

# Lam 函数和非线性演化方程的多级准确解的不变性\*

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利用小扰动方法对非线性演化方程作展开得到原始方程的各级近似方程.并在 Lam 方程和 Lam 函数的基础上,应用 Jacobi 椭圆函数展开法求出了非线性演化方程的多级准确解,从而得到了多级准确解中存在的守恒形式.

关键词: Lam 函数, Jacobi 椭圆函数, 多级准确解, 非线性演化方程, 扰动方法, 不变性

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## 1. 引 言

寻找非线性演化方程的准确解在非线性问题中占有很重要的地位.为了求得更多的准确解和求得更多非线性演化方程的各级准确解,众多的求解方法被陆续提出,如齐次平衡法<sup>[1-3]</sup>、双曲正切函数展开法<sup>[4]</sup>、非线性变换法<sup>[5,6]</sup>、试探函数法<sup>[7,8]</sup>、sine-cosine方法<sup>[9]</sup>和 Jacobi 椭圆函数展开法<sup>[10-12]</sup>等.应用这些求解非线性演化方程准确解的方法,求得了非线性演化方程的解主要有孤立波解、冲击波解<sup>[11-9,13-21]</sup>和周期波解(包括椭圆函数解)等<sup>[10-12,22-24]</sup>.为了讨论这些解的稳定性,必须在这些解的基础上叠加一个小扰动<sup>[25,26]</sup>,并分析小扰动的演化.这种做法实质上是将非线性演化方程的解展开为小参数的幂级数,并力求获得它的各级准确解.同时,我们知道非线性演化方程中存在很多守恒量或不变量,在方程和方程之间也存在着众多的不变性,这也是非线性研究的一个重点.本文在小扰动方法<sup>[25,26]</sup>和 Jacobi 椭圆函数展开法<sup>[10-12]</sup>的基础上,应用 Lam 方程和 Lam 函数<sup>[27]</sup>求得了某些非线性演化方程的多级准确解,并在此基础上研究了它们的极限性质和不变性.

## 2. BBM 方程的多级准确解

Benjamin-Bona-Mahoney (BBM) 方程为

$$\frac{\partial u}{\partial t} + c_0 \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^2 \partial t} = 0. \quad (1)$$

设它的行波解为

$$u = u(\xi), \quad \xi = k(x - ct), \quad (2)$$

式中  $k$  和  $c$  分别为波数和波速.

将(2)式代入方程(1),求得

$$k^2 c \frac{d^3 u}{d\xi^3} - u \frac{du}{d\xi} + c \frac{du}{d\xi} - c_0 \frac{du}{d\xi} = 0. \quad (3)$$

(3)式对  $\xi$  积分一次,取积分常数为零,得到

$$k^2 c \frac{d^2 u}{d\xi^2} - \frac{1}{2} u^2 + (c - c_0) u = 0. \quad (4)$$

对方程(4)应用小扰动方法,设

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots, \quad (5)$$

式中  $\epsilon$  为小参数( $0 < \epsilon \ll 1$ ),  $u_0, u_1, u_2, \dots$  分别代表  $u$  的零级、一级、二级等各级解.

将(5)式代入方程(4),求得它的零级方程、一级方程和二级方程分别为

$$\begin{aligned} &0: \\ &k^2 c \frac{d^2 u_0}{d\xi^2} - \frac{1}{2} u_0^2 + (c - c_0) u_0 = 0, \quad (6) \\ &1: \end{aligned}$$

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$$k^2 c \frac{d^2 u_1}{d^2} - [u_0 - (c - c_0)] u_1 = 0, \quad (7)$$

2 :

$$k^2 \frac{d^2 u_2}{d^2} - [u_0 - (c - c_0)] u_2 = \frac{1}{2} u_1^2. \quad (8)$$

对于零级方程(6),应用 Jacobi 椭圆正弦函数展开法<sup>[10-12]</sup>,令

$$u_0 = a_0 + a_1 \text{sn}^2 + a_2 \text{sn}^4, \quad (9)$$

式中 sn x 为 Jacobi 椭圆正弦函数, m 为模数 (0 < m < 1). 将(9)式代入方程(6),很容易定得

$$\begin{aligned} a_0 &= c - c_0 - 4(1 + m^2) k^2 c, \\ a_1 &= 0, \\ a_2 &= 12 m^2 k^2 c. \end{aligned} \quad (10)$$

因而,BBM 方程(1)的零级准确解为

$$u_0 = c - c_0 - 4(1 + m^2) k^2 c + 12 m^2 k^2 c \text{sn}^2. \quad (11)$$

对于一级方程(7),将(11)式代入得到

$$\frac{d^2 u_1}{d^2} + [4(1 + m^2) - 12 m^2 \text{sn}^2] u_1 = 0. \quad (12)$$

这是一类典型的 Lam 方程<sup>[27]</sup>.

函数 y(x) 的 Lam 方程<sup>[27]</sup>通常可以写为

$$\frac{d^2 y}{dx^2} + [\mu - n(n+1) m^2 \text{sn}^2 x] y = 0, \quad (13)$$

式中 μ 为本征值, n 通常为正整数.

若作自变量变换

$$x = \text{sn}^2 x, \quad (14)$$

则 Lam 方程(13)化为

$$\begin{aligned} \frac{d^2 y}{d^2} + \frac{1}{2} \left( \frac{1}{1} + \frac{1}{-1} + \frac{1}{-h} \right) \frac{dy}{d} \\ - \frac{\mu + n(n+1)}{4(-1)(-h)} y = 0, \end{aligned} \quad (15)$$

其中

$$\begin{aligned} h &= m^{-2} > 1, \\ \mu &= -h. \end{aligned} \quad (16)$$

方程(15)是包含 4 个正则奇点的 μ = 0, 1, h 和 ∞ 的 Fuch 型方程,它的解称为 Lam 函数.

例如,当 n = 3, μ = 4(1 + m^2), [μ = -4(1 + m^2)] 时, Lam 函数为

$$\begin{aligned} L_3(x) &= {}^{1/2} (1 - x)^{1/2} (1 - h^{-1} x)^{1/2} \\ &= \text{sn} x \text{cn} x \text{dn} x, \end{aligned} \quad (17)$$

式中 cn x 和 dn x 分别为 Jacobi 椭圆余弦函数和第三类 Jacobi 椭圆函数.

很明显 Lam 方程(13)和 BBM 方程的一级近似方程(12)形式一样.因此,方程(13)的解为

$$u_1 = AL_3(x) = A \text{sn} x \text{cn} x \text{dn} x, \quad (18)$$

式中 A 为任意常数,这就是 BBM 方程的一级准确解.

对于二级方程(8),以(11)和(18)式代入得到

$$\begin{aligned} \frac{d^2 u_2}{d^2} + [4(1 + m^2) - 12 m^2 \text{sn}^2] u_2 \\ = \frac{A^2}{2 k^2 c} \text{sn}^2 \text{cn}^2 \text{dn}^2. \end{aligned} \quad (19)$$

考虑 cn^2 = 1 - sn^2, dn^2 = 1 - m^2 sn^2, 则二级方程(19)可以写为

$$\begin{aligned} \frac{d^2 u_2}{d^2} + [4(1 + m^2) - 12 m^2 \text{sn}^2] u_2 \\ = \frac{A^2}{2 k^2 c} [\text{sn}^2 - (1 + m^2) \text{sn}^4 + m^2 \text{sn}^6]. \end{aligned} \quad (20)$$

由于方程(20)的齐次方程与方程(12)的形式相同,所以,二级方程(20)为非齐次的 Lam 方程,关键在于方程(20)中非齐次项的特解.考虑方程(20)中非齐次项的形式,我们设

$$u_2 = b_0 + b_2 \text{sn}^2 + b_4 \text{sn}^4, \quad (21)$$

将(21)式代入方程(20),定得

$$\begin{aligned} b_0 &= \frac{A^2}{48 m^2 k^2 c}, \\ b_2 &= -\frac{(1 + m^2) A^2}{24 m^2 k^2 c}, \\ b_4 &= \frac{A^2}{16 k^2 c}. \end{aligned} \quad (22)$$

因而求得 BBM 方程的二级准确解为

$$u_2 = \frac{A^2}{48 m^2 k^2 c} - \frac{(1 + m^2) A^2}{24 m^2 k^2 c} \text{sn}^2 + \frac{A^2}{16 k^2 c} \text{sn}^4, \quad (23)$$

即

$$u_2 = \frac{A^2}{48 m^2 k^2 c} [1 - 2(1 + m^2) \text{sn}^2 + 3 m^2 \text{sn}^4]. \quad (24)$$

### 3. 两类非线性方程的多级准确解

#### 3.1. 第一类非线性方程的多级准确解

类似于 BBM 方程,还有很多这类方程可以类似地得到各级准确解,下面列出几个方程及其一级准确解和二级准确解.

##### 3.1.1. 正规长波(regular long wave)方程

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{\partial^2 u^2}{\partial x \partial t} = 0. \quad (25)$$

其一级准确解为

$$u_1 = AL_3(\ ) = A \operatorname{sn} \operatorname{cn} \operatorname{dn} . \quad (26)$$

二级准确解为

$$u_2 = \frac{A^2}{24m^2 k^2 c} [1 - 2(1 + m^2) \operatorname{sn}^2 + 3m^2 \operatorname{sn}^4] . \quad (27)$$

3.1.2. KdV 方程

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 . \quad (28)$$

其一级准确解为

$$u_1 = AL_3(\ ) = A \operatorname{sn} \operatorname{cn} \operatorname{dn} . \quad (29)$$

二级准确解为

$$u_2 = - \frac{A^2}{48m^2 k^2} [1 - 2(1 + m^2) \operatorname{sn}^2 + 3m^2 \operatorname{sn}^4] . \quad (30)$$

3.1.3. Boussinesq 方程

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u^2}{\partial x^2} = 0 . \quad (31)$$

其一级准确解为

$$u_1 = AL_3(\ ) = A \operatorname{sn} \operatorname{cn} \operatorname{dn} . \quad (32)$$

二级准确解为

$$u_2 = - \frac{A^2}{24m^2 k^2} [1 - 2(1 + m^2) \operatorname{sn}^2 + 3m^2 \operatorname{sn}^4] . \quad (33)$$

3.1.4. 非线性 Klein-Gordon 方程( )

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} + u - u^2 = 0 . \quad (34)$$

其一级准确解为

$$u_1 = AL_3(\ ) = A \operatorname{sn} \operatorname{cn} \operatorname{dn} . \quad (35)$$

二级准确解为

$$u_2 = \frac{A^2}{24m^2 k^2 (c^2 - c_0^2)} \times [1 - 2(1 + m^2) \operatorname{sn}^2 + 3m^2 \operatorname{sn}^4] . \quad (36)$$

### 3.2. 第二类非线性方程的多级准确解

除了前面得到的 5 个方程都是由当  $n = 3, \mu = 4(1 + m^2), [\mu = -4(1 + m^2)]$  时满足的 Lam 型方程得到的一级准确解和二级准确解. 另外, 当  $n = 2, \mu = 1 + m^2, [\mu = -(1 + m^2)]$  时, Lam 函数为

$$L_2(x) = (1 - h^{-1})^{1/2} (1 - h^{-1} x^2)^{1/2} = \operatorname{cn} x \operatorname{dn} x . \quad (37)$$

这时的 Lam 方程的形式为

$$\frac{d^2 y}{dx^2} + [(1 + m^2) - 6m^2 \operatorname{sn}^2 x] y = 0 . \quad (38)$$

对应着也可以求得相应的非线性演化方程的多级准

准确解.

3.2.1. mBBM 方程

$$\frac{\partial u}{\partial t} + c_0 \frac{\partial u}{\partial x} + u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^2 \partial t} = 0 . \quad (39)$$

其一级准确解为

$$u_1 = AL_2(\ ) = A \operatorname{cn} \operatorname{dn} . \quad (40)$$

二级准确解为

$$u_2 = \mp \sqrt{\frac{6}{c}} \frac{(1 + m^2) A^2}{12mk} \operatorname{sn} \left[ 1 - \frac{2m^2}{1 + m^2} \operatorname{sn}^2 \right] . \quad (41)$$

3.2.2. mKdV 方程

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 . \quad (42)$$

其一级准确解为

$$u_1 = AL_2(\ ) = A \operatorname{cn} \operatorname{dn} . \quad (43)$$

二级准确解为

$$u_2 = \mp \sqrt{\frac{6}{c}} \frac{(1 + m^2) A^2}{12mk} \operatorname{sn} \left[ 1 - \frac{2m^2}{1 + m^2} \operatorname{sn}^2 \right] . \quad (44)$$

3.2.3. 非线性 Klein-Gordon 方程( )

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} + u - u^3 = 0 . \quad (45)$$

其一级准确解为

$$u_1 = AL_2(\ ) = A \operatorname{cn} \operatorname{dn} . \quad (46)$$

二级准确解为

$$u_2 = \mp \frac{1 + m^2}{2mk} \sqrt{2(c^2 - c_0^2)} A^2 \operatorname{sn} \left[ 1 - \frac{2m^2}{1 + m^2} \operatorname{sn}^2 \right] . \quad (47)$$

## 4. 多级准确解的退化

当  $m = 1$  时,  $\operatorname{sn} = \tanh, \operatorname{cn} = \operatorname{sech}, \operatorname{dn} = \operatorname{sech}$ , 所以第一类非线性方程的一级准确解退化为

$$u_1 = AL_3(\ ) = A \operatorname{tanh} \operatorname{sech}^2 = A \operatorname{tanh} (1 - \operatorname{tanh}^2) . \quad (48)$$

二级准确解为

$$u_2 = f_1(\nu_1, \nu_2, \dots, k, c) (1 - 4 \operatorname{tanh}^2 + 3 \operatorname{tanh}^4) , \quad (49)$$

式中  $f_1(\nu_1, \nu_2, \dots, k, c)$  仅是原方程各项的系数和波数及波速的函数.

同样, 第二类非线性方程的一级准确解退化为

$$u_1 = AL_2(\ ) = A \operatorname{sech}^2 = A (1 - \operatorname{tanh}^2) . \quad (50)$$

二级准确解为

$$u_2 = f_2(\alpha_1, \alpha_2, \dots, k, c) \tanh^2(1 - \tanh^2) \\ = f_2(\alpha_1, \alpha_2, \dots, k, c) \tanh \operatorname{sech}^2, \quad (51)$$

式中  $f_2(\alpha_1, \alpha_2, \dots, k, c)$  仅是原方程各项的系数和波数及波速的函数。

可以看出,这两种类型的非线性方程的一级准确解和二级准确解分别对应着不同类型的孤立波。

### 5. 总结及结论

从以上所述不难发现,所有的第一类方程的一级准确解都是一样的,同样第二类方程的一级准确解也是相同的.再就是,所有第一类方程的二级准确解的基本形式都是一样的,只是系数不同.同样,对于第二类方程的二级准确解,情况也是一样的.这就是说,在不同的非线性演化方程的多级准确解中存在着不变的量或形式。

对于第一类非线性演化方程,在行波解的框架下,方程最后可以改写成如下各级近似方程:

零级近似方程为

$$e_1 \frac{d^2 u_0}{d^2} + e_2 u_0 + e_3 u_0^2 = 0. \quad (52)$$

其基本解为

$$u_0 = -\frac{e_2}{2e_3} + \frac{2(1+m^2)e_1}{e_3} - \frac{6e_1}{e_3} m^2 \operatorname{sn}^2. \quad (53)$$

相应的一级近似方程为

$$e_1 \frac{d^2 u_1}{d^2} + (e_2 + 2e_3 u_0) u_1 = 0, \quad (54)$$

即

$$\frac{d^2 u_1}{d^2} + [4(1+m^2) - 12m^2 \operatorname{sn}^2] u_1 = 0. \quad (55)$$

其基本解为

$$u_1 = AL_3(\ ) = A \operatorname{sn} \operatorname{cn} \operatorname{dn} \frac{d u_0}{d}. \quad (56)$$

相应的二级近似方程为

$$e_1 \frac{d^2 u_2}{d^2} + (e_2 + 2e_3 u_0) u_2 = -e_3 u_1^2, \quad (57)$$

即

$$\frac{d^2 u_2}{d^2} + [4(1+m^2) - 12m^2 \operatorname{sn}^2] u_2 \\ = -\frac{e_3}{e_1} A^2 \operatorname{sn}^2 \operatorname{cn}^2 \operatorname{dn}^2. \quad (58)$$

其基本解为

$$u_2 = -\frac{e_3 A^2}{24 m^2 e_1} [1 - 2(1+m^2)$$

$$\times \operatorname{sn}^2 + 3 m^2 \operatorname{sn}^4]. \quad (59)$$

对于第二类非线性演化方程,同样可以得到:零级近似方程为

$$e_1 \frac{d^2 u_0}{d^2} + e_2 u_0 + e_3 u_0^3 = 0. \quad (60)$$

其基本解为

$$u_0 = \pm \sqrt{-\frac{2m^2 e_1}{e_3}} \operatorname{sn}, \\ e_2 = (1+m^2) e_1. \quad (61)$$

相应的一级近似方程为

$$e_1 \frac{d^2 u_1}{d^2} + (e_2 + 3e_3 u_0^2) u_1 = 0, \quad (62)$$

即

$$\frac{d^2 u_1}{d^2} + [(1+m^2) - 6m^2 \operatorname{sn}^2] u_1 = 0. \quad (63)$$

其基本解为

$$u_1 = AL_2(\ ) = A \operatorname{cn} \operatorname{dn} \frac{d u_0}{d}. \quad (64)$$

相应的二级近似方程为

$$e_1 \frac{d^2 u_2}{d^2} + (e_2 + 3e_3 u_0^2) u_2 = -3e_3 u_0 u_1^2, \quad (65)$$

即

$$\frac{d^2 u_2}{d^2} + [(1+m^2) - 6m^2 \operatorname{sn}^2] u_2 \\ = \pm 3 \sqrt{-\frac{2m^2 e_3}{e_1}} A^2 \operatorname{sn} \operatorname{cn}^2 \operatorname{dn}^2. \quad (66)$$

其基本解为

$$u_2 = \pm \frac{e_3 A^2 (1+m^2)}{4 m^2 e_1} \sqrt{-\frac{2m^2 e_1}{e_3}} \\ \times \operatorname{sn} \left[ 1 - \frac{2m^2}{1+m^2} \operatorname{sn}^2 \right]. \quad (67)$$

对两类不同的非线性演化方程的多级准确解,不同点仅在于得到的最后系数  $e_1, e_2$  和  $e_3$  的组合不同,这导致了方程解的结构不同,而且对不同级的解,其差异是不一样的.对两类非线性方程,其一级解对每类非线性方程都是相同的.二级解的基本结构是相同的,不同点在于振幅和相位(即解的系数)是不同的.对于第一类非线性演化方程,其零级解的结构相对而言更为丰富(解的形式和 3 个系数都有关系).而第二类非线性演化方程的零级解的基本结构也是相同的,不同点在于振幅和相位(即解的系数).这里所有存在的共性构成了非线性演化方程多级准确解的不变量或不变形式。

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# Lamè function and invariants of multi-order exact solutions among nonlinear evolution equations<sup>\*</sup>

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## Abstract

Applying the perturbation method, the nonlinear evolution equations are expanded as multirorder approximate equations. And based on Lam equation and Lam function, these multi-order approximate equations can be solved by Jacobi elliptic function expansion method, where multi-order exact solutions of nonlinear evolution equations are derived. Then the invariants of the multi-order exact solutions are found among different nonlinear evolution equations.

**Keywords:** Lam function, Jacobi elliptic function, multi-order exact solution, nonlinear evolution equation, perturbation method, invariants

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