformations and the knowledge of elliptic equation and Jacobian elliptic functions [13–17], we will utilize the above bridge to construct more novel solutions to SPE.

3. Exact traveling wave solutions to SPE

First of all, we will apply the solutions of the sine-Gordon equation expressed by the first kind of transformation used in Ref. [11] $X = 2\tan^{-1} v$, where three cases will be discussed in details, other cases can be considered similarly.

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Case 1a. $X = 2\tan^{-1}(\mathrm{csch}\xi)$, where $\xi = ky + \frac{1}{k}t$. Then we have

$$u = X_t = \mathrm{sech}\left(ky + \frac{1}{k}t\right), \tag{14}$$

$$w_t = -\frac{1}{2}X_t^2 = -\frac{1}{2}\mathrm{sech}^2\left(ky + \frac{1}{k}t\right), \tag{15}$$

$$w_y = \cos X = 1 - 2\mathrm{sech}^2\left(ky + \frac{1}{k}t\right), \tag{16}$$

from which we can find the final solution to SPE is

$$\begin{aligned} u &= \mathrm{sech}\left(2y + \frac{1}{2}t\right),\\ x &= y - \tanh\left(2y + \frac{1}{2}t\right), \end{aligned} \tag{17}$$

this is a novel loop solitary solution to SPE different from that given in Ref. [9], and it moves from the right to the left with unchanged shape as shown in Fig. 1. Actually, loop soliton solutions have been found in many different nonlinear systems, such as loop soliton found by Matsuno in short-wave models for the Camassa–Holm and Degasperis-Procesi equations [18].

Case 1b. $X = 2\tan^{-1}\left(\sqrt{\frac{m^2}{1-m^2}}\mathrm{cn}\xi\right)$, where $\mathrm{cn}\xi$ is Jacobian elliptic cosine function [15,16], $\xi = ky - \frac{1}{k}t$. Then we have

$$u = X_t = \frac{2\sqrt{m^2(1-m^2)}}{k}\mathrm{sd}\left(ky - \frac{1}{k}t, m\right), \tag{18}$$

with $\mathrm{sd}\xi = \frac{\mathrm{sn}\xi}{\mathrm{dn}\xi}$, where $\mathrm{sn}\xi$ and $\mathrm{dn}\xi$ are Jacobian elliptic sine function and Jacobian elliptic function of the third kind [15,16].

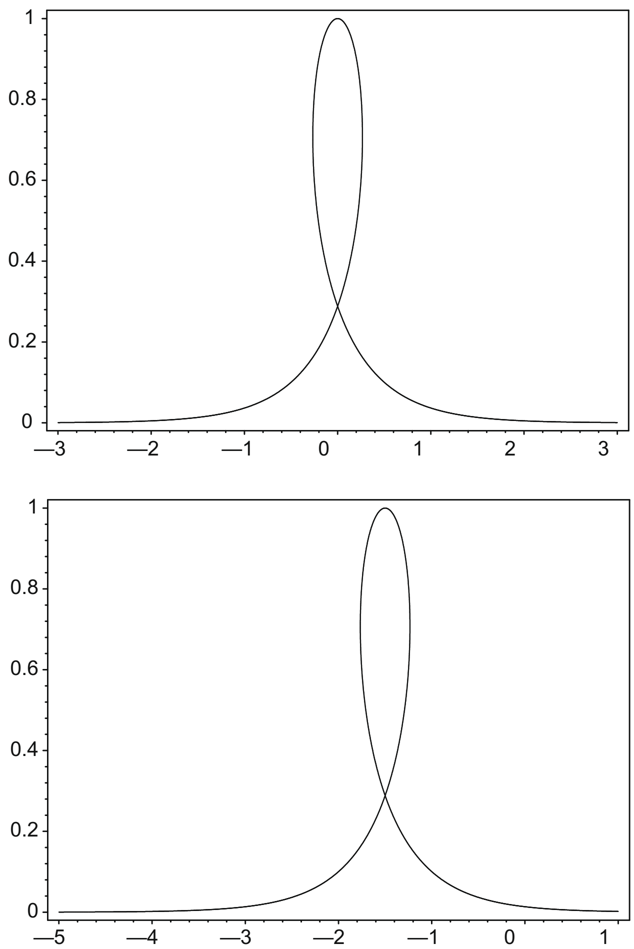

Fig. 1. The loop solitary solution $u(x,t)$ (17). (a) $t = 0, y \in [-4,4]$ (top) and (b) $t = 6.0, y \in [-4,4]$ (bottom).

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$$w_t = -\frac{1}{2}X_t^2 = -\frac{1}{k^2}\left[2(1-m^2)\mathrm{nd}^2(ky-\frac{1}{k}t,m)-1\right], \tag{19}$$

$$w_y = \cos X = 2(1-m^2)\mathrm{nd}^2\left(ky-\frac{1}{k}t,m\right)-1, \tag{20}$$

with $\mathrm{nd}\xi = \frac{1}{\mathrm{dn}\xi}$, from which we can find the final solution to SPE is

$$\begin{aligned} u &= \frac{2\sqrt{m^2(1-m^2)}}{k}\mathrm{sd}\left(ky-\frac{1}{k}t,m\right), \\ x &= \frac{1}{k}\left[2E\left(ky-\frac{1}{k}t,m\right)-2m^2\mathrm{sn}\left(ky-\frac{1}{k}t,m\right)\mathrm{cd}\left(ky-\frac{1}{k}t,m\right)-F\left(ky-\frac{1}{k}t,m\right)\right], \end{aligned} \tag{21}$$

with $\mathrm{cd}\xi = \frac{\mathrm{cn}\xi}{\mathrm{dn}\xi}$ and

$$F(\varphi,m) = \int_0^{\varphi}\frac{d\theta}{\sqrt{1-m^2\sin^2\theta}}, \tag{22}$$

called the normal elliptic integral of the first kind, and

$$E(\varphi,m) = \int_0^{\varphi}\sqrt{1-m^2\sin^2\theta}d\theta, \tag{23}$$

called the normal elliptic integral of the second kind [17], where $0 \leqslant m \leqslant 1$ is called modulus of Jacobi elliptic functions [15–17]. Solution (21) is a novel solution expressed in terms of special function to SPE which has not been reported, and it can be found that its shape depends on the modulus m, when the m is less than a critical value, the solution is single-valued (see from Fig. 2a to c); when it is larger than this critical value, the solution is multiple-valued(see from Fig. 2d to f). At the same time, when m is fixed, the shape of this solution will be constant when the time changes (see Fig. 2b and c).

Case 1c. $X = 2\tan^{-1}(m\mathrm{sd}\xi)$, where $\xi = ky - \frac{1}{k}t$. Then we have

$$u = X_t = -\frac{2m}{k}\mathrm{cn}\left(ky-\frac{1}{k}t,m\right), \tag{24}$$

$$w_t = -\frac{1}{2}X_t^2 = -\frac{2m^2}{k^2}\mathrm{cn}^2\left(ky-\frac{1}{k}t,m\right), \tag{25}$$

$$w_y = \cos X = 1-2m^2\mathrm{sn}^2\left(ky-\frac{1}{k}t,m\right), \tag{26}$$

from which we can find the final solution to SPE is

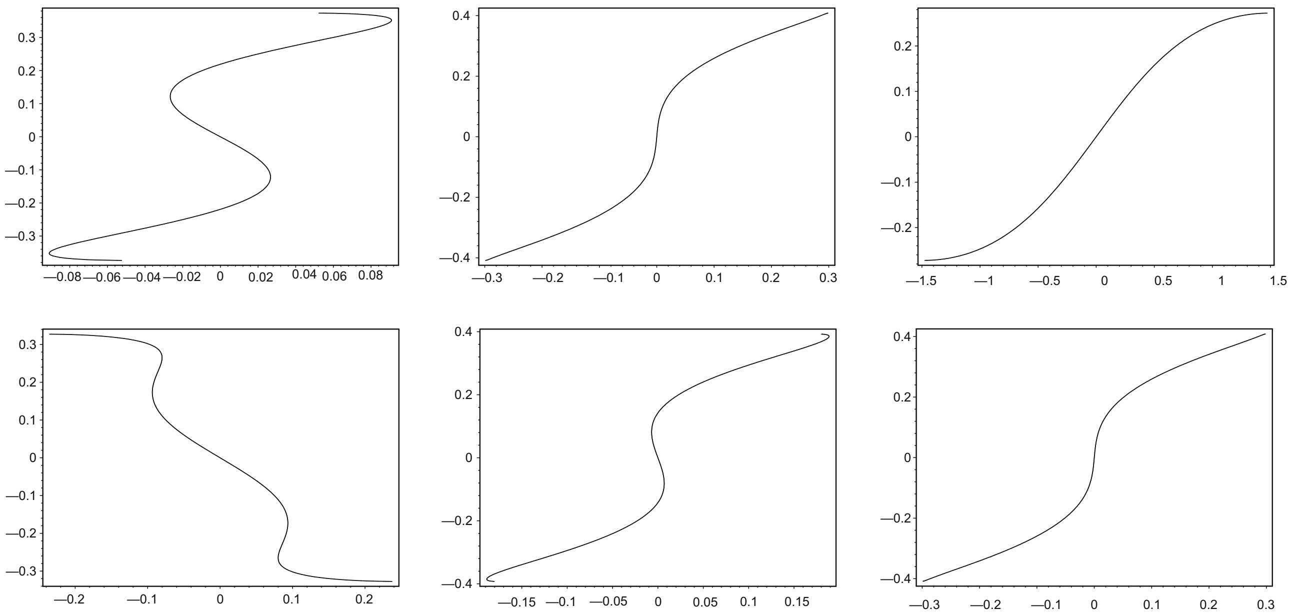

Fig. 2. Novel solution $u(x,t)$ (21) expressed in terms of Jacobian elliptic function and its integrals. (a) $k=1, t=1, m=0.2,\ y\in[-6,10]$ (top and right), (b) $k=1,\ t=1,\ m=0.7,\ y\in[-6,10]$ (bottom and right), (c) $k=1, t=0.0, m=0.7, y\in[-6,10]$ (middle and top), (d) $k=1,\ t=0.0,\ m=0.725,\ y\in[-6,10]$ (middle and bottom), (e) $k=1, t=0.0, m=0.75, y\in[-6,10]$ (top and left) and (f) $k=1,\ t=1.0,\ m=0.8,\ y\in[-6,10]$ (bottom and left).

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$$\begin{aligned} u &= -\frac{\sqrt{2}}{k}\mathrm{cn}\left(ky - \frac{1}{k}t, \frac{\sqrt{2}}{2}\right), \\ x &= \frac{1}{k}\left[2E\left(ky - \frac{1}{k}t, \frac{\sqrt{2}}{2}\right) - F\left(ky - \frac{1}{k}t, \frac{\sqrt{2}}{2}\right)\right], \end{aligned} \tag{27}$$

this is another novel solution to SPE expressed in terms of special function.

Secondly, we will apply the solutions of the sine-Gordon equation expressed by the fourth kind of transformation used in Ref. [11] $X = 2\cos^{-1} v$, where five cases will be discussed in details, other cases can be also considered similarly.

Case 2a. $X = 2\cos^{-1}(m\mathrm{sn}\xi)$, where $\xi = ky + \frac{1}{k}t$. Then we have

$$u = X_t = -\frac{2m}{k}\mathrm{cn}\left(ky + \frac{1}{k}t, m\right), \tag{28}$$

$$w_t = -\frac{1}{2}X_t^2 = -\frac{2m^2}{k^2}\mathrm{cn}^2\left(ky + \frac{1}{k}t, m\right), \tag{29}$$

$$w_y = \cos X = 2m^2\mathrm{sn}^2\left(ky + \frac{1}{k}t, m\right) - 1, \tag{30}$$

from which we can find the final solution to SPE is

$$\begin{aligned} u &= -\frac{\sqrt{2}}{k}\mathrm{cn}\left(ky + \frac{1}{k}t, \frac{\sqrt{2}}{2}\right), \\ x &= -\frac{1}{k}\left[2E\left(ky + \frac{1}{k}t, \frac{\sqrt{2}}{2}\right) - F\left(ky + \frac{1}{k}t, \frac{\sqrt{2}}{2}\right)\right], \end{aligned} \tag{31}$$

this is also a novel solution to SPE expressed in terms of special function.

Case 2b. $X = 2\cos^{-1}(\tanh\xi)$, where $\xi = ky + \frac{1}{k}t$. Then we have the final solution to SPE is

$$\begin{aligned} u &= -\frac{2}{k}\mathrm{sech}\left(ky + \frac{1}{k}t\right), \\ x &= y - \frac{2}{k}\tanh\left(ky + \frac{1}{k}t\right), \end{aligned} \tag{32}$$

this is another loop (for $k < 0$) or anti-loop (for $k > 0$) solitary solution to SPE different from that given in Ref. [9], when $k = -1$, solution (32) recovers solution given in Ref. [9].

Case 2c. $X = 2\cos^{-1}(\mathrm{sech}\xi)$, where $\xi = ky - \frac{1}{k}t$. Then we have the final solution to SPE is

$$\begin{aligned} u &= -\frac{2}{k}\mathrm{sech}\left(ky - \frac{1}{k}t\right), \\ x &= \frac{2}{k}\tanh\left(ky - \frac{1}{k}t\right) - 2y, \end{aligned} \tag{33}$$

this is also a loop or anti-loop solitary solution to SPE.

Case 2d. $X = 2\cos^{-1}(\mathrm{dn}\xi)$, where $\xi = ky - \frac{1}{k}t$. For this case, the solution to the SPE will recover the solution given by (27).

Case 2e. $X = 2\cos^{-1}\left(\frac{\tanh\xi}{1+\mathrm{sech}\xi}\right)$, where $\xi = ky + \frac{4}{k}t$. Then we have the final solution to SPE is

$$\begin{aligned} u &= -\frac{4}{k}\mathrm{sech}\left(\frac{k}{2}y + \frac{2}{k}t\right), \\ x &= y - \frac{4}{k}\tanh\left(\frac{k}{2}y + \frac{2}{k}t\right), \end{aligned} \tag{34}$$

this is still another loop or anti-loop solitary solution to SPE different from that given in Ref. [9], and it moves from the right to the left (when k is positive) with unchanged shape as shown in Fig. 3 or moves from the left to the right (when k is negative) with unchanged shape (figure is not shown).

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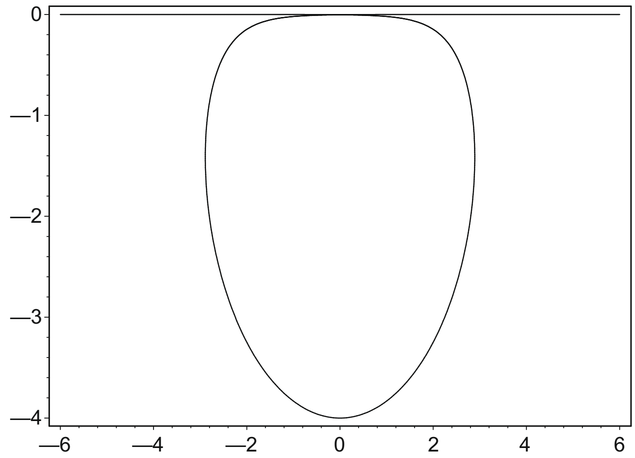

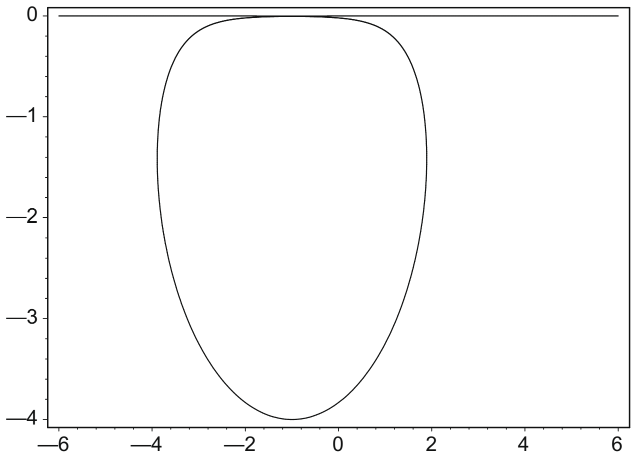

Fig. 3. The anti-loop solitary solution $u(x,t)$ (34). (a) $k = 1, t = 0, y \in [-10, 10]$ (top) and (b) $k = 1, t = 1.0, y \in [-10, 10]$ (bottom).

4. Conclusion

In this paper, we presented the process to find exact solutions for the SPE with the help from the bridge connecting SPE to the sine-Gordon equation and obtained some novel types of solutions, these solutions may be applied to describe and/or explain some phenomena found in the nonlinear optical fibres, since the model has been proposed to model short optical pulse. Since we only considered the solutions of the sine-Gordon equation expressed in terms of single Jacobian elliptic function, more solutions of the sine-Gordon equation found by other methods can be applied to obtain more types solutions to SPE, this will be reported in our next paper.

Acknowledgements

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