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Elliptic Equation and New Solutions to Nonlinear Wave Equations*

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Abstract *The new solutions to elliptic equation are shown, and then the elliptic equation is taken as a transformation and is applied to solve nonlinear wave equations. It is shown that more kinds of solutions are derived, such as periodic solutions of rational form, solitary wave solutions of rational form, and so on.*

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1 Introduction

We have taken elliptic equation as an intermediate transformation to solve nonlinear wave equations,[1−3] and obtained many periodic solutions and solitary wave solutions. However, there are still more researches needed to find more solutions of different forms. In Ref. [4], we derived periodic solutions of rational forms, which are due to external forcing. It is an interesting issue to apply different methods to obtain this kind of solutions of rational forms. In this paper, we will revisit the elliptic equation methods,[1] and show that we can construct this kind of solutions of rational forms just by the elliptic equation methods.[1]

2 KdV Equation

KdV equation[5] reads

$$u_t + uu_x + \beta u_{xxx} = 0\,. \tag{1}$$

We seek its travelling wave solutions in the following frame

$$u = u(\xi), \quad \xi = k(x - ct)\,, \tag{2}$$

where c is wave velocity, and k is wave number.

Substituting Eq. (2) into Eq. (1) and integrating once yields

$$-cu + \frac{1}{2}u^2 + \beta k^2 u'' = C\,, \tag{3}$$

where C is an integration constant. And then we suppose that equation (1) has the following solution

$$u = u(y) = \sum_{j=0}^{n} b_j y^j, \quad y = y(\xi)\,, \tag{4}$$

where y satisfies the elliptic equation[5]

$$y'^2 = \sum_{i=0}^{i=4} a_i y^i, \quad a_4 \neq 0\,, \tag{5}$$

where $y' = \mathrm{d}y/\mathrm{d}\xi$, then

$$y'' = \frac{a_1}{2} + a_2 y + \frac{3a_3}{2}y^2 + 2a_4 y^3\,. \tag{6}$$

Obviously, two special cases of Eq. (5) are

$$\frac{\mathrm{d}y}{\mathrm{d}\xi} = b + y^2 \tag{7}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}\xi} = R(1 + \mu y^2)\,, \tag{8}$$

which were introduced by Fan[6] and Yan *et al.*,[7] respectively.

Here n in Eq. (4) can be determined by the partial balance between the highest order derivative terms and the highest degree nonlinear term in Eq. (1). Here we know that the degree of u is

$$O(u) = O(y^n) = n\,, \tag{9}$$

and from Eqs. (5) and (6), one has

$$O(y'^2) = O(y^4) = 4, \quad O(y'') = O(y^3) = 3\,, \tag{10}$$

and actually one can have

$$O(y^{(l)}) = l + 1\,. \tag{11}$$

So one has

$$O(u) = n, \quad O(u') = n + 1\,, \\ O(u'') = n + 2, \quad O(u^{(l)}) = n + l\,. \tag{12}$$

For KdV equation (1), we have $n = 2$, so the ansatz solution of Eq. (4) can be rewritten as

$$u = b_0 + b_1 y + b_2 y^2, \quad b_2 \neq 0\,, \tag{13}$$

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then

$$u^2 = b_0^2 + 2b_0b_1y + (2b_0b_2 + b_1^2)y^2 + 2b_1b_2y^3 + b_2^2y^4\,, \tag{14}$$

$$u'' = \Big(\frac{1}{2}a_1b_1 + 2a_0b_2\Big) + (a_2b_1 + 3a_1b_2)y + \Big(\frac{3}{2}a_3b_1 + 4a_2b_2\Big)y^2 + (2a_4b_1 + 5a_3b_2)y^3 + 6a_4b_2y^4\,. \tag{15}$$

Substituting Eqs. (13) $\sim$ (15) into Eq. (3) and collecting each order of y yields algebraic equations about coefficients b_j $(j = 0, 1, 2)$ and a_i $(i = 0, 1, 2, 3, 4)$, i.e.,

$$-cb_0 + \frac{1}{2}b_0^2 + \beta k^2\Big(\frac{1}{2}a_1b_1 + 2a_0b_2\Big) - C = 0\,, \tag{16a}$$

$$-cb_1 + b_0b_1 + \beta k^2(a_2b_1 + 3a_1b_2) = 0\,, \tag{16b}$$

$$-cb_2 + \frac{1}{2}(2b_0b_2 + b_1^2) + \beta k^2\Big(\frac{3}{2}a_3b_1 + 4a_2b_2\Big) = 0\,, \tag{16c}$$

$$b_1b_2 + \beta k^2(2a_4b_1 + 5a_3b_2) = 0\,, \tag{16d}$$

$$\frac{1}{2}b_2^2 + 6\beta k^2a_4b_2 = 0\,, \tag{16e}$$

from which we have

$$b_2 = -12\beta k^2a_4, \quad b_1 = -6\beta k^2a_3\,, \quad b_0 = c - 4\beta k^2a_2 + \frac{3\beta k^2a_3^2}{4a_4}\,. \tag{17}$$

At the same time there is

$$a_1 = \frac{a_3}{2a_4}\Big(a_2 - \frac{a_3^2}{4a_4}\Big)\,. \tag{18}$$

So if $a_3 = 0$, then

$$b_1 = a_1 = 0, \quad b_2 = -12\beta k^2a_4, \quad b_0 = c - 4\beta k^2a_2\,, \tag{19}$$

and transformation (5) takes the following form

$$y'^2 = a_0 + a_2y^2 + a_4y^4\,, \tag{20}$$

which has many more kinds of solutions, some of which we have shown in Refs. [1] $\sim$ [3]. Actually, there are more other kinds of solutions to Eq. (20). Next we will show some solutions of rational forms expressed in terms of different elliptic functions.[5]

(i) If $a_0 = (1 - m^2)/4$, $a_2 = (1 + m^2)/2$ and $a_4 = (1 - m^2)/4$ (where $0 \le m \le 1$, and m is called modulus of Jacobi elliptic functions, see Refs. [5] and [8] $\sim$ [10], then the solutions to Eq. (20) are

$$y_1 = \frac{\mathrm{cn}(\xi, m)}{1 + \mathrm{sn}(\xi, m)}\,, \tag{21}$$

where $\mathrm{sn}(\xi, m)$ and $\mathrm{cn}(\xi, m)$ are sine and cosine Jacobi elliptic functions, respectively,[5,8–10] and

$$y_2 = \frac{\mathrm{cn}(\xi, m)}{1 - \mathrm{sn}(\xi, m)}\,. \tag{22}$$

These are two new solutions to Eq. (20) which are not shown in Refs. [1] $\sim$ [3]. So based on the above results, we can derive new solutions to Eq. (1),

$$u_1 = b_0 + b_2y^2 = c - 2\beta k^2(1 + m^2) - \frac{3\beta k^2(1 - m^2)\mathrm{cn}^2(\xi, m)}{[1 + \mathrm{sn}(\xi, m)]^2}\,, \tag{23}$$

and

$$u_2 = b_0 + b_2y^2 = c - 2\beta k^2(1 + m^2) - \frac{3\beta k^2(1 - m^2)\mathrm{cn}^2(\xi, m)}{[1 - \mathrm{sn}(\xi, m)]^2}\,. \tag{24}$$

(ii) If $a_0 = -(1 - m^2)/4$, $a_2 = (1 + m^2)/2$, and $a_4 = -(1 - m^2)/4$, then the solutions to Eq. (20) are

$$y_3 = \frac{\mathrm{dn}(\xi, m)}{1 + m\,\mathrm{sn}(\xi, m)}\,, \tag{25}$$

and

$$y_4 = \frac{\mathrm{dn}(\xi, m)}{1 - m\,\mathrm{sn}(\xi, m)}\,, \tag{26}$$

where $\mathrm{dn}(\xi, m)$ is Jacobi elliptic function of the third kind[5,8–10] and new solutions to Eq. (1) are

$$u_3 = b_0 + b_2y^2 = c - 2\beta k^2(1 + m^2) + \frac{3\beta k^2(1 - m^2)\mathrm{dn}^2(\xi, m)}{[1 + m\,\mathrm{sn}(\xi, m)]^2}\,, \tag{27}$$

and

$$u_4 = b_0 + b_2y^2 = c - 2\beta k^2(1 + m^2) + \frac{3\beta k^2(1 - m^2)\mathrm{dn}^2(\xi, m)}{[1 - m\,\mathrm{sn}(\xi, m)]^2}\,. \tag{28}$$

(iii) If $a_0 = m^2/4$, $a_2 = -(2-m^2)/2$, and $a_4 = m^2/4$, then the solutions to Eq. (20) are

$$y_5 = \frac{m\,\mathrm{sn}(\xi, m)}{1 + \mathrm{dn}(\xi, m)}\,, \tag{29}$$

and

$$y_6 = \frac{m\,\mathrm{sn}(\xi, m)}{1 - \mathrm{dn}(\xi, m)}\,. \tag{30}$$

New solutions to Eq. (1) are

$$u_5 = b_0 + b_2y^2 = c + 2\beta k^2(2 - m^2) - \frac{3\beta k^2m^4\mathrm{sn}^2(\xi, m)}{[1 + \mathrm{dn}(\xi, m)]^2}\,, \tag{31}$$

and

$$u_6 = b_0 + b_2y^2 = c + 2\beta k^2(2 - m^2) - \frac{3\beta k^2m^4\mathrm{sn}^2(\xi, m)}{[1 - \mathrm{dn}(\xi, m)]^2}\,. \tag{32}$$

It is known that when $m \to 1$, $\mathrm{sn}(\xi, m) \to \tanh\xi$, $\mathrm{cn}(\xi, m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}(\xi, m) \to \mathrm{sech}\,\xi$, so new solutions to Eq. (1) are

$$u_{5'} = b_0 + b_2y^2 = c + 2\beta k^2 - \frac{3\beta k^2\tanh^2(\xi)}{[1 + \mathrm{sech}(\xi)]^2}\,, \tag{33}$$

and

$$u_{6'} = b_0 + b_2y^2 = c + 2\beta k^2 - \frac{3\beta k^2\tanh^2(\xi)}{[1 - \mathrm{sech}(\xi)]^2}\,. \tag{34}$$

(iv) If $a_0 = 1/4$, $a_2 = (1-2m^2)/2$, and $a_4 = 1/4$, then the solutions to Eq. (20) are

$$y_7 = \frac{\mathrm{sn}(\xi,m)}{1+\mathrm{cn}(\xi,m)}, \tag{35}$$

and

$$y_8 = \frac{\mathrm{sn}(\xi,m)}{1-\mathrm{cn}(\xi,m)}. \tag{36}$$

New solutions to Eq. (1) are

$$u_7 = b_0 + b_2 y^2 = c - 2\beta k^2(1-2m^2) - \frac{3\beta k^2 \mathrm{sn}^2(\xi,m)}{[1+\mathrm{cn}(\xi,m)]^2}, \tag{37}$$

and

$$u_8 = b_0 + b_2 y^2 = c - 2\beta k^2(1-2m^2) - \frac{3\beta k^2 \mathrm{sn}^2(\xi,m)}{[1-\mathrm{cn}(\xi,m)]^2}. \tag{38}$$

Similarly, when $m \to 1$, the solutions u_7 and u_8 reduce to solutions $u_{5'}$ and $u_{6'}$.

(v) If $a_0 = 1/4$, $a_2 = -(2-m^2)/2$, and $a_4 = m^4/4$, then the solutions to Eq. (20) are

$$y_9 = \frac{\mathrm{sn}(\xi,m)}{1+\mathrm{dn}(\xi,m)}, \tag{39}$$

and

$$y_{10} = \frac{\mathrm{sn}(\xi,m)}{1-\mathrm{dn}(\xi,m)}. \tag{40}$$

New solutions to Eq. (1) are

$$u_9 = b_0 + b_2 y^2 = c + 2\beta k^2(2-m^2) - \frac{3\beta k^2 m^4 \mathrm{sn}^2(\xi,m)}{[1+\mathrm{dn}(\xi,m)]^2}, \tag{41}$$

and

$$u_{10} = b_0 + b_2 y^2 = c + 2\beta k^2(2-m^2) - \frac{3\beta k^2 m^4 \mathrm{sn}^2(\xi,m)}{[1-\mathrm{dn}(\xi,m)]^2}, \tag{42}$$

which are the same as Eqs. (31) and (32), respectively.

3 Klein–Gordon Equation

Nonlinear Klein–Gordon equation reads

$$u_{tt} - c_0^2 u_{xx} + \alpha u - \beta u^3 = 0. \tag{43}$$

Substituting Eq. (2) into Eq. (43) leads to

$$u'' + \alpha_1 u - \beta_1 u^3 = 0, \tag{44}$$

where

$$\alpha_1 = \frac{\alpha}{k^2(c^2-c_0^2)}, \quad \beta_1 = \frac{\beta}{k^2(c^2-c_0^2)}. \tag{45}$$

Similarly, assuming that the solutions of Eq. (43) take the form of Eq. (4), we can get $n = 1$ for Eq. (43), i.e.,

$$u = b_0 + b_1 y, \quad b_1 \neq 0, \tag{46}$$

where y satisfies elliptic equation (5). Then substituting Eq. (46) into Eq. (44) leads to

$$b_1 = \pm\sqrt{\frac{2a_4}{\beta_1}}, \quad b_0 = \pm\frac{a_3}{2\beta_1}\sqrt{\frac{\beta_1}{2a_4}}, \tag{47}$$

and

$$a_2 = -\alpha_1 + \frac{3a_3^2}{8a_4}, \quad a_1 = \frac{(a_3^2 - 8\alpha_1 a_4)a_3}{16a_4^2}. \tag{48}$$

If $a_3 = 0$, then $b_0 = a_1 = 0$ and

$$b_1 = \pm\sqrt{\frac{2a_4}{\beta_1}}, \quad a_2 = -\alpha_1, \tag{49}$$

then the transformation takes the following form

$$y'^2 = a_0 + a_2 y^2 + a_4 y^4. \tag{50}$$

This is an elliptic equation, which also has many kinds of solutions. There exist many kinds of solutions expressed in terms of different Jacobi elliptic functions.[5] We show some solutions just like what we have done in the former section next.

(i) If $a_0 = (1-m^2)/4$, $a_2 = -\alpha_1 = (1+m^2)/2$, and $a_4 = (1-m^2)/4$, then the solutions to Eq. (20) are Eqs. (21) and (22), so the solutions to Eq. (43) are

$$u_1 = b_1 y = \pm\sqrt{\frac{1-m^2}{2\beta_1}}\frac{\mathrm{cn}(\xi,m)}{1+\mathrm{sn}(\xi,m)}, \tag{51}$$

and

$$u_2 = b_1 y = \pm\sqrt{\frac{1-m^2}{2\beta_1}}\frac{\mathrm{cn}(\xi,m)}{1-\mathrm{sn}(\xi,m)} \tag{52}$$

with constraints $\alpha_1 < 0$ and $\beta_1 > 0$.

(ii) If $a_0 = -(1-m^2)/4$, $a_2 = -\alpha_1 = (1+m^2)/2$, and $a_4 = -(1-m^2)/4$, then the solutions to Eq. (20) are Eqs. (25) and (26), so the solutions to Eq. (43) are

$$u_3 = b_1 y = \pm\sqrt{-\frac{1-m^2}{2\beta_1}}\frac{\mathrm{dn}(\xi,m)}{1+m\,\mathrm{sn}(\xi,m)}, \tag{53}$$

and

$$u_4 = b_1 y = \pm\sqrt{-\frac{1-m^2}{2\beta_1}}\frac{\mathrm{dn}(\xi,m)}{1-m\,\mathrm{sn}(\xi,m)} \tag{54}$$

with constraints $\alpha_1 < 0$ and $\beta_1 < 0$.

(iii) If $a_0 = m^2/4$, $a_2 = -\alpha_1 = -(2-m^2)/2$, and $a_4 = m^2/4$, then the solutions to Eq. (20) are Eqs. (29) and (30), so the solutions to Eq. (43) are

$$u_5 = b_1 y = \pm\sqrt{\frac{m^2}{2\beta_1}}\frac{m\,\mathrm{sn}(\xi,m)}{1+\mathrm{dn}(\xi,m)}, \tag{55}$$

and

$$u_6 = b_1 y = \pm\sqrt{\frac{m^2}{2\beta_1}}\frac{m\,\mathrm{sn}(\xi,m)}{1-\mathrm{dn}(\xi,m)} \tag{56}$$

with constraints $\alpha_1 > 0$ and $\beta_1 > 0$.

(iv) If $a_0 = 1/4$, $a_2 = -\alpha_1 = (1-2m^2)/2$, and $a_4 = 1/4$, then the solutions to Eq. (20) are Eqs. (35)

and (36), so the solutions to Eq. (43) are

$$u_7 = b_1 y = \pm\sqrt{\frac{1}{2\beta_1}}\frac{\mathrm{sn}(\xi,m)}{1+\mathrm{cn}(\xi,m)}\,, \tag{57}$$

and

$$u_8 = b_1 y = \pm\sqrt{\frac{1}{2\beta_1}}\frac{\mathrm{sn}(\xi,m)}{1-\mathrm{cn}(\xi,m)} \tag{58}$$

with constraint $\beta_1 > 0$.

Of course, we can have more solutions if we do not take $a_3 = 0$. We do not discuss this here.

4 Conclusion

In this paper, we consider elliptic equation as a transformation to solve nonlinear wave equations. More kinds of solutions can be got from there for more new solutions to elliptic equation, including periodic solutions of rational forms and solitary wave solutions constructed in terms of hyperbolic functions of rational forms. And applying of transformation (5) to some nonlinear wave equations, the obtained solutions have not been obtained by the sine-cosine method,[11] the homogeneous balance method,[12−14] the hyperbolic function expansion method,[6,15,15] the Jacobi elliptic function expansion method,[17,18] the nonlinear transformation method,[19,20] the trial function method,[21,22] and others.[23−26] So more applications of new solutions of elliptic equation to solve other nonlinear wave equations are also applicable and deserved.

References

[1] Z.T. Fu, S.D. Liu, and S.K. Liu, Commun. Theor. Phys. (Beijing, China) **39** (2003) 531.

[2] Z.T. Fu, S.K. Liu, and S.D. Liu, Commun. Theor. Phys. (Beijing, China) **40** (2003) 285.

[3] Z.T. Fu, Z. Chen, S.K. Liu, and S.D. Liu, Commun. Theor. Phys. (Beijing, China) **41** (2004) 675.

[4] Z.T. Fu, Z. Chen, S.D. Liu, and S.K. Liu, Commun. Theor. Phys. (Beijing, China) **42** (2004) 43.

[5] S.K. Liu and S.D. Liu, *Nonlinear Equations in Physics*, Peking University Press, Beijing (2000).

[6] E.G. Fan, Phys. Lett. **A277** (2000) 212.

[7] Z.Y. Yan and H.Q. Zhang. Phys. Lett. **A285** (2001) 355.

[8] F. Bowman, *Introduction to Elliptic Functions with Applications*, Universities, London (1959).

[9] V. Prasolov and Y. Solovyev, *Elliptic Functions and Elliptic Integrals*, American Mathematical Society, Providence, R.I. (1997).

[10] Z.X. Wang and D.R. Guo, *Special Functions*, World Scietific, Singapore (1989).

[11] C.T. Yan, Phys. Lett. **A224** (1996) 77.

[12] M.L. Wang, Phys. Lett. **A199** (1995) 169.

[13] M.L. Wang, Y.B. Zhou, and Z.B. Li, Phys. Lett. **A216** (1996) 67.

[14] L. Yang, Z. Zhu, and Y. Wang, Phys. Lett. **A260** (1999) 55.

[15] E.J. Parkes and B.R. Duffy, Phys. Lett. **A229** (1997) 217.

[16] L. Yang, J. Liu, and K. Yang, Phys. Lett. **A278** (2001) 267.

[17] Z.T. Fu, S.K. Liu, S.D. Liu, and Q. Zhao, Phys. Lett. **A290** (2001) 72.

[18] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Phys. Lett. **A289** (2001) 69.

[19] R. Hirota, J. Math. Phys. **14** (1973) 810.

[20] N.A. Kudryashov, Phys. Lett. **A147** (1990) 287.

[21] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Appl. Math. Mech. **22** (2001) 326.

[22] M. Otwinowski, R. Paul, and W.G. Laidlaw, Phys. Lett. **A128** (1988) 483.

[23] Z.T. Fu, S.K. Liu, and S.D. Liu, Phys. Lett. **A299** (2002) 507.

[24] A.V. Porubov, Phys. Lett. **A221** (1996) 391.

[25] A.V. Porubov and M.G. Velarde, J. Math. Phys. **40** (1999) 884.

[26] A.V. Porubov and D.F. Parker, Wave Motion **29** (1999) 97.