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Lame Function and Its Application to Some Nonlinear Evolution Equations*

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Abstract *In this paper, based on the Lame function and Jacobi elliptic function, the perturbation method is applied to some nonlinear evolution equations to derive their multi-order solutions.*

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1 Introduction

It plays an important role to find exact solutions of nonlinear evolution equations in the nonlinear studies. Many new methods, such as the homogeneous balance method,[1−3] the hyperbolic tangent function expansion method,[4−6] the nonlinear transformation method,[7,8] the trial function method,[9,10] sine-cosine method,[11] the Jacobi elliptic function expansion method,[12,13] and so on,[14−16] have been proposed and applied to get many exact solutions, from which the richness of structures is shown to exist in the different nonlinear wave equations. Furthermore, in order to discuss the stability of these solutions, it is necessary to superimpose a small disturbance on these solutions and analyze the evolution of the small disturbance.[17,18] This is equivalent to that the solutions of nonlinear evolution equations are expanded as a power series in terms of a small parameter ϵ, and multi-order exact solutions are derived. In this paper, on the basis of the Jacobi elliptic function expansion method, the multi-order exact solutions of some nonlinear evolution equations are obtained by means of the Jacobi elliptic functions and Lame function.[18,19]

2 Lame Equation and Lame Functions

Usually, Lame equation[19] in terms of $y(x)$ can be written as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + [\lambda - n(n+1)m^2\mathrm{sn}^2 x]y = 0\,, \tag{1}$$

where λ is an eigenvalue, n is a positive integer, $\mathrm{sn}\,x$ is the Jacobi elliptic sin function with its modulus being m $(0 < m < 1)$.

Set

$$\eta = \mathrm{sn}^2 x\,, \tag{2}$$

then the Lame equation (1) becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\eta^2} + \frac{1}{2}\Big(\frac{1}{\eta} + \frac{1}{\eta-1} + \frac{1}{\eta-h}\Big)\frac{\mathrm{d}y}{\mathrm{d}\eta} - \frac{\mu + n(n+1)\eta}{4\eta(\eta-1)(\eta-h)}y = 0\,, \tag{3}$$

where

$$h = m^{-2} > 1\,, \qquad \mu = -h\lambda\,. \tag{4}$$

Equation (3) is a kind of Fuchs-typed equations with four regular singular points $\eta = 0, 1, h$, and $\eta = \infty$, and its solution is known as Lame function.

For example, when $n = 3$, $\lambda = 4(1 + m^2)$, i.e. $\mu = -4(1 + m^{-2})$, the Lame function is

$$L_3(x) = \eta^{1/2}(1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2} = \mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x\,. \tag{5}$$

When $n = 2$, $\lambda = 1 + m^2$, i.e. $\mu = -(1 + m^{-2})$, the Lame function is

$$L_2(x) = (1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2} = \mathrm{cn}\,x\,\mathrm{dn}\,x\,. \tag{6}$$

In Eqs. (5) and (6), $\mathrm{cn}\,x$ and $\mathrm{dn}\,x$ are the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind,[18,19] respectively. In the next sections, we will apply these two kinds of Lame functions $L_3(x)$ and $L_2(x)$ and their corresponding Lame equations to solve nonlinear evolution systems and to derive their corresponding multi-order exact solutions.

3 Application to (1 + 1)-Dimensional Nonlinear Evolution Equation

In this section, we consider an application of Lame equation to $(1+1)$-dimensional nonlinear evolution equations. Here we use combined mKdV-KdV equation to illustrate this case.

The combined mKdV-KdV equation reads

$$\frac{\partial u}{\partial t} + (\alpha + \gamma u)u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0\,. \tag{7}$$

We seek its travelling wave solutions of the following form:

$$u = u(\xi)\,, \qquad \xi = k(x - ct)\,, \tag{8}$$

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where k and c are wave number and wave speed, respectively.

Substituting Eq. (8) into Eq. (9), we have

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi}+(\alpha+\gamma u)u\frac{\mathrm{d}u}{\mathrm{d}\xi}+\beta k^2\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}=0\,. \tag{9}$$

Integrating Eq. (9) once with respect to ξ and taking the integration constants as zero, we get

$$\beta k^2\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}+\frac{\gamma}{3}u^3+\frac{\alpha}{2}u^2-cu=0\,. \tag{10}$$

Here we consider perturbation method and setting

$$u=u_0+\epsilon u_1+\epsilon^2u_2+\cdots\,, \tag{11}$$

where ϵ ($0<\epsilon\ll 1$) is a small parameter, u_0, u_1, and u_2 represent the zeroth-order, first-order, and second-order solutions, respectively.

Substituting Eq. (11) into Eq. (10), we derive the following systems of the zeroth-order, the first-order, and the second-order equations

$$\epsilon^0:\quad \beta k^2\frac{\mathrm{d}^2u_0}{\mathrm{d}\xi^2}+\frac{\gamma}{3}u_0^3+\frac{\alpha}{2}u_0^2-cu_0=0\,, \tag{12}$$

$$\epsilon^1:\quad \beta k^2\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2}+[\gamma u_0^2+\alpha u_0-c]u_1=0\,, \tag{13}$$

and

$$\epsilon^2:\ \beta k^2\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2}+[\gamma u_0^2+\alpha u_0-c]u_2=-\Big(\gamma u_0+\frac{\alpha}{2}\Big)u_1^2\,. \tag{14}$$

The zeroth-order equation (12) can be solved by the Jacobi elliptic sine function expansion method. The ansatz solution

$$u_0=a_0+a_1\,\mathrm{sn}\,\xi \tag{15}$$

can be assumed.

Substituting Eq. (15) into Eq. (12), the expansion coefficients a_0 and a_1 can be easily determined as

$$a_0=-\frac{\alpha}{2\gamma}\,,\qquad a_1=\pm\sqrt{-\frac{6\beta}{\gamma}}mk\,,$$
$$c=-\frac{\alpha^2}{6\gamma}\,,\qquad k^2=-\frac{\alpha^2}{12\beta\gamma(1+m^2)}\,, \tag{16}$$

so the zeroth-order exact solution is

$$u_0=-\frac{\alpha}{2\gamma}\pm\sqrt{-\frac{6\beta}{\gamma}}\,mk\,\mathrm{sn}\,\xi\,. \tag{17}$$

Substituting the zeroth-order exact solution (17) into the first-order equation Eq. (13) yields

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2}+[(1+m^2)-6m^2\mathrm{sn}^2\xi]u_1=0\,, \tag{18}$$

which obviously is just a Lame equation as Eq. (1) with $n=2$ and $\lambda=(1+m^2)$, then the Lame equation Eq. (1) reduces to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+[(1+m^2)-6m^2\mathrm{sn}^2x]y=0\,. \tag{19}$$

So the solution of Eq. (18) is

$$u_1=AL_2(\xi)=A\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi\,, \tag{20}$$

where A is an arbitrary constant, and equation (20) is the first-order exact solution of combined mKdV-KdV equation (7).

In order to solve the second-order equation (14), the zeroth-order exact solution Eq. (17) and the first-order exact solution Eq. (20) have to be substituted into Eq. (14), thus the second-order equation (14) is rewritten as

$$\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2}+[(1+m^2)-6m^2\mathrm{sn}^2\xi]u_2$$
$$=\pm\sqrt{-\frac{6\gamma}{\beta}}\frac{mA^2}{k}\,\mathrm{sn}\,\xi\,\mathrm{cn}^2\xi\,\mathrm{dn}^2\xi\,. \tag{21}$$

It is obvious that this is an inhomogeneous Lame equation with $n=2$ and $\lambda=(1+m^2)$. Its solution of homogeneous equation is just the same as Eq. (20) and its special solution of inhomogeneous terms can be assumed to be

$$u_2=b_1\,\mathrm{sn}\,\xi+b_3\,\mathrm{sn}^3\xi\,. \tag{22}$$

Substituting Eq. (22) into Eq. (21), we can determine the expansion coefficients b_1 and b_3 as

$$b_1=\mp\frac{1+m^2}{12m}\sqrt{-\frac{6\gamma}{\beta}}\frac{A^2}{k}\,,\quad b_3=\pm\frac{1}{6}\sqrt{-\frac{6\gamma}{\beta}}\frac{mA^2}{k}\,, \tag{23}$$

so the second-order exact solution of combined mKdV-KdV equation Eq. (7) can be written as

$$u_2=\mp\sqrt{-\frac{6\gamma}{\beta}}\frac{(1+m^2)A^2}{12mk}\,\mathrm{sn}\,\xi\Big[1-\frac{2m^2}{1+m^2}\mathrm{sn}^2\xi\Big]. \tag{24}$$

4 Application to (1 + 2)-Dimensional Nonlinear Evolution Equation

In the above section, we discuss the application of the Lame equation under the condition of $n=2$ and $\lambda=(1+m^2)$ to the $(1+1)$-dimensional nonlinear evolution equation and get the multi-order exact solutions to combined mKdV-KdV equation. We know that the Lame equation Eq. (1) has another form under the condition of $n=3$ and $\lambda=4(1+m^2)$. There it reduces to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+[4(1+m^2)-12m^2\mathrm{sn}^2x]y=0\,, \tag{25}$$

and the solution to Eq. (25) is Eq. (5). Next, we will illustrate the application of Eq. (25) to solve $(1+2)$-dimensional nonlinear evolution equations. Here we use KP equation as an example.

KP equation reads

$$\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\beta\frac{\partial^3u}{\partial x^3}\Big)+\frac{c_0}{2}\frac{\partial^2u}{\partial y^2}=0\,. \tag{26}$$

We seek its travelling wave solution in the following frame:

$$u=u(\xi)\,,\qquad \xi=kx+ly-\omega t\,, \tag{27}$$

where k and l are wave number in the directions of x and y, respectively and ω is angular frequency. Then equation (27) can be rewritten as

$$k\frac{\mathrm{d}}{\mathrm{d}\xi}\Big(-\omega\frac{\mathrm{d}u}{\mathrm{d}\xi}+ku\frac{\mathrm{d}u}{\mathrm{d}\xi}+\beta k^3\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}\Big)+\frac{c_0}{2}l^2\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}=0\,, \tag{28}$$

which can be integrated twice with respect to ξ and if the integration constants are set at zero, then equation (28) becomes

$$\beta k^4 \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + \frac{k^2}{2} u^2 - \Big(\omega k - \frac{c_0}{2} l^2\Big) u = 0 . \tag{29}$$

Set

$$c_1 = \frac{\omega}{k} - \frac{c_0}{2}\frac{l^2}{k^2} , \tag{30}$$

thus equation (29) is rewritten as

$$\beta k^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + \frac{1}{2} u^2 - c_1 u = 0 . \tag{31}$$

Combining Eqs. (11) and (31), we can get the multi-order expansion equations, for example, the zeroth-order equation is

$$\epsilon^0 : \quad \beta k^2 \frac{\mathrm{d}^2 u_0}{\mathrm{d}\xi^2} + \frac{1}{2} u_0^2 - c_1 u_0 = 0 , \tag{32}$$

the first-order equation is

$$\epsilon^1 : \quad \beta k^2 \frac{\mathrm{d}^2 u_1}{\mathrm{d}\xi^2} + (u_0 - c_1) u_1 = 0 , \tag{33}$$

and the second-order equation is

$$\epsilon^2 : \quad \beta k^2 \frac{\mathrm{d}^2 u_2}{\mathrm{d}\xi^2} + (u_0 - c_1) u_2 = -\frac{1}{2} u_1^2 . \tag{34}$$

The zeroth-order equation (32) can be solved by the Jacobi elliptic sine function expansion method when the ansatz solution

$$u_0 = a_0 + a_1 \,\mathrm{sn}\,\xi + a_2 \mathrm{sn}^2\xi \tag{35}$$

is introduced.

Substituting Eq. (35) into Eq. (32) leads to

$$u_0 = c_1 + 4(1+m^2)\beta k^2 - 12 m^2 \beta k^2 \mathrm{sn}^2\xi , \tag{36}$$

which is the zeroth-order exact solution of KP equation.

Substituting the zeroth-order exact solution (36) into the first-order equation (33) results in

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\xi^2} + [4(1+m^2) - 12 m^2 \mathrm{sn}^2 x] u_1 = 0 , \tag{37}$$

which is just the same as Eq. (25), so its solution is

$$u_1 = A L_3(\xi) = A \,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi , \tag{38}$$

where A is an arbitrary constant.

In order to solve the second-order equation (34), it is necessary to substitute the zeroth-order solution (36) and the first-order solution (38) into Eq. (34), then we can get

$$\frac{\mathrm{d}^2 u_2}{\mathrm{d}\xi^2} + [4(1+m^2) - 12 m^2 \mathrm{sn}^2 x] u_2 = -\frac{A^2}{2} \mathrm{sn}^2\xi \mathrm{cn}^2\xi \mathrm{dn}^2\xi , \tag{39}$$

from which the second-order exact solution can be determined as

$$u_2 = -\frac{A^2}{48 m^2 \beta k^2} [1 - 2(1+m^2)\mathrm{sn}^2\xi + 3m^2 \mathrm{sn}^4\xi] . \tag{40}$$

5 Application to Coupled Nonlinear System

In the above two sections, we discussed the application of the Lame equation and Lame functions to single nonlinear equations, and their different multi-order solutions are given. In this section, we will apply the Lame equation and Lame functions to coupled nonlinear systems. Here we consider coupled KdV equations, which reads

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} + \mu v \frac{\partial v}{\partial x} = 0 , \tag{41a}$$

$$\frac{\partial v}{\partial t} + \gamma v \frac{\partial v}{\partial x} + \delta \frac{\partial (uv)}{\partial x} = 0 . \tag{41b}$$

We solve it in the following frame:

$$u = u(\xi) , \qquad v = v(\xi) , \qquad \xi = k(x - ct) , \tag{42}$$

thus equation (41) becomes

$$\beta k^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + \frac{\alpha}{2} u^2 - cu + \frac{\mu}{2} v^2 = 0 , \quad \delta uv + \frac{\gamma}{2} v^2 - cv = 0 , \tag{43}$$

where integration has been taken once with respect to ξ, and integration constants are set at zero.

The solutions to Eq. (43) can be expanded as a multi-order power series by applying perturbation method, i.e.,

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots , \tag{44a}$$

$$v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots , \tag{44b}$$

where ϵ $(0 < \epsilon \ll 1)$ is a small parameter.

Substituting Eq. (43) into Eq. (44) leads to the multi-order equations, for example, the first three-order equations are

$$\beta k^2 \frac{\mathrm{d}^2 u_0}{\mathrm{d}\xi^2} + \frac{\alpha}{2} u_0^2 - c u_0 + \frac{\mu}{2} v_0^2 = 0 , \tag{45a}$$

$$\delta u_0 v_0 + \frac{\gamma}{2} v_0^2 - c v_0 = 0 , \tag{45b}$$

$$\beta k^2 \frac{\mathrm{d}^2 u_1}{\mathrm{d}\xi^2} + (\alpha u_0 - c) u_1 + \mu v_0 v_1 = 0 , \tag{46a}$$

$$\delta (u_0 v_1 + v_0 u_1) + \gamma v_0 v_1 - c v_1 = 0 , \tag{46b}$$

and

$$\beta k^2 \frac{\mathrm{d}^2 u_2}{\mathrm{d}\xi^2} + (\alpha u_0 - c) u_2 + \mu v_0 v_2 = -\frac{\alpha}{2} u_1^2 - \frac{\mu}{2} v_1^2 , \tag{47a}$$

$$\delta (u_0 v_2 + v_0 u_2) + \gamma v_0 v_2 - c v_2 = -\delta u_1 v_1 - \frac{\gamma}{2} v_1^2 . \tag{47b}$$

The zeroth-order equation Eq. (45) can be solved by Jacobi elliptic sine function expansion method when the ansatz solution

$$u_0 = a_0 + a_1 \,\mathrm{sn}\,\xi + a_2 \mathrm{sn}^2\xi , \quad v_0 = b_0 + b_1 \mathrm{sn}\xi + b_2 \mathrm{sn}^2\xi . \tag{48}$$

Substituting Eq. (48) into Eq. (45) yields

$$u_0 = \frac{(4\mu\delta + \gamma^2)c}{4\mu\delta^2 + \alpha\gamma^2} + \frac{4(1+m^2)\beta\gamma^2 k^2}{4\mu\delta^2 + \alpha\gamma^2} - \frac{12 m^2 \beta \gamma^2 k^2}{4\mu\delta^2 + \alpha\gamma^2} \mathrm{sn}^2\xi , \tag{49a}$$

$$v_0 = -\frac{2(\delta - \alpha)\gamma c}{4\mu\delta^2 + \alpha\gamma^2} - \frac{8(1+m^2)\beta\gamma\delta k^2}{4\mu\delta^2 + \alpha\gamma^2} + \frac{24 m^2 \beta\gamma\delta k^2}{4\mu\delta^2 + \alpha\gamma^2} \mathrm{sn}^2\xi . \tag{49b}$$

It is obvious that there exists the following relation between u_0 and v_0,

$$\gamma v_0 + 2\delta u_0 = 2c . \tag{50}$$

Considering the relation Eq. (50), we substitute the zeroth-order solution Eq. (49) into the first-order equation (46) to get

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2}+[4(1+m^2)-12m^2\mathrm{sn}^2\xi]u_1=0\,, \tag{51a}$$

$$v_1=-\frac{2\delta}{\gamma}u_1\,. \tag{51b}$$

Obviously, equation (51a) is the Lame-typed equation Eq. (25), the first-order exact solution to Eq. (51) is

$$u_1=AL_3(\xi)\,,\qquad v_1=-\frac{2\delta}{\gamma}AL_3(\xi)\,. \tag{52}$$

In order to solve the second-order equations of coupled KdV equations (47), we have to substitute the zeroth-order solution (49) and the first-order solution (52) into Eq. (47), then we can obtain the rewritten second-order equations

$$\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2}+[4(1+m^2)-12m^2\mathrm{sn}^2\xi]u_2 =-\frac{(4\mu\delta^2+\alpha\gamma^2)A^2}{2\beta\gamma^2k^2}\mathrm{sn}^2\xi\mathrm{cn}^2\xi\mathrm{dn}^2\xi\,, \tag{53a}$$

$$v_2=-\frac{2\delta}{\gamma}u_2\,. \tag{53b}$$

Similarly, its solution is

$$u_2=-\frac{(4\mu\delta^2+\alpha\gamma^2)A^2}{48\beta\gamma^2k^2}\times[1-2(1+m^2)\mathrm{sn}^2\xi+3m^2\,\mathrm{sn}^4\xi]\,, \tag{54a}$$

$$v_2=-\frac{2\delta}{\gamma}u_2=\frac{(4\mu\delta^2+\alpha\gamma^2)\delta A^2}{24\beta\gamma^3k^2}\times[1-2(1+m^2)\mathrm{sn}^2\xi+3m^2\,\mathrm{sn}^4\xi]\,. \tag{54b}$$

6 Conclusion and Discussion

In this paper, the Lame equation and Lame functions are applied to solve nonlinear $(1+1)$-dimensional, $(1+2)$-dimensional and coupled evolution equations. When perturbation method and two kinds of Lame functions $L_3(x)$ and $L_2(x)$ are considered, then the multi-order solutions to these nonlinear evolution systems are obtained. The results got in this paper are very important for nonlinear instability analysis of nonlinear waves.

References

[1] M.L. Wang, Phys. Lett. **A199** (1995) 169.

[2] M.L. Wang, Y.B. Zhou, and Z.B. Li, Phys. Lett. **A216** (1996) 67.

[3] L. Yang, Z. Zhu, and Y. Wang, Phys. Lett. **A260** (1999) 55.

[4] E.J. Parkes and B.R. Duffy, Phys. Lett. **A229** (1997) 217.

[5] L. Yang, J. Liu, and K. Yang, Phys. Lett. **A278** (2001) 267.

[6] E.G. Fan, Phys. Lett. **A277** (2000) 212.

[7] R. Hirota, J. Math. Phys. **14** (1973) 810.

[8] M. Otwinowski, R. Paul, and W.G. Laidlaw, Phys. Lett. **A128** (1988) 483.

[9] N.A. Kudryashov, Phys. Lett. **A147** (1990) 287.

[10] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Appl. Math. Mech. **22** (2001) 326.

[11] C.T. Yan, Phys. Lett. **A224** (1996) 77.

[12] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Phys. Lett. **A289** (2001) 69.

[13] Z.T. Fu, S.K. Liu, S.D. Liu, and Q. Zhao, Phys. Lett. **A290** (2001) 72.

[14] A.V. Porubov, Phys. Lett. **A221** (1996) 391.

[15] A.V. Porubov and M.G. Velarde, J. Math. Phys. **40** (1999) 884.

[16] A.V. Porubov and D.F. Parker, Wave Motion **29** (1999) 97.

[17] A.H. Nayfeh, *Perturbation Methods*, John Wiley and Sons, New York (1973).

[18] S.K. Liu and S.D. Liu, *Nonlinear Equations in Physics*, Peking University Press, Beijing (2000).

[19] Z.X. Wang and D.R. Guo, *Special Functions*, World Scientific, Singapore (1989).