Applied Mathematics and Mechanics
(English Edition, Vol 25, No 1, Jan 2004)

Published by Shanghai University,
Shanghai, China

c Editorial Committee of Appl. Math. Mech., ISSN 0253-4827 Article ID: 0253-4827(2003)01-0073-07

# NEW EXACT SOLUTIONS TO KdV EQUATIONS WITH VARIABLE COEFFICIENTS OR FORCING*

FU Zun-tao (付遵涛)[1,2], LIU Shi-da (刘式达)[1,2],
LIU Shi-kuo (刘式适)[1], ZHAO Qiang (赵 强)[1]

(1. School of Physics, Peking University, Beijing 100871, P. R. China;
2. State Key Laboratory for Turbulence and Complex System,
Peking University, Beijing 100871, P. R. China)

(Communicated by HUANG Yong-nian)

Abstract: Jacobi elliptic function expansion method is extended to construct the exact solutions to another kind of KdV equations, which have variable coefficients or forcing terms. And new periodic solutions obtained by this method can be reduced to the soliton-typed solutions under the limited condition.

Key words: Jacobi elliptic function; soliton-typed solution; cnoidal wave-typed solution

Chinese Library Classification: O175; O411 Document code: A

**2000** Mathematics Subject Classification: 35G25

## Introduction

The variable-coefficient KdV equation

$$u_t + \alpha(t)\, u u_x + \beta(t)\, u_{xxx} = 0 \tag{1}$$

was originally proposed in Ref. [1], where $\alpha(t)$ and $\beta(t)$ are arbitrary analytic functions. And it is also rewritten as the general variable-coefficient KdV equation[2]

$$u_t + 2\beta(t)\, u + [\alpha(t) + \beta(t)\, x]\, u_x - 3c\gamma(t)\, u u_x + \gamma(t)\, u_{xxx} = 0 \tag{2}$$

which can be reduced to other more physical forms, for example, the cylinder KdV equation[3] reads

$$u_t + \frac{1}{2t} u + 6 u u_x + u_{xxx} = 0 \tag{3}$$

which has been widely applied in plasma physics and other specific physics.

Many methods have been proposed to solve constant-coefficient nonlinear equations and much more exact solitary wave solutions or periodic solutions were obtained[4~19]. But we know that the constant coefficients are just highly idealized assumptions, which care only some degree

* Received date: 2002-08-28; Revised date: 2003-07-31

Foundation items: the National Natural Science Foundation of China (40175016); the State Key Project of National Natural Science Foundation of China (40035010)

Biography: FU Zun-tao (1970~), Associate Professor, Doctor (Tel: 86-010-62767184; Fax: 86-010-62751615; E-mail: fuzt@mech.pku.edu.cn)

about realistic physical importance. So more attention[20~26] has been paid to studying the integrability and symmetry of variable-coefficient nonlinear equations, since numerous application in physical sciences and engineering deal with variable-coefficient nonlinear equations. Actually, variable-coefficient nonlinear equations are seldom considered for their complexity. In this paper, we will extend the Jacobi elliptic function expansion method[19] and apply it to get the periodic solutions and corresponding shock or solitary wave solutions to variable-coefficient or forced KdV equations.

## 1 Extended Jacobi Elliptic Function Expansion

Considering the general variable-coefficient nonlinear equation

$$N(u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0. \tag{4}$$

We seek its general travelling wave solution

$$u = u(\xi), \quad \xi = f(t)x + g(t), \tag{5}$$

where $f(t)$ and $g(t)$ are undetermined functions of $t$. Assuming that $u(\xi)$ has the following ansatz solution:

$$u(\xi) = \sum_{j=0}^{n} a_j(t)\,\mathrm{sn}^j\xi, \tag{6}$$

we can select $n$ to balance the derivative term of the highest order and nonlinear term in (4), then we have the final determined expansion form.

When $m \to 1$, $\mathrm{sn}\xi \to \tanh\xi$, so (6) degenerates to

$$u(\xi) = \sum_{j=0}^{n} a_j(t)\tanh^j\xi. \tag{7}$$

Notice that

$$\mathrm{cn}^2\xi = 1 - \mathrm{sn}^2\xi \tag{8}$$

and when $m \to 1$, $\mathrm{cn}\xi \to \mathrm{sech}\xi$, so we get cnoidal wave solution and its corresponding solitary wave solution.

In the following sections, we will apply (5) and (6) to solve another kind of KdV equations.

## 2 Solutions to Another Kind of KdV Equations

### 2.1 Solutions to a kind of KdV equation

Here, the considered KdV equation takes the following form:

$$v_t + avv_y + bv_{yyy} + \frac{\delta}{t}v = 0, \tag{9}$$

where $a$ and $b$ are the constants. It is obvious that this is a generalized kind of variant KdV equation. When $\delta = 0$, Eq. (9) is just the constant coefficient KdV equation, i.e.,

$$v_t + avv_y + bv_{yyy} = 0. \tag{10}$$

While when $\delta = 1$, Eq. (9) is just spherical KdV equation, i.e.,

$$v_t + avv_y + bv_{yyy} + \frac{1}{t}v = 0. \tag{11}$$

While when $\delta = 1/2$, Eq. (9) is just cylindrical KdV equation, i.e.,

$$v_t + avv_y + bv_{yyy} + \frac{1}{2t}v = 0, \tag{12}$$

which can be re-scaled to the form of Eq. (3) .

In order to solve Eq. (9) , first the transformation

$$u = t^{\delta} v \tag{13}$$

is taken , then Eq. (9) can be rewritten as

$$u_t + a t^{-\delta} u u_y + b u_{yyy} = 0 \tag{14}$$

and then the independent variable takes the following transformation:

$$x = t^{-\delta/2} y. \tag{15}$$

Equation (14) is re-scaled as

$$u_t + a t^{-3\delta/2} u u_x + b t^{-3\delta/2} u_{xxx} = 0. \tag{16}$$

Setting the coefficients as

$$a t^{-3\delta/2} = \alpha(t) , \quad b t^{-3\delta/2} = \beta(t) , \tag{17}$$

then Eq. (16) takes the same form as Eq. (1) , so one can solve Eq. (1) in order to solve Eq. (9) .

Substituting (5) and (6) into (1) and balancing the derivative term of the highest order and nonlinear term to determine $n$ yield the ansatz solution

$$u = a_0(t) + a_1(t)\,\mathrm{sn}\,\xi + a_2(t)\,\mathrm{sn}^2\xi. \tag{18}$$

Notice that

$$u_t = a_0' + a_1'\mathrm{sn}\,\xi + a_2'\mathrm{sn}^2\xi + (a_1 + 2a_2\mathrm{sn}\,\xi)(f'x + g')\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi, \tag{19}$$

$$u_x = f(a_1 + 2a_2\mathrm{sn}\,\xi)\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi, \tag{20}$$

$$u u_x = f[a_0 a_1 + (a_1^2 + 2a_0 a_2)\mathrm{sn}\,\xi + 3a_1 a_2 \mathrm{sn}^2\xi + 2a_2^2\mathrm{sn}^3\xi]\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi, \tag{21}$$

$$\begin{aligned} u_{xx} = {} & f^2[2a_2 - (1+m^2)a_1\mathrm{sn}\,\xi - 4(1+m^2)a_2\mathrm{sn}^2\xi + \\ & 2m^2 a_1\mathrm{sn}^3\xi + 6m^2 a_2\mathrm{sn}^4\xi] , \end{aligned} \tag{22}$$

$$\begin{aligned} u_{xxx} = {} & f^3[-(1+m^2)a_1 - 8(1+m^2)a_2\mathrm{sn}\,\xi + \\ & 6m^2 a_1\mathrm{sn}^2\xi + 24m^2 a_2\mathrm{sn}^3\xi]\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi, \end{aligned} \tag{23}$$

where $m(0 < m < 1)$ is modulus.

Substituting (19) , (21) and (23) into (1) yields

$$\begin{aligned} & a_0' + a_1'\mathrm{sn}\,\xi + a_2'\mathrm{sn}^2\xi + a_1[f'x + g' + \alpha f a_0 - \\ & (1+m^2)\beta f^3 a_2]\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi + [2a_2(f'x + g') + \alpha f(a_1^2 + 2a_0 a_2) - \\ & 8(1+m^2)\beta f^3 a_2]\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi + 3a_1 f[\alpha a_2 + 2m^2\beta f^2]\mathrm{sn}^2\xi\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi + \\ & 2a_2 f[\alpha a_2 + 12m^2\beta f^2]\mathrm{sn}^3\xi\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi = 0. \end{aligned} \tag{24}$$

Thus we have

$$a_0'(t) = a_1'(t) = a_2'(t) = 0 , \tag{25}$$

$$a_1[f'x + g' + \alpha f a_0 - (1+m^2)\beta f^3 a_2] = 0 , \tag{26}$$

$$2a_2(f'x + g') + \alpha f(a_1^2 + 2a_0 a_2) - 8(1+m^2)\beta f^3 a_2 = 0 , \tag{27}$$

$$a_1 f[\alpha a_2 + 2m^2\beta f^2] = 0 , \tag{28}$$

$$a_2 f[\alpha a_2 + 12m^2\beta f^2] = 0. \tag{29}$$

From which we can determine the constraint between variable coefficients

$$\frac{\beta(t)}{\alpha(t)} = \gamma \qquad (\gamma = \text{const.} \neq 0) \tag{30}$$

and

$$f(t) = k, \; g(t) = -kc\int^t \alpha(\tau)\,\mathrm{d}\tau \qquad (k = \text{const.}, \; c = \text{const.}), \tag{31}$$

$$a_0 = c + 4(1+m^2)\gamma k^2, \; a_1 = 0, \; a_2 = -12m^2\gamma k^2. \tag{32}$$

It is obvious that the constraint (30) requires that the variable coefficients are linearly dependent, just the same as the assumption given in Ref. [26]. From (17), one can see that this constraint is satisfied and $\gamma = b/a$.

So the exact solution is

$$u = c + 4(1+m^2)\gamma k^2 - 12m^2\gamma k^2 \mathrm{sn}^2\xi = (c + 4(1-2m^2)\gamma k^2 + 12m^2\gamma k^2\mathrm{cn}^2\xi \tag{33}$$

which is the cnoidal wave-like solution to (1), where $\xi = k\left[x - c\int^t \alpha(\tau)\,\mathrm{d}\tau\right]$.

When $m \to 1$, (33) reduces to

$$u = c + 8\gamma k^2 - 12\gamma k^2\tanh^2\xi = c - 4\gamma k^2 + 12\gamma k^2\mathrm{sech}^2\xi \tag{34}$$

which is siliton-type solution to (1).

So the cnoidal wave-like solution to (9) is

$$\begin{aligned} v = {} & t^{-\delta}[c + 4(1+m^2)\gamma k^2 - 12m^2\gamma k^2\mathrm{sn}^2\xi] = \\ & t^{-\delta}[c + 4(1-2m^2)\gamma k^2 + 12m^2\gamma k^2\mathrm{cn}^2\xi], \end{aligned} \tag{35}$$

its corresponding siliton-type solution is

$$v = t^{-\delta}[c + 8\gamma k^2 - 12\gamma k^2\tanh^2\xi] = t^{-\delta}[c - 4\gamma k^2 + 12\gamma k^2\mathrm{sech}^2\xi], \tag{36}$$

where

$$\xi = kt^{-\delta/2}\left(y - \frac{2ac}{2-3\delta}t^{1-\delta}\right). \tag{37}$$

We consider three special cases:

Case A: $\delta = 0$, the constant coefficient KdV equation, the cnoidal wave solution is

$$\begin{aligned} v = {} & c + 4(1+m^2)\gamma k^2 - 12m^2\gamma k^2\mathrm{sn}^2\xi = \\ & c + 4(1-2m^2)\gamma k^2 + 12m^2\gamma k^2\mathrm{cn}^2\xi, \end{aligned} \tag{38}$$

its corresponding siliton solution is

$$v = c + 8\gamma k^2 - 12\gamma k^2\tanh^2\xi = c - 4\gamma k^2 + 12\gamma k^2\mathrm{sech}^2\xi, \tag{39}$$

where

$$\xi = k(y - act). \tag{40}$$

Case B: $\delta = 1$, the spherical KdV equation, the cnoidal wave-like solution is

$$\begin{aligned} v = {} & t^{-1}[c + 4(1+m^2)\gamma k^2 - 12m^2\gamma k^2\mathrm{sn}^2\xi] = \\ & t^{-1}[c + 4(1-2m^2)\gamma k^2 + 12m^2\gamma k^2\mathrm{cn}^2\xi], \end{aligned} \tag{41}$$

its corresponding siliton-typed solution is

$$v = t^{-1}[c + 8\gamma k^2 - 12\gamma k^2\tanh^2\xi] = t^{-1}[c - 4\gamma k^2 + 12\gamma k^2\mathrm{sech}^2\xi], \tag{42}$$

where

$$\xi = kt^{-1/2}(y + 2ac). \tag{43}$$

Case C: $\delta = 1/2$, the cylindrical KdV equation, the cnoidal wave-like solution is

$$\begin{aligned} v = {} & t^{-1/2}[c + 4(1+m^2)\gamma k^2 - 12m^2\gamma k^2\mathrm{sn}^2\xi] = \\ & t^{-1/2}[c + 4(1-2m^2)\gamma k^2 + 12m^2\gamma k^2\mathrm{cn}^2\xi], \end{aligned} \tag{44}$$

its corresponding siliton-typed solution is

$$v = t^{-1/2}[c + 8\gamma k^2 - 12\gamma k^2 \tanh^2 \xi] = t^{-1/2}[c - 4\gamma k^2 + 12\gamma k^2 \mathrm{sech}^2 \xi], \tag{45}$$

where

$$\xi = kt^{-1/4}(y - 4act^{1/2}). \tag{46}$$

**2.2** Solutions to the forced KdV equation

The forced KdV equation reads

$$v_t + \alpha v v_x + \beta v_{xxx} = F(t), \tag{47}$$

where $F(t)$ is forcing varying with time $t$; $\alpha$ and $\beta$ are the constants.

First, we make a transformation about $v$, i,e.,

$$v = u + \Gamma(t), \quad \Gamma(t) = \int^t F(\tau)\,d\tau, \tag{48}$$

then we have

$$u_t + \alpha[\Gamma(t) + u]u_x + \beta u_{xxx} = 0 \tag{49}$$

It can be easily obtained that the equation (49) has the ansatz solution (18). Sub-stituting (18) into (49) yields

$$\begin{aligned} & a_0' + a_1'\mathrm{sn}\xi + a_2'\mathrm{sn}^2\xi + a_2[f'x + g' + \alpha f a_0 + \alpha f\Gamma a_1 - \\ & (1 + m^2)\beta f^3]\mathrm{cn}\xi\mathrm{dn}\xi + [2a_2(f'x + g') + \alpha f(a_1^2 + 2a_0a_2) + \\ & 2\alpha\Gamma f a_2 - 8(1 + m^2)\beta f^3 a_2]\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi + 3a_1 f[\alpha a_2 + \\ & 2m^2\beta f^2]\mathrm{sn}^2\xi\mathrm{cn}\xi\mathrm{dn}\xi + 2a_2 f(\alpha a_2 + 12m^2\beta f^2)\mathrm{sn}^3\xi\mathrm{cn}\xi\mathrm{dn}\xi = 0 \end{aligned} \tag{50}$$

from which the undetermined parameters and functions can be determined

$$f = k, \quad g = -kct - k\alpha\int^t \Gamma(\tau)\,d\tau \tag{51}$$

and

$$a_0 = \frac{c}{\alpha} + 4(1 + m^2)k^2\frac{\beta}{\alpha}, \quad a_1 = 0, \quad a_2 = -12m^2k^2\frac{\beta}{\alpha}, \tag{52}$$

where $k$ and $c$ are the constants.

So the solution to the forced KdV equation can be written as

$$v = \frac{c}{\alpha} - 4(2m^2 - 1)k^2\frac{\beta}{\alpha} + \int^t F(\tau)\,d\tau + 12m^2k^2\frac{\beta}{\alpha}\mathrm{cn}^2\xi \tag{53}$$

and its corresponding soliton-like solution is

$$v = \frac{c}{\alpha} - 4k^2\frac{\beta}{\alpha} + \int^t F(\tau)\,d\tau + 12k^2\frac{\beta}{\alpha}\mathrm{sech}^2\xi, \tag{54}$$

where

$$\xi = k\left[x - ct - \alpha\int^t\int^\tau F(\psi)\,d\psi\,d\tau\right].$$

## 3 Conclusion

In this paper, the exact periodic-like solutions to some variable-coefficient or forced KdV equations are obtained by use of Jacobi elliptic sine function expansion method. The periodic-like solutions got by this method can degenerate to the soliton-like solutions. Similarly, this solving process can be applied to other variable-coefficient nonlinear equations , such as variable-coefficient KP ( Kadomtsev-Petviashvili) equation and some others.

References :


[1] Grimshaw R H J. Slowly varying solitary waves[J]. Proc Roy Soc Lon A , 1979 ,**368** (1734) : 359 - 375.

[2] Chan W L ,ZHANG Xiao. Symmetries , conservation-laws and Hamiltonian structures of the non-isospectral and variable-coefficient KdV and mKdV equations[J]. J Phys A ,1995 ,**28** (2) :407 - 419.

[3] TIAN Chou. Symmetries and a hierarchy of the general KdV equation[J]. J Phys A ,1987 ,**20** (2) : 359 - 366.

[4] WANG Ming-liang. Solitary wave solutions for variant Boussinesq equations[J]. Phys Lett A , 1995 ,**199** (3/4) :169 - 172.

[5] WANG Ming-liang ,ZHOU Yu-bin ,LI Zhi-bin. Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics[J]. Phys Lett A , 1996 ,**216** (1/5) : 67 - 75.

[6] YANG Lei ,ZHU Zheng-gang , WANG Ying-hai. Exact solutions of nonlinear equations[J]. Phys Lett A , 1999 ,**260** (1/2) :55 - 59.

[7] YANG Lei ,LIU Jiang , YANG Kong-qing. Exact solutions of nonlinear PDE , nonlinear transformations and reduction of nonlinear PDE to a quadrature[J]. Phys Lett A , 2001 ,**278** (5) : 267 - 270.

[8] Parkes E J ,Duffy B R. Travelling solitary wave solutions to a compound KdV-Burgers equation [J]. Phys Lett A ,1997 ,**229** (4) :217 - 220.

[9] FAN En-gui. Extended tanh-function method and its applications to nonlinear equations[J]. Phys Lett A , 2000 ,**277** (45) :212 - 218.

[10] Hirota R. Exact N-solutions of the wave equation of long waves in shallow water and in nonlinear lattices[J]. J Math Phys ,1973 ,**14** (7) :810 - 814.

[11] Kudryashov N A. Exact solutions of the generalized Kuramoto-Sivashinsky equation[J]. Phys Lett A , 1990 ,**147** (5/6) :287 - 291.

[12] Otwinowski M ,Paul R ,Laidlaw W G. Exact travelling wave solutions of a class of nonlinear diffusion equations by reduction to a quadrature[J]. Phys Lett A ,1988 ,**128** (9) :483 - 487.

[13] LIU Shi-kuo ,FU Zun-tao ,LIU Shi-da , et al. A simple fast mehod in finding particular solutions of some nonlinear PDE[J]. Appl Math Mech ( English Ed) ,2001 ,**22** ,326 - 331.

[14] YAN Chun-tao. A simple transformation for nonlinear waves[J]. Phys Lett A ,1996 ,**224** (1/2) : 77 - 84.

[15] ZHANG Jie-fang ,WU Feng-min. Simple soliton solution method for the (2 + 1) dimensional long disperive equation[J]. Chinese Physics ,1999 ,**8** (5) :326 - 331.

[16] Porubov A V. Periodical solution to the nonlinear dissipative equation for surface waves in a convecting liquid layer[J]. Phys Lett A ,1996 ,**221** (6) :391 - 394.

[17] Porubov A V ,Velarde M G. Exact periodic solutions of the complex Ginzburg-Landau equation[J]. J Math Phys ,1999 ,**40** (2) :884 - 896.

[18] Porubov A V , Parker D F. Some general periodic solutions to coupled nonlinear Schrödinger equations[J]. Wave Motion ,1999 ,**29** (2) :97 - 108.

[19] LIU Shi-kuo ,FU Zun-tao ,LIU Shi-da , et al. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations[J]. Phys Lett A ,2001 ,**289** (1/2) :69 - 74.

[20] Nirmala N ,Vedan M J ,Baby B V. Auto-Backland transformation ,Lax pairs ,Painleve property of a

variable coefficient Korteweg-de Vries equation[J]. J Math Phys, 1986, **27**(10): 2640 - 2648.

[21] Oevel W H, Steeb W H. Painleve analysis for a time-dependent Kadomtsev-Petviashvili equation [J]. Phys A, 1984, **103**(2): 239 - 242.

[22] Steeb W H, Spicker B M. Kadomtsev-Petviashvili equation with explicit x and t dependence[J]. Phys Rev A, 1985, **31**(3): 1952 - 1960.

[23] ZHU Zuo-nong. Lax pairs, Backland transformation, solitary wave solution and infinite conservation laws of the general KP equation and MKP equation with variable coefficients[J]. Phys Lett A, 1993, **180**(6): 409 - 412.

[24] ZHU Zuo-nong. Painleve property, Backland transformation, Lax pairs and soliton-like solutions for a variable coefficient KP equation[J]. Phys Lett A, 1993, **182**(2/3): 277 - 281.

[25] Hong W, Jung Y D. Auto-Bäckland transformation and analytic solutions for general variable-coefficient KdV equation[J]. Phys Lett A, 1999, **257**(3/4): 149 - 152.

[26] WANG Ming-liang, WAMG Yue-ming. A new Bäckland transformation and multi-solitons to the KdV equations with general variable coefficients[J]. Phys Lett A, 2001, **287**(3/4): 211 - 216.