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New Solutions to Generalized mKdV Equation*

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Abstract *Trial function method is applied to solve generalized mKdV (GmKdV for short) equations. It is shown that GmKdV equations with a real number parameter can be solved directly by this method without a transformation, and more new kinds of solitary wave solutions are obtained.*

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1 Introduction

In Ref. [1], a transformation is introduced to transform the generalized mKdV (GmKdV for short) equations to be solvable for Jacobi elliptic function expansion method (Actually this kind of transformation is needed for all expansion methods to solve GmKdV equation). And there for specific values of γ, periodic solutions and solitary wave solutions are derived. In fact, as more problems in branches of physics, mathematics, and other interdisciplinary sciences are described in terms of suitable nonlinear models, such as nonlinear Schrödinger equations in plasma physics,[2] KdV equation in shallow water model,[3] and so on. Recently, special attention has been devoted to solving nonlinear evolution equations, and many methods have been proposed to construct exact solutions to nonlinear equations. Among them are the function transformation method,[4,5] the homogeneous balance method,[6,7] the hyperbolic function expansion method,[8,9] the Jacobi elliptic function expansion method,[10,11] the nonlinear transformation method,[12,13] the trial function method,[14,15] and others.[16−19] But not all these methods are suitable for directly solving some special kinds of nonlinear evolution equations, such as GmKdV equation. For expansion methods, such as the function transformation method,[4,5] the homogeneous balance method,[6,7] the hyperbolic function expansion method,[8,9] and the Jacobi elliptic function expansion method,[10,11] the expansion order must be a positive integer. However, for more nonlinear evolution equations, the expansion order (obtained from the partial balance between the highest degree nonlinear terms and the highest order derivative terms) is not a positive integer, it may be a negative integer, or it may be just a real number. When the expansion order is not a positive integer, the expansion method cannot be applied to solving the corresponding nonlinear equation directly. Then some kinds of transformations or some other methods are needed.

In this paper, we will consider this case. Trial function method is applied to solving GmKdV equation directly without a transformation, and more new kinds of solitary wave solutions are obtained.

The GmKdV equation considered here is introduced by Fedele,[20] which reads

$$\frac{\partial u}{\partial t} + \alpha u^\gamma \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0\,, \tag{1}$$

where u is a real function, and α, β, and γ are real numbers. We seek its travelling wave solution, i.e.

$$u = u(\xi) \qquad \xi = x - ct\,, \tag{2}$$

where c is the wave speed. Substitution Eq. (2) into Eq. (1) yields

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi} + \alpha u^\gamma \frac{\mathrm{d}u}{\mathrm{d}\xi} + \beta \frac{\mathrm{d}^3 u}{\mathrm{d}\xi^3} = 0\,. \tag{3}$$

Integrating Eq. (3) with respect to ξ once yields

$$-cu + \frac{\alpha}{\gamma+1} u^{\gamma+1} + \beta \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} = A\,, \tag{4}$$

where A is an integration constant.

In order to simplify computation, without loss of generality, we take $A = 0$, then equation (4) can be rewritten as

$$-cu + \frac{\alpha}{\gamma+1} u^{\gamma+1} + \beta \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} = 0\,. \tag{5}$$

In the next sections, we will apply trial function method to solve Eq. (5), and then obtain solutions to GmKdV equation (1).

2 Solutions to GmKdV Equation

According to the trial function method,[14,21] the ansatz solution to Eq. (5) can be taken as

$$u = \frac{B\,\mathrm{e}^{b\xi}}{(1 + d\,\mathrm{e}^{a\xi})^p}\,, \qquad 0 \le b \le ap\,, \qquad ad \ne 0\,, \tag{6}$$

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where B, a, b, d, and p are undetermined constants. Differently from the assumption given in Refs. [14] and [21], here constant p can be any real number, and d is not always set as 1, but can take other values.

Because of $0 \le b \le ap$, we can take the order of u as

$$O(u) = p\,, \tag{7}$$

then, it is easily derived that

$$O\Big(\frac{\mathrm{d}u}{\mathrm{d}\xi}\Big) = p+1\,,$$
$$O\Big(\frac{\mathrm{d}^n u}{\mathrm{d}\xi^n}\Big) = p+n \quad (n=1,2,3,\ldots)\,. \tag{8}$$

Partial balance between the highest degree nonlinear terms and the highest order derivative terms arrives at

$$O\Big(\frac{\mathrm{d}^3 u}{\mathrm{d}\xi^3}\Big) = O\Big(u^\gamma \frac{\mathrm{d}u}{\mathrm{d}\xi}\Big), \tag{9}$$

i.e.,

$$p+3 = \gamma p + p + 1\,, \tag{10}$$

from which p can be determined as

$$p = \frac{2}{\gamma}\,. \tag{11}$$

So the ansatz solution (6) can be rewritten as

$$u = \frac{B\,\mathrm{e}^{b\xi}}{(1+d\,\mathrm{e}^{a\xi})^{2/\gamma}}\,, \qquad 0 \le b \le \frac{2a}{\gamma}\,, \qquad ad \ne 0\,. \tag{12}$$

From Eq. (12), one has

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{B\,\mathrm{e}^{b\xi}}{(1+d\,\mathrm{e}^{a\xi})^{2/\gamma}}\Big(b - \frac{2ad}{\gamma}\frac{\mathrm{e}^{a\xi}}{1+d\,\mathrm{e}^{a\xi}}\Big)\,, \tag{13}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} = \frac{B\,\mathrm{e}^{b\xi}}{(1+d\,\mathrm{e}^{a\xi})^{2/\gamma}}\Big[b^2 - \frac{4abd}{\gamma}\frac{\mathrm{e}^{a\xi}}{1+d\,\mathrm{e}^{a\xi}} - \frac{2a^2d}{\gamma}\frac{\mathrm{e}^{a\xi}}{(1+d\,\mathrm{e}^{a\xi})^2} + \frac{4a^2d^2}{\gamma^2}\frac{\mathrm{e}^{2a\xi}}{(1+d\,\mathrm{e}^{a\xi})^2}\Big]\,, \tag{14}$$

and

$$u^{\gamma+1} = \frac{B\,\mathrm{e}^{b\xi}}{(1+d\,\mathrm{e}^{a\xi})^{2/\gamma}}\frac{B^\gamma\,\mathrm{e}^{b\gamma\xi}}{(1+d\,\mathrm{e}^{a\xi})^2}\,. \tag{15}$$

Substituting Eqs. (12), (14), and (15) into Eq. (5) yields

$$-c + \frac{\alpha}{\gamma+1}\frac{B^\gamma\,\mathrm{e}^{b\gamma\xi}}{(1+d\,\mathrm{e}^{a\xi})^2} + \beta\Big[b^2 - \frac{4abd}{\gamma}\frac{\mathrm{e}^{a\xi}}{1+d\,\mathrm{e}^{a\xi}} - \frac{2a^2d}{\gamma}\frac{\mathrm{e}^{a\xi}}{(1+d\,\mathrm{e}^{a\xi})^2} + \frac{4a^2d^2}{\gamma^2}\frac{\mathrm{e}^{2a\xi}}{(1+d\,\mathrm{e}^{a\xi})^2}\Big] = 0\,, \tag{16}$$

i.e.,

$$(\beta b^2 - c) + \frac{\alpha}{\gamma+1}B^\gamma\,\mathrm{e}^{b\gamma\xi} + \Big[\beta\Big(2b^2 - \frac{4ab}{\gamma} - \frac{2a^2}{\gamma}\Big) - 2c\Big]d\,\mathrm{e}^{a\xi} + \Big[\beta\Big(b^2 - \frac{4ab}{\gamma} + \frac{4a^2}{\gamma^2}\Big) - c\Big]d^2\,\mathrm{e}^{2a\xi} = 0\,. \tag{17}$$

For Eq. (17), there are three cases needed to be considered: (a) $b\gamma = 0$, (b) $b\gamma = a$, and (c) $b\gamma = 2a$, we will give detailed discussions below.

Case (a) $b\gamma = 0$

From $b\gamma = 0$, one has $b = 0$, then equation (17) can be rewritten as

$$\Big(\frac{\alpha}{\gamma+1}B^\gamma - c\Big) + \Big(-\frac{2\beta a^2}{\gamma} - 2c\Big)d\,\mathrm{e}^{a\xi} + \Big(\frac{4\beta a^2}{\gamma^2} - c\Big)d^2\,\mathrm{e}^{2a\xi} = 0\,. \tag{18}$$

Due to the arbitrariness of ξ, equation (18) results in the following algebraic equations,

$$\frac{\alpha}{\gamma+1}B^\gamma - c = 0\,,$$
$$\Big(-\frac{2\beta a^2}{\gamma} - 2c\Big)d = 0\,,$$
$$\Big(\frac{4\beta a^2}{\gamma^2} - c\Big)d^2 = 0\,. \tag{19}$$

From Eqs. (19), one can determine the undetermined constants as

$$B = \Big(-\frac{\alpha}{3c}\Big)^{1/4}\,, \qquad \gamma = -4\,, \qquad a = \pm\sqrt{\frac{4c}{\beta}}\,, \tag{20}$$

and $d \ne 0$ is an arbitrary constant. So the solution to Eq. (1) is

$$u_1 = \Big(-\frac{\alpha}{3c}\Big)^{1/4}\Big(1 + d\,\mathrm{e}^{\pm\sqrt{4c/\beta}\xi}\Big)^{1/2}\,. \tag{21}$$

This is a new generalized solution to Eq. (1) that we did not given in Ref. [1]. When $d = 1$, solution (21) reduces to

$$u_2 = \Big(-\frac{\alpha}{3c}\Big)^{1/4}\Big(\frac{2}{1 \pm \tanh\sqrt{c/\beta}\xi}\Big)^{1/2}\,, \tag{22}$$

and when $d = -1$, solution (21) reduces to

$$u_3 = \Big(-\frac{\alpha}{3c}\Big)^{1/4}\Big(\frac{2}{\coth\sqrt{c/\beta}\xi \pm 1}\Big)^{1/2}\,. \tag{23}$$

To our knowledge, solutions (22) and (23) have not been obtained before.

Case (b) $b\gamma = a$

If $b\gamma = a$, then $b = a/\gamma$. Substituting $b = a/\gamma$ into Eq. (17) leads to

$$\Big(\beta\frac{a^2}{\gamma^2} - c\Big) + \Big[\frac{\alpha}{\gamma+1}B^\gamma - \beta\Big(\frac{2a^2}{\gamma^2} + \frac{2a^2}{\gamma}\Big)d - 2cd\Big]\,\mathrm{e}^{a\xi} + \Big(\beta\frac{a^2}{\gamma^2} - c\Big)d^2\,\mathrm{e}^{2a\xi} = 0\,. \tag{24}$$

The arbitrariness of ξ makes Eq. (24) become the following algebraic equations,

$$\beta\frac{a^2}{\gamma^2} - c = 0\,, \qquad \frac{\alpha}{\gamma+1}B^\gamma - \beta\Big(\frac{2a^2}{\gamma^2} + \frac{2a^2}{\gamma}\Big)d - 2cd = 0\,, \qquad \Big(\beta\frac{a^2}{\gamma^2} - c\Big)d^2 = 0\,. \tag{25}$$

From Eqs. (25), the undetermined constants can be determined as

$$B = \Big[\frac{2dc(\gamma+1)(\gamma+2)}{\alpha}\Big]^{1/\gamma}, \qquad a = \pm\sqrt{\frac{c\gamma^2}{\beta}}\,, \tag{26}$$

where $d \neq 0$ is an arbitrary constant and $\gamma \neq 0$ is any real number. So the solution to Eq. (10 is

$$u_4 = \Big[\frac{2dc(\gamma+1)(\gamma+2)}{\alpha}\Big]^{1/\gamma}\frac{\mathrm{e}^{\pm\sqrt{c\gamma^2/\beta}\;\xi/\gamma}}{[1 + d\,\mathrm{e}^{\pm\sqrt{c\gamma^2/\beta}\;\xi}]^{2/\gamma}}\,. \tag{27}$$

This is another new generalized solution to Eq. (1) we did not give in Ref. [1]. When $d = 1$, solution (27) reduces to

$$u_5 = \Big[\frac{c(\gamma+1)(\gamma+2)}{2\alpha}\Big]^{1/\gamma}\Big(\mathrm{sech}\sqrt{\frac{c\gamma^2}{4\beta}}\xi\Big)^{2/\gamma}\,. \tag{28}$$

This kind of solution has been given in Ref. [1]. When $d = -1$, solution (27) reduces to

$$u_6 = \Big[-\frac{c(\gamma+1)(\gamma+2)}{2\alpha}\Big]^{1/\gamma}\Big(\mathrm{csch}\sqrt{\frac{c\gamma^2}{4\beta}}\xi\Big)^{2/\gamma}\,. \tag{29}$$

This is a new solution to Eq. (1) that we did not give in Ref. [1].

Case (c) $b\gamma = 2a$

If $b\gamma = 2a$, then $b = 2a/\gamma$. Similarly, from Eq. (17) one has

$$\Big(\beta\frac{4a^2}{\gamma^2} - c\Big) + \Big(-\frac{2\beta a^2}{\gamma} - 2c\Big)d\,\mathrm{e}^{a\xi} + \Big(\frac{\alpha}{\gamma+1}B^\gamma - cd^2\Big)\,\mathrm{e}^{2a\xi} = 0\,. \tag{30}$$

The arbitrariness of ξ makes Eq. (30) become the following algebraic equations,

$$\begin{aligned} &\beta\frac{4a^2}{\gamma^2} - c = 0\,,\\ &\frac{\beta a^2}{\gamma} + c = 0\,,\\ &\frac{\alpha}{\gamma+1}B^\gamma - cd^2 = 0\,. \end{aligned} \tag{31}$$

From Eqs. (31), the undetermined constants can be determined as

$$\gamma = -4\,, \quad B = \Big(-\frac{\alpha}{3d^2c}\Big)^{1/4}, \quad a = \pm\sqrt{\frac{4c}{\beta}}\,, \tag{32}$$

where $d \neq 0$ is an arbitrary constant. So the solution to Eq. (1) is

$$u_7 = \Big(-\frac{\alpha}{3d^2c}\Big)^{1/4}\Big(d + \mathrm{e}^{\pm\sqrt{4c/\beta}\xi}\Big)^{1/2}\,. \tag{33}$$

This is another new generalized solution to Eq. (1) that we did not give in Ref. [1]. When $d = 1$, solution (33) reduces to

$$u_8 = \Big(-\frac{\alpha}{3c}\Big)^{1/4}\Big(\frac{2}{1 \pm \tanh\sqrt{c/\beta}\xi}\Big)^{1/2}\,, \tag{34}$$

which is the same one as solution (22). When $d = -1$, solution (33) reduces to

$$u_9 = \Big(-\frac{\alpha}{3c}\Big)^{1/4}\Big(\frac{2}{\pm 1 - \coth\sqrt{c/\beta}\xi}\Big)^{1/2}\,, \tag{35}$$

which is a new solution to Eq. (1) that we did not give in Ref. [1], either.

3 Conclusion

In this paper, we apply trial function method to solve GmKdV equation directly without a transformation. Many solutions are obtained for this generalized mKdV equation, including solitary wave solutions constructed in terms of hyperbolic functions, which are some special cases of the generalized solutions, and some solutions are not given in literatures to our knowledge. Of course, the similar proceedings can be applied to other nonlinear wave equations that cannot be solved directly by expansion methods.

References

[1] Z.T. Fu, L.T. Deng, S.K. Liu, and S.D. Liu, Commun. Theor. Phys. (Beijing, China) **40** (2003) 641.

[2] P.K. Shukla, N.N. Rao, M.Y. Yu, *et al.*, Phys. Rep. **138** (1986) 1.

[3] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Tranforms*, Philadelphia, SIAM (1981).

[4] C.L. Bai, Phys. Lett. **A288** (2001) 191.

[5] Z.T. Fu, S.K. Liu, and S.D. Liu, Phys. Lett. **A299** (2002) 507.

[6] M.L. Wang, Phys. Lett. **A199** (1995) 169.

[7] M.L. Wang, Y.B. Zhou, and Z.B. Li, Phys. Lett. **A216** (1996) 67.

[8] E.G. Fan, Phys. Lett. **A277** (2000) 212.

[9] E.J. Parkes and B.R. Duffy, Phys. Lett. **A229** (1997) 217.

[10] Z.T. Fu, S.K. Liu, S.D. Liu, and Q. Zhao, Phys. Lett. **A290** (2001) 72.

[11] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Phys. Lett. **A289** (2001) 69.

[12] R. Hirota, J. Math. Phys. **14** (1973) 810.

[13] N.A. Kudryashov, Phys. Lett. **A147** (1990) 287.

[14] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Appl. Math. Mech. **22** (2001) 326.

[15] M. Otwinowski, R. Paul, and W.G. Laidlaw, Phys. Lett. **A128** (1988) 483.

[16] C.T. Yan, Phys. Lett. **A224** (1996) 77.

[17] A.V. Porubov, Phys. Lett. **A221** (1996) 391.

[18] A.V. Porubov and M.G. Velarde, J. Math. Phys. **40** (1999) 884.

[19] A.V. Porubov and D.F. Parker, Wave Motion **29** (1999) 97.

[20] R. Fedele, Physica Scripta **65** (2002) 502.

[21] S.K. Liu and S.D. Liu, *Nonlinear Equations in Physics*, Peking University Press, Beijing (2000).