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Elliptic Equation and Its Direct Applications to Nonlinear Wave Equations*

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Abstract *Elliptic equation is taken as an ansatz and applied to solve nonlinear wave equations directly. More kinds of solutions are directly obtained, such as rational solutions, solitary wave solutions, periodic wave solutions and so on. It is shown that this method is more powerful in giving more kinds of solutions, so it can be taken as a generalized method.*

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1 Introduction

Since more and more problems have to involve nonlinearity, how to solve these nonlinear models attracts much attention. Many methods have been proposed to construct exact solutions to nonlinear equations up to now. Among them are the sine-cosine method,[1] the homogeneous balance method,[2,3] the hyperbolic tangent expansion method,[4,5] the Jacobi elliptic function expansion method,[6,7] the nonlinear transformation method,[8,9] the trial function method,[10,11] and others.[12−14]

Apart from methods mentioned above, direct algebra method[15,16] has its own advantages: it is simple and has a strong operability. Hu *et al.*[15] introduced an ansatz

$$u' = (Au - a)(Bu - b)\,, \tag{1}$$

where A, B, a, and b are constants to be determined, and u' is derivative in terms of its argument. They gave some examples to illustrate the applications of Eq. (1) and the solutions they obtained are mainly different rational solutions and solutions of twisted kink forms.

In this paper, we will consider elliptic equation,[17]

$$y'^2 = \sum_{i=0}^{i=4} a_i y^i\,, \tag{2}$$

and take it as a new ansatz to solve nonlinear wave equations. Obviously, equation (1) is just a special case of Eq. (2), so application of Eq. (2) to nonlinear wave equations will lead to more kinds of solutions. In the following sections, applications of ansatz (2) to some well-known equations will be given.

2 KdV Equation

KdV equation reads[17]

$$u_t + uu_x + \beta u_{xxx} = 0\,, \tag{3}$$

and it is met in many fields, such as shallow water model, plasma science, biophysics, etc.

We seek its travelling wave solutions in the following frame,

$$u = u(\xi)\,, \qquad \xi = k(x - ct)\,, \tag{4}$$

where c is wave velocity, and k is wave number.

Substituting Eq. (4) into Eq. (3) and integrating once yields

$$-cu + \frac{1}{2}u^2 + \beta k^2 u'' = D\,, \tag{5}$$

where D is integration constant. And we consider elliptic equation (2) and take it as ansatz

$$u'^2 = \sum_{i=0}^{i=4} a_i u^i\,, \tag{6}$$

where a_i $(i = 0, 1, 2, 3, 4)$ are constants determined by specific nonlinear model, and then

$$u'' = \frac{a_1}{2} + a_2 u + \frac{3a_3}{2}u^2 + 2a_4 u^3\,. \tag{7}$$

From Eqs. (5) and (7), one has

$$\frac{\beta k^2 a_1}{2} = D\,, \quad 2\beta k^2 a_4 = 0\,, \quad a_2 - c = 0\,, \quad \frac{1}{2} + \frac{3}{2}a_3\beta k^2 = 0\,. \tag{8}$$

So the coefficients of ansatz (6) can be determined as

$$a_0 = C_0\,, \quad a_1 = C_1\,, \quad a_2 = c\,, \quad a_3 = -\frac{1}{3\beta k^2}\,, \qquad a_4 = 0\,, \tag{9}$$

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where C_0, C_1, and c are arbitrary constants. Thus we derive the ansatz suitable for KdV equation,

$$u'^2 = \sum_{i=0}^{i=3} a_i u^i \,, \tag{10}$$

from which more kinds of solutions can be obtained. We will give some discussions about some of these solutions.

Case A Consider $a_0 = a_1 = a_2 = 0$, i.e. $c = 0$. This is a non-travelling solution, then

$$u'^2 = a_3 u^3 \,, \tag{11}$$

from which one has

$$u = -\frac{12\beta}{x^2} \,. \tag{12}$$

This is a rational solution. The rational solutions are a disjoint union of manifolds and the particle system describing the motion of pole of rational solutions, which have been discussed in many literatures, such as Refs. [18] and [19].

Case B Consider $a_0 = a_1 = 0$, then

$$u'^2 = a_2 u^2 + a_3 u^3 \,, \tag{13}$$

from which if $a_2 = c > 0$, one has

$$u = 3c\beta k^2 \mathrm{sech}^2\Big(\frac{\sqrt{c}}{2}\,\xi\Big). \tag{14}$$

This is a bell-soliton solution.

If $a_2 = c < 0$, one has

$$u = 3c\beta k^2 \sec^2\Big(\frac{\sqrt{-c}}{2}\,\xi\Big) \,. \tag{15}$$

This is a kind of solutions dealing with "hot spots" or "blow-up" of solutions,[20−23] which can develop singularity at a finite point.

Case C Consider $a_0 = 0$, then

$$u'^2 = a_1 u + a_2 u^2 + a_3 u^3 \,, \tag{16}$$

from which many more solutions expressed in terms of different elliptic functions[17] can be reached.

(i) If $a_1 = 4$, $a_2 = c = -4(1+m^2)$, and $a_3 = 4m^2$ (where $0 \le m \le 1$, is called modulus of Jacobi elliptic functions, see Refs. [17], and [24] $\sim$ [26], one has

$$u = \mathrm{sn}^2(\xi, m) \,, \tag{17}$$

where $\mathrm{sn}(\xi, m)$ is the Jacobi elliptic sine function, see Refs. [17], and [24] $\sim$ [26].

(ii) If $a_1 = 4(1-m^2)$, $a_2 = c = 4(2m^2-1)$, and $a_3 = -4m^2$, one has

$$u = \mathrm{cn}^2(\xi, m) \,, \tag{18}$$

where $\mathrm{cn}(\xi, m)$ is the Jacobi elliptic cosine function, see Refs. [17], and [24] $\sim$ [26].

(iii) If $a_1 = -4(1-m^2)$, $a_2 = c = 4(2-m^2)$, and $a_3 = -4$, one has

$$u = \mathrm{dn}^2(\xi, m) \,, \tag{19}$$

where $\mathrm{dn}(\xi, m)$ is the Jacobi elliptic function of the third kind, see Refs. [17], and [24] $\sim$ [26].

Of course, we can have more generalized solutions, for example,

(iv) If $a_1 = 4A$, $a_2 = c = -4(1+m^2)$, and $a_3 = 4m^2/A$, one has

$$u = A\mathrm{sn}^2(\xi, m) = -12m^2\beta k^2 \mathrm{sn}^2(\xi, m) \,. \tag{20}$$

(v) If $a_1 = 4(1-m^2)A$, $a_2 = c = 4(2m^2-1)$, and $a_3 = -4m^2/A$, one has

$$u = A\mathrm{cn}^2(\xi, m) = 12m^2\beta k^2 \mathrm{cn}^2(\xi, m) \,. \tag{21}$$

(vi) If $a_1 = -4(1-m^2)A$, $a_2 = c = 4(2-m^2)$, and $a_3 = -4/A$, one has

$$u = A\mathrm{dn}^2(\xi, m) = 12\beta k^2 \mathrm{dn}^2(\xi, m) \,. \tag{22}$$

(vii) If $a_1 = 4A\mu^2$, $a_2 = c = -4\mu^2(1+m^2)$, and $a_3 = 4\mu^2 m^2/A$, one has

$$u = A\mathrm{sn}^2(\mu\xi, m) = \frac{3c\beta m^2 k^2}{1+m^2}\mathrm{sn}^2\Big(\pm\sqrt{-\frac{c}{4(1+m^2)}}\,\xi, m\Big) \,. \tag{23}$$

(viii) If $a_1 = 4\mu^2(1-m^2)A$, $a_2 = c = 4\mu^2(2m^2-1)$, and $a_3 = -4\mu^2 m^2/A$, one has

$$u = A\mathrm{cn}^2(\mu\xi, m) = \frac{3c\beta m^2 k^2}{2m^2-1}\mathrm{cn}^2\Big(\pm\sqrt{\frac{c}{4(2m^2-1)}}\,\xi, m\Big) \,. \tag{24}$$

(ix) If $a_1 = -4\mu^2(1-m^2)A$, $a_2 = c = 4\mu^2(2-m^2)$, and $a_3 = -4\mu^2/A$, one has

$$u = A\mathrm{dn}^2(\mu\xi, m) = \frac{3c\beta k^2}{2-m^2}\mathrm{dn}^2\Big(\pm\sqrt{\frac{c}{4(2-m^2)}}\,\xi, m\Big) \,. \tag{25}$$

There still exist many other kinds of Jacobi elliptic functions, which we do not show here. It is known that when $m \to 1$, $\mathrm{sn}(\xi, m) \to \tanh\xi$, $\mathrm{cn}(\xi, m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}(\xi, m) \to \mathrm{sech}\,\xi$, and when $m \to 0$, $\mathrm{sn}(\xi, m) \to \sin\xi$, $\mathrm{cn}(\xi, m) \to \cos\xi$, so we also can derive solutions expressed in terms of hyperbolic functions and trigonometric functions.

Case D If the ansatz is just the same one as Eq. (10), then we have two possible solutions:

(i) If $a_0 = -Au_1u_2u_3$, $a_1 = A(u_1u_2 + u_2u_3 + u_3u_1)$, $a_2 = c = -A(u_1 + u_2 + u_3)$, and $a_3 = -1/3\beta k^2 = A$ ($A >$

$0, u_3 \le u_2 \le u_1$), then

$$u = u_3 + (u_2 - u_3)\mathrm{sn}^2\Big[\sqrt{-\frac{u_1-u_3}{12\beta k^2}}\,\xi, m\Big], \qquad \Big(u_3 \le u \le u_2, m = \sqrt{\frac{u_2-u_3}{u_1-u_3}}\,\Big). \tag{26}$$

(ii) If $a_0 = Au_1u_2u_3$, $a_1 = -A(u_1u_2 + u_2u_3 + u_3u_1)$, $a_2 = c = -A(u_1 + u_2 + u_3)$, and $a_3 = -1/3\beta k^2 = -A$ ($A > 0, u_3 \le u_2 \le u_1$), then

$$u = u_2 + (u_1 - u_2)\mathrm{cn}^2\Big[\sqrt{\frac{u_1-u_2}{12\beta k^2}}\,\xi, m\Big], \qquad \Big(u_3 \le u \le u_2, m = \sqrt{\frac{u_1-u_2}{u_1-u_3}}\,\Big). \tag{27}$$

(iii) If $a_0 = -g_3$, $a_1 = -g_2$, $a_2 = c = 0$, and $a_3 = -1/3\beta k^2 = 4$, then

$$u = \wp(\xi; g_2; g_3)\,, \tag{28}$$

where $\wp(\xi; g_2; g_3)$ is Weierstrass function, see Refs. [17] and [24] ∼ [26].

We have given an example to show the application of the elliptic equation taking the form of Eq. (10) to solve KdV equation. Many more kinds of solutions have been obtained there, too. Maybe one thinks it unnecessary to introduce Eq. (2) as an ansatz to solve nonlinear evolution equations. Actually, this is right for some cases where the nonlinear evolution equations taking simple forms to allow us to transform nonlinear equations to the elliptic equation (2) directly. For more cases, one cannot transform nonlinear equations to the elliptic equation (2) directly, then the ansatz solution is needed, for example, the fifth-order dispersion equation,

$$u_t + \alpha u_x u_{xx} + \beta u_{xxxxx} = 0\,. \tag{29}$$

In the frame of Eq. (4), it can be transformed to

$$\beta k^4 u^{(4)} + \frac{\alpha k^2}{2}(u')^2 - cu = D\,, \tag{30}$$

where D is integration constant. Applying Eq. (6) as its ansatz solution, then

$$u^{(4)} = \Big(3a_0a_3 + \frac{a_1a_2}{2}\Big) + \Big(12a_0a_4 + a_2^2 + \frac{9a_1a_3}{2}\Big)u + \Big(\frac{15a_2a_3}{2} + 15a_1a_4\Big)u^2$$
$$+ \Big(\frac{15a_3^2}{2} + 20a_2a_4\Big)u^3 + 30a_3a_4u^4 + 24a_4^2u^5\,. \tag{31}$$

So we can have

$$a_4 = 0\,, \quad a_3 = -\frac{\alpha}{15\beta k^2}\,, \quad a_2 = c_1\,, \quad a_1 = \frac{5c}{\alpha k^2} - \frac{5\beta k^2 c_1^2}{\alpha} = c_2\,, \quad a_0 = D - \frac{5\beta k^2 c_1 c_2}{3\alpha} = c_3\,, \tag{32}$$

where D, c_1, c_2, and c_3 are arbitrary constants. Obviously, we can get many kinds of solutions as done to KdV equation, here we omit these details.

3 Klein–Gordon Equation

In the last section, we get a kind of elliptic equation of the form (10), which can be applied to solve some nonlinear wave equations. In this section, we will show that another kind of elliptic equation (2) can be applied to solve some other nonlinear wave equations. Here we consider nonlinear Klein–Gordon equation as an example.

Nonlinear Klein–Gordon equation[17] reads

$$u_{tt} - c_0^2 u_{xx} + \alpha u - \beta u^3 = 0\,. \tag{33}$$

Substituting Eq. (4) into Eq. (33) leads to

$$u'' + \alpha_1 u - \beta_1 u^3 = 0\,, \tag{34}$$

where

$$\alpha_1 = \frac{\alpha}{k^2(c^2 - c_0^2)}, \quad \beta_1 = \frac{\beta}{k^2(c^2 - c_0^2)}\,. \tag{35}$$

Similarly, we assume that the solutions of Eq. (33) takes the form of ansatz Eq. (6), then substituting Eq. (7) into Eq. (34) leads to

$$a_1 = 0\,, \quad a_2 = -\alpha_1\,, \quad a_3 = 0\,, \quad a_4 = \frac{\beta_1}{2}\,, \tag{36}$$

so the ansatz suitable for Klein-Gordon equation takes the following form

$$u'^2 = a_0 + a_2u^2 + a_4u^4\,. \tag{37}$$

This is another kind of elliptic equation, which also has many more kinds of solutions (some discussions about Eq. (37) are given in Ref. [27]). We will show some next.

Case A Consider $a_0 = 0$, then we have two kinds of solutions.

(i) If $a_2 = -\alpha_1 > 0$ and $a_4 = \beta_1/2 > 0$, the solution is

$$u = \pm\sqrt{-\frac{2\alpha_1}{\beta_1}}\mathrm{csch}(\sqrt{-\alpha_1}\xi)$$

$$= \pm\sqrt{-\frac{2\alpha}{\beta}}\mathrm{csch}\Big(\sqrt{-\frac{\alpha}{k^2(c^2-c_0^2)}}\xi\Big). \tag{38}$$

(ii) If $a_2 = -\alpha_1 > 0$ and $a_4 = \beta_1/2 < 0$, the solution is

$$u = \pm\sqrt{\frac{2\alpha_1}{\beta_1}}\mathrm{sech}(\sqrt{-\alpha_1}\xi)$$
$$= \pm\sqrt{\frac{2\alpha}{\beta}}\mathrm{sech}\Big(\sqrt{-\frac{\alpha}{k^2(c^2-c_0^2)}}\xi\Big). \tag{39}$$

Case B The ansatz just takes the form of Eq. (37), there will exist many more kinds of solutions expressed in terms of different Jacobi elliptic functions.[17] We show some next.

(i) If $a_0 = 1$, $a_2 = -\alpha_1 = -(1+m^2)$, and $a_4 = \beta_1/2 = m^2$, then the solution is

$$u = \mathrm{sn}(\xi, m)\,. \tag{40}$$

(ii) If $a_0 = 1 - m^2$, $a_2 = -\alpha_1 = 2m^2 - 1$, and $a_4 = \beta_1/2 = -m^2$, then the solution is

$$u = \mathrm{cn}(\xi, m)\,. \tag{41}$$

(iii) If $a_0 = 1 - m^2$, $a_2 = -\alpha_1 = 2 - m^2$, and $a_4 = \beta_1/2 = -1$, then the solution is

$$u = \mathrm{dn}(\xi, m)\,. \tag{42}$$

(iv) If $a_0 = m^2$, $a_2 = -\alpha_1 = -(1 + m^2)$, and $a_4 = \beta_1/2 = 1$, then the solution is

$$u = \mathrm{ns}(\xi, m) \equiv \frac{1}{\mathrm{sn}(\xi, m)}\,. \tag{43}$$

(v) If $a_0 = -m^2$, $a_2 = -\alpha_1 = 2m^2 - 1$, and $a_4 = \beta_1/2 = 1 - m^2$, then the solution is

$$u = \mathrm{nc}(\xi, m) \equiv \frac{1}{\mathrm{cn}(\xi, m)}\,. \tag{44}$$

(vi) If $a_0 = -1$, $a_2 = -\alpha_1 = 2 - m^2$, and $a_4 = \beta_1/2 = m^2 - 1$, then the solution is

$$u = \mathrm{nd}(\xi, m) \equiv \frac{1}{\mathrm{dn}(\xi, m)}\,. \tag{45}$$

(vii) If $a_0 = 1$, $a_2 = -\alpha_1 = 2 - m^2$, and $a_4 = \beta_1/2 = 1 - m^2$, then the solution is

$$u = \mathrm{sc}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)}\,. \tag{46}$$

(viii) If $a_0 = 1$, $a_2 = -\alpha_1 = 2m^2 - 1$, and $a_4 = \beta_1/2 = (m^2 - 1)m^2$, then the solution is

$$u = \mathrm{sd}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}\,. \tag{47}$$

(ix) If $a_0 = 1 - m^2$, $a_2 = -\alpha_1 = 2 - m^2$, and $a_4 = \beta_1/2 = 1$, then the solution is

$$u = \mathrm{cs}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)}\,. \tag{48}$$

(x) If $a_0 = 1$, $a_2 = -\alpha_1 = -(1+m^2)$, and $a_4 = \beta_1/2 = m^2$, then the solution is

$$u = \mathrm{cd}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)}\,. \tag{49}$$

(xi) If $a_0 = m^2(m^2 - 1)$, $a_2 = -\alpha_1 = 2m^2 - 1$, and $a_4 = \beta_1/2 = 1$, then the solution is

$$u = \mathrm{ds}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}\,. \tag{50}$$

(xii) If $a_0 = m^2$, $a_2 = -\alpha_1 = -(1 + m^2)$, and $a_4 = \beta_1/2 = 1$, then the solution is

$$u = \mathrm{dc}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)}\,. \tag{51}$$

Of course, we can get more generalized solutions just like what we have done in the former section.

(xiii) If $a_0 = \mu^2A^2$, $a_2 = -\alpha_1 = -\mu^2(1 + m^2)$, and $a_4 = \beta_1/2 = \mu^2m^2/A^2$, then the solution is

$$u = \pm\sqrt{\frac{2m^2\alpha}{(1+m^2)\beta}}\,\mathrm{sn}\Big(\pm\sqrt{\frac{\alpha}{(1+m^2)k^2(c^2-c_0^2)}}\,\xi, m\Big). \tag{52}$$

(xiv) If $a_0 = \mu^2(1 - m^2)A^2$, $a_2 = -\alpha_1 = \mu^2(2m^2 - 1)$, and $a_4 = \beta_1/2 = -\mu^2m^2/A^2$, then the solution is

$$u = \pm\sqrt{\frac{2m^2\alpha}{(2m^2-1)\beta}}\,\mathrm{cn}\Big(\pm\sqrt{-\frac{\alpha}{(2m^2-1)k^2(c^2-c_0^2)}}\,\xi, m\Big)\,. \tag{53}$$

(xv) If $a_0 = \mu^2(1 - m^2)A^2$, $a_2 = -\alpha_1 = \mu^2(2 - m^2)$, and $a_4 = \beta_1/2 = -\mu^2/A^2$, then the solution is

$$u = \pm\sqrt{\frac{2\alpha}{(2-m^2)\beta}}\,\mathrm{dn}\Big(\pm\sqrt{-\frac{\alpha}{(2-m^2)k^2(c^2-c_0^2)}}\,\xi, m\Big)\,. \tag{54}$$

(xvi) If $a_0 = \mu^2m^2A^2$, $a_2 = -\alpha_1 = -\mu^2(1 + m^2)$, and $a_4 = \beta_1/2 = \mu^2/A^2$, then the solution is

$$u = \pm\sqrt{\frac{2\alpha}{(1+m^2)\beta}}\,\mathrm{ns}\Big(\pm\sqrt{\frac{\alpha}{(1+m^2)k^2(c^2-c_0^2)}}\,\xi, m\Big)\,. \tag{55}$$

(xvii) If $a_0 = -\mu^2 m^2 A^2$, $a_2 = -\alpha_1 = \mu^2(2m^2-1)$, and $a_4 = \beta_1/2 = \mu^2(1-m^2)/A^2$, then the solution is

$$u = \pm\sqrt{-\frac{2(1-m^2)\alpha}{(2m^2-1)\beta}}\,\mathrm{nc}\Big(\pm\sqrt{-\frac{\alpha}{(2m^2-1)k^2(c^2-c_0^2)}}\,\xi, m\Big)\,. \tag{56}$$

(xviii) If $a_0 = -\mu^2 A^2$, $a_2 = -\alpha_1 = \mu^2(2-m^2)$, and $a_4 = \beta_1/2 = \mu^2(m^2-1)/A^2$, then the solution is

$$u = \pm\sqrt{\frac{2(1-m^2)\alpha}{(2-m^2)\beta}}\,\mathrm{nd}\Big(\pm\sqrt{-\frac{\alpha}{(2-m^2)k^2(c^2-c_0^2)}}\,\xi, m\Big)\,. \tag{57}$$

(xix) If $a_0 = \mu^2 A^2$, $a_2 = -\alpha_1 = \mu^2(2-m^2)$, and $a_4 = \beta_1/2 = \mu^2(1-m^2)/A^2$, then the solution is

$$u = \pm\sqrt{-\frac{2(1-m^2)\alpha}{(2-m^2)\beta}}\,\mathrm{sc}\Big(\pm\sqrt{-\frac{\alpha}{(2-m^2)k^2(c^2-c_0^2)}}\,\xi, m\Big). \tag{58}$$

(xx) If $a_0 = \mu^2 A^2$, $a_2 = -\alpha_1 = \mu^2(2m^2-1)$, and $a_4 = \beta_1/2 = \mu^2(m^2-1)m^2/A^2$, then the solution is

$$u = \pm\sqrt{\frac{2m^2(1-m^2)\alpha}{(2m^2-1)\beta}}\,\mathrm{sd}\Big(\pm\sqrt{-\frac{\alpha}{(2m^2-1)k^2(c^2-c_0^2)}}\,\xi, m\Big)\,. \tag{59}$$

(xxi) If $a_0 = \mu^2(1-m^2)A^2$, $a_2 = -\alpha_1 = \mu^2(2-m^2)$, and $a_4 = \beta_1/2 = \mu^2/A^2$, then the solution is

$$u = \pm\sqrt{-\frac{2\alpha}{(2-m^2)\beta}}\,\mathrm{cs}\Big(\pm\sqrt{-\frac{\alpha}{(2-m^2)k^2(c^2-c_0^2)}}\,\xi, m\Big)\,. \tag{60}$$

(xxii) If $a_0 = \mu^2 A^2$, $a_2 = -\alpha_1 = -\mu^2(1+m^2)$, and $a_4 = \beta_1/2 = \mu^2 m^2/A^2$, then the solution is

$$u = \pm\sqrt{\frac{2m^2\alpha}{(1+m^2)\beta}}\,\mathrm{cd}\Big(\pm\sqrt{\frac{\alpha}{(1+m^2)k^2(c^2-c_0^2)}}\,\xi, m\Big)\,. \tag{61}$$

(xxiii) If $a_0 = \mu^2 m^2(m^2-1)A^2$, $a_2 = -\alpha_1 = \mu^2(2m^2-1)$, and $a_4 = \beta_1/2 = \mu^2/A^2$, then the solution is

$$u = \pm\sqrt{-\frac{2\alpha}{(2m^2-1)\beta}}\,\mathrm{ds}\Big(\pm\sqrt{-\frac{\alpha}{(2m^2-1)k^2(c^2-c_0^2)}}\,\xi, m\Big)\,. \tag{62}$$

(xxiv) If $a_0 = \mu^2 m^2 A^2$, $a_2 = -\alpha_1 = -\mu^2(1+m^2)$, and $a_4 = \beta_1/2 = \mu^2/A^2$, then the solution is

$$u = \pm\sqrt{\frac{2\alpha}{(1+m^2)\beta}}\,\mathrm{dc}\Big(\pm\sqrt{\frac{\alpha}{(1+m^2)k^2(c^2-c_0^2)}}\,\xi, m\Big)\,. \tag{63}$$

It is known that when $m \to 1$, $\mathrm{sn}(\xi, m) \to \tanh\xi$, $\mathrm{cn}\,(\xi, m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}\,(\xi, m) \to \mathrm{sech}\,\xi$, and when $m \to 0$, $\mathrm{sn}(\xi, m) \to \sin\xi$, $\mathrm{cn}(\xi, m) \to \cos\xi$. And among the Jacobi elliptic functions, Jacobi elliptic sine function, Jacobi elliptic cosine function, and Jacobi elliptic function of the third kind are three basic ones, and all other Jacobi elliptic functions can be expressed in terms of them. So we can also arrive at more solutions expressed in terms of hyperbolic functions and trigonometric functions.

4 Conclusion

In this paper, we consider elliptic equation as a new ansatz to solve nonlinear wave equations directly. More kinds of solutions are derived, including rational solutions, solitary wave solutions constructed in terms of hyperbolic functions, and periodic solutions expressed by trigonometric functions and periodic solutions dealing with elliptic functions. It is obvious that ansatz (1) is just one special case of ansatz (2) (it has two cases: ansatz (10) and ansatz (37)). Ansatz (1) cannot be applied to solve that kind of nonlinear equations solved by ansatz (10). From this point, nonlinear wave equations can be classified into two categories: one is solvable by ansatz (10), and the other is solvable by ansatz (37).

For application of ansatz (2) to some nonlinear wave equations, the obtained solutions consist of those from the hyperbolic tangent expansion method,[4,5] the Jacobi elliptic function expansion method,[6,7] the nonlinear transformation method,[8,9] and the trial function method,[10,11] so it can be taken as a generalized method. More applications to solve other nonlinear wave equations are applicable.

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