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Equatorial Rossby Solitary Wave Under the External Forcing*

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Abstract *A simple shallow-water model with influence of external forcing on a β-plane is applied to investigate the nonlinear equatorial Rossby waves in a shear flow. By the perturbation method, the extended variable-coefficient KdV equation under an external forcing is derived for large amplitude equatorial Rossby wave in a shear flow. And then various periodic-like structures for these equatorial Rossby waves are obtained with the help of Jacobi elliptic functions. It is shown that the external forcing plays an important role in various periodic-like structures.*

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1 Introduction

In the last decades, the theory of equatorial waves has attracted much more attention on equatorial atmospheric dynamics and nonlinear dynamics. It provides a dynamical frame to analyze the slowly evolving large-scale phenomena in low latitudes and underlining dynamics. These theories of equatorial waves have been used for various purposes, especially in explaining some fundamental features of tropical climate and global changes, such as Walker circulation,[1] the low-frequency Madden–Julian oscillation,[2] and ENSO.[3]

Among the nonlinear theories for equatorial waves, many are related to nonlinear Rossby wave activity, for it can manifest some of the prime events of geophysical fluid flows, and this activity often leads to a large-scale localized coherent structures that have remarkable permanence and stability. When the zonal flow shear is taken to be nonuniform, one can derive Rossby solitary waves and envelope Rossby solitary waves. Benney,[4] Yamagata,[5] and Zhao[6] investigated envelope Rossby solitary waves in barotropic shear and uniform or nonuniform flows, independently. However, none of them considered the effect of external sources, especially the influence of diabatic heating from oceans. In this paper, we will address this issue by the method of perturbation expansion to derive the extended variable-coefficient KdV equation under an external forcing satisfied by the large-amplitude equatorial Rossby waves. And then the basic structures of this extended variable-coefficient KdV equation without and with external source are obtained by using knowledge of Jacobi elliptic functions.

2 Derivation of Extended KdV Equation with an External Forcing

The governing equation is quasi-geostrophic potential vorticity equation of shallow-water model on an equatorial β-plane with an external forcing, i.e.

$$\Big(\frac{\partial}{\partial t_*}-\frac{\partial\psi_*}{\partial y_*}\frac{\partial}{\partial x_*}+\frac{\partial\psi_*}{\partial x_*}\frac{\partial}{\partial y_*}\Big)\Big(\beta y_*+\nabla_*^2\psi_*-\frac{\beta^2y_*^2}{c_0^2}\psi_*\Big)=Q_*(x_*,y_*,t_*)\,,\tag{1}$$

where ψ_* is the stream function, $\beta>0$ is the planetary-vorticity gradient, c_0 is velocity of pure gravity waves, and $Q_*(x_*,y_*,t_*)$ is the diabatic heating due to the tropical ocean. And ∇_*^2 is the horizontal Laplacian operator, which is defined as

$$\nabla_*^2=\frac{\partial^2}{\partial x_*^2}+\frac{\partial^2}{\partial y_*^2}\,.\tag{2}$$

Since the equatorial waves are trapped near the equator, the appropriate boundary condition can be given as

$$\frac{\partial\psi_*}{\partial x_*}\to 0\,,\qquad \text{as}\quad y_*\to\pm\infty\,.\tag{3}$$

Equation (3) can be nondimensionalized as

$$\frac{\partial}{\partial t}\bar\nabla^2\psi+\varepsilon J(\psi,\bar\nabla^2\psi)+\frac{\partial\psi}{\partial x}=\mu Q(x,y,t)\tag{4}$$

with

$$J(a,b)=\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}\,,\qquad \bar\nabla^2=\nabla^2-y^2\,,$$
$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\,,\tag{5}$$

and ε is the equatorial Rossby number, μ is an amplitude parameter, and they are all small number, near the equator $\varepsilon\sim O(10^{-2})$. Here it is obvious that ε denotes the

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magnitude of nonlinearity and μ represents the strength of external forcing.

First of all, the stream function ψ can be decomposed as

$$\psi = \bar{\psi} + \psi' = -\int^{y} [\bar{u}(s) - c]\,\mathrm{d}s + \psi' \,, \tag{6}$$

then equation (4) is rewritten as

$$\Big[\frac{\partial}{\partial t} + (\bar{u} - c)\frac{\partial}{\partial x}\Big]\bar{\nabla}^2\psi' + \varepsilon J(\psi', \bar{\nabla}^2\psi') + (1 - \bar{u}'')\frac{\partial \psi'}{\partial x} = \mu Q \,. \tag{7}$$

Due to the existence of these small parameters, the perturbation expansion method[7] can be applied to solve the problem (4), where the stretching coordinates of the following form

$$X = \varepsilon^{1/2} x \,, \qquad T = \varepsilon^{3/2} t \tag{8}$$

is introduced, which is known as Gardner–Morikawa transformation.

Applying Eq. (8) to Eq. (7) leads to

$$\wp_0(\psi') + \varepsilon\Big\{\wp_1(\psi') + J\Big[\psi', \Big(\frac{\partial^2\psi'}{\partial y^2} - y^2\psi'\Big)\Big]\Big\} = \mu\varepsilon^{-1/2} Q \tag{9}$$

with

$$\wp_0() \equiv (\bar{u}-c)\frac{\partial}{\partial X}\Big(\frac{\partial^2}{\partial y^2} - y^2\Big) + (1-\bar{u}'')\frac{\partial}{\partial X} \,, \quad \wp_1() \equiv \frac{\partial}{\partial T}\Big(\frac{\partial^2}{\partial y^2} - y^2\Big) + (\bar{u}-c)\frac{\partial^3}{\partial X^3} \,, \quad J[a,b] \equiv \frac{\partial a}{\partial X}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial X}\frac{\partial a}{\partial y} \,. \tag{10}$$

In order to solve Eq. (9), ψ' can be expanded as the following form:

$$\psi' = \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \cdots \tag{11}$$

If the external forcing is weak, i.e. $O(\mu) \sim O(\varepsilon^{5/2})$, then combining Eq. (11) with Eq. (9) leads to

$$O(\varepsilon): \qquad \wp_0(\psi_1) = (\bar{u} - c)\frac{\partial}{\partial X}\Big(\frac{\partial^2\psi_1}{\partial y^2} - y^2\psi_1\Big) + (1 - \bar{u}'')\frac{\partial \psi_1}{\partial X} = 0 \,, \tag{12}$$

$$O(\varepsilon^2): \qquad \wp_0(\psi_2) = -\Big\{\wp_1\psi_1 + J\Big[\psi_1, \Big(\frac{\partial^2\psi_1}{\partial y^2} - y^2\psi'\Big)\Big]\Big\} + Q \equiv F \tag{13}$$

with F being defined as

$$F = -\Big\{\frac{\partial}{\partial T}\Big(\frac{\partial^2\psi_1}{\partial y^2} - y^2\psi_1\Big) + (\bar{u} - c)\frac{\partial^3\psi_1}{\partial X^3} + \frac{\partial\psi_1}{\partial X}\frac{\partial}{\partial y}\Big(\frac{\partial^2\psi_1}{\partial y^2} - y^2\psi_1\Big) - \frac{\partial\psi_1}{\partial y}\frac{\partial}{\partial X}\Big(\frac{\partial^2\psi_1}{\partial y^2} - y^2\psi_1\Big)\Big\} + Q \,. \tag{14}$$

Obviously, equation (12) has the following variable-separation solution:

$$\psi_1 = A(X,T)\Phi_1(T,y) \tag{15}$$

with

$$\Big(\frac{\partial^2}{\partial y^2} + \frac{1 - \bar{u}''}{\bar{u} - c} - y^2\Big)\Phi_1 = 0 \,, \qquad \Phi_1|_{y\to\pm\infty} \to 0 \,. \tag{16}$$

Substitution ψ_1 into Eq. (13) yields

$$F = \frac{1 - \bar{u}''}{\bar{u} - c}\Phi_1\frac{\partial A}{\partial T} + \frac{1 - \bar{u}''}{\bar{u} - c}\frac{\partial \Phi_1}{\partial T}A + \Big[\Phi_1^2\frac{\mathrm{d}}{\mathrm{d}y}\frac{1 - \bar{u}''}{\bar{u} - c}\Big]A\frac{\partial A}{\partial X} - (\bar{u} - c)\Phi_1\frac{\partial^3 A}{\partial X^3} + Q \,. \tag{17}$$

In order to show the evolution of A, we have to seek the solution to Eq. (13). Similarly, the solution to ψ_2 can be written as

$$\psi_2 = B(X,T)\Phi_2(T,y) \,, \tag{18}$$

then

$$\frac{\partial B}{\partial X}\Big(\frac{\partial^2}{\partial y^2} + \frac{1 - \bar{u}''}{\bar{u} - c} - y^2\Big)\Phi_2 = \frac{F}{\bar{u} - c} \,, \qquad \Phi_2|_{y\to\pm\infty} \to 0 \,. \tag{19}$$

Multiplying Eq. (19) with Φ_1 and then integrating it with respect to y, the orthogonality condition can avoid secular growth, i.e.

$$\int_{-\infty}^{+\infty} \frac{F\Phi_1}{\bar{u} - c}\mathrm{d}y = 0 \,, \tag{20}$$

from which one has

$$\frac{\partial A}{\partial T} + \alpha A\frac{\partial A}{\partial X} + \beta\frac{\partial^3 A}{\partial X^3} + \gamma A = \eta Q_1(X,T) \tag{21}$$

with

$$\alpha(T) = \frac{I_1}{I}, \quad \beta(T) = \frac{I_2}{I}, \quad \gamma(T) = \frac{I_3}{I}, \quad \eta(T) = \frac{I_4}{I}, \tag{22}$$

where

$$I = \int_{-\infty}^{+\infty} \frac{1 - \bar{u}''}{(\bar{u} - c)^2}\mathrm{d}y \,,$$

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}y}\Big(\frac{1-\bar{u}''}{\bar{u}-c}\Big)\frac{\Phi_1^3}{\bar{u}-c}\mathrm{d}y\,,\\
I_2 &= \int_{-\infty}^{+\infty} \Phi_1^2 \mathrm{d}y\,,\\
I_3 &= \int_{-\infty}^{+\infty} \frac{1-\bar{u}''}{(\bar{u}-c)^2}\Phi_1\frac{\partial\Phi_1}{\partial T}\mathrm{d}y\,,\\
I_4 &= \int_{-\infty}^{+\infty} \frac{Q\Phi_1}{\bar{u}-c}\mathrm{d}y\,.
\end{aligned}
\tag{23}
$$

Obviously, equation (21) is a variable-coefficient nonlinear evolution model with an external forcing, we call it extended KdV equation with an external forcing. If $\partial\Phi_1/\partial T = 0$, then $\gamma = 0$, α, β, and η are constants, so equation (21) is a general constant-coefficient KdV equation with an external forcing. For variable-coefficient KdV equation without an external forcing, many variants have been reported.[8−9] And much attention[11−17] has been paid to study the integrability and symmetry of variable-coefficient nonlinear equations, since numerous application in physical sciences and engineering deal with variable-coefficient nonlinear equations.

3 Solutions to the Extended KdV Equation

Actually, variable-coefficient nonlinear equations are seldom considered for their complexity. In this section, we will show some solutions to some special cases of Eq. (21). We will extend the Jacobi elliptic function expansion method[18,19] and apply it to get the periodic solutions and corresponding shock or solitary wave solutions to variable-coefficient or forced KdV equations.

First of all, we consider the case where $\partial\Phi_1/\partial T = 0$, i.e., $\gamma = 0$, α, β, η are constants, and the external forcing varies only with time T, i.e. $\eta Q_1 = S(T)$, so equation (21) can be rewritten as

$$
\frac{\partial A}{\partial T} + \alpha A\frac{\partial A}{\partial X} + \beta\frac{\partial^3 A}{\partial X^3} = S(T)\,. \tag{24}
$$

In order to solve Eq. (24), the following transformation is introduced,

$$
A = v + \Gamma(T)\,, \qquad \Gamma(T) = \int^T S(\tau)\mathrm{d}\tau\,, \tag{25}
$$

then we have

$$
v_T + \alpha[\Gamma(T) + v]v_X + \beta v_{XXX} = 0\,. \tag{26}
$$

We seek its general travelling wave solution

$$
v = v(\xi)\,, \qquad \xi = f(T)X + g(T)\,, \tag{27}
$$

where $f(T)$ and $g(T)$ are undetermined functions of T. Assuming that $v(\xi)$ has the following ansatz solution

$$
v(\xi) = \sum_{j=0}^{n} a_j(T)\mathrm{sn}^j\xi\,, \tag{28}
$$

where $\mathrm{sn}(\xi, m)$ is Jacobi elliptic sine function and $0 \le m \le 1$ is called modulus of Jacobi elliptic functions, see Refs. [20] $\sim$ [24]. We can select n to balance the derivative term of the highest order and nonlinear term in Eq. (26), then we have the final determined expansion form.

When $m \to 1$, $\mathrm{sn}\,\xi \to \tanh\xi$, so equation (28) degenerates to

$$
v(\xi) = \sum_{j=0}^{n} a_j(T)\tanh^j\xi\,. \tag{29}
$$

Notice that

$$
\mathrm{cn}^2\xi = 1 - \mathrm{sn}^2\xi\,, \tag{30}
$$

where $\mathrm{cn}\,(\xi, m)$ is Jacobi elliptic cosine function.[20−24] When $m \to 1$, $\mathrm{cn}\,\xi \to \mathrm{sech}\,\xi$, so we get cnoidal wave solution and its corresponding solitary wave solution.

For Eq. (26), the ansatz solution can be written as

$$
v = a_0(T) + a_1(T)\mathrm{sn}\,\xi + a_2(T)\mathrm{sn}^2\xi\,. \tag{31}
$$

Substituting Eq. (31) into Eq. (26) yields

$$
\begin{aligned}
&a_0' + a_1'\mathrm{sn}\,\xi + a_2'\mathrm{sn}^2\xi + a_2[f'X + g' + \alpha f a_0 + \alpha f\Gamma a_1 - (1+m^2)\beta f^3]\mathrm{cn}\,\xi\mathrm{dn}\,\xi\\
&\quad + [2a_2(f'X + g') + \alpha f(a_1^2 + 2a_0a_2) + 2\alpha\Gamma f a_2 - 8(1+m^2)\beta f^3 a_2]\mathrm{sn}\,\xi\mathrm{cn}\,\xi\mathrm{dn}\,\xi\\
&\quad + 3a_1 f[\alpha a_2 + 2m^2\beta f^2]\mathrm{sn}^2\xi\mathrm{cn}\,\xi\mathrm{dn}\,\xi + 2a_2 f(\alpha a_2 + 12m^2\beta f^2)\mathrm{sn}^3\xi\mathrm{cn}\,\xi\mathrm{dn}\,\xi = 0\,,
\end{aligned}
\tag{32}
$$

from which the undetermined parameters and functions can be determined

$$
f = k, \quad g = -kcT - k\alpha\int^T \Gamma(\tau)\mathrm{d}\tau\,, \tag{33}
$$

and

$$
\begin{aligned}
a_0 &= \frac{c}{\alpha} + 4(1+m^2)k^2\frac{\beta}{\alpha}\,,\\
a_1 &= 0\,, \qquad a_2 = -12m^2k^2\frac{\beta}{\alpha}\,,
\end{aligned}
\tag{34}
$$

where k and c are constants.

So the solution to the forced KdV equation can be written as

$$
A = \frac{c}{\alpha} - 4(2m^2-1)k^2\frac{\beta}{\alpha} + \int^T S(\tau)\mathrm{d}\tau + 12m^2k^2\frac{\beta}{\alpha}\mathrm{cn}^2\xi\,, \tag{35}
$$

which is a periodic-like solution and its corresponding soliton-like solution is

$$
A = \frac{c}{\alpha} - 4k^2\frac{\beta}{\alpha} + \int^T S(\tau)\mathrm{d}\tau + 12k^2\frac{\beta}{\alpha}\mathrm{sech}^2\xi\,, \tag{36}
$$

where $\xi = k[X - cT - \alpha \int^T \int^\tau S(\psi)\,\mathrm{d}\psi\,\mathrm{d}\tau]$.

Secondly, we consider the case without external forcing, i.e.

$$\frac{\partial A}{\partial T} + \alpha(T)A\frac{\partial A}{\partial X} + \beta(T)\frac{\partial^3 A}{\partial X^3} + \gamma(T)A = 0\,, \tag{37}$$

which is a variable-coefficient extended KdV equation, whose ansatz solution is

$$A = a_0(T) + a_1(T)\mathrm{sn}\,\xi + a_2(T)\mathrm{sn}^2\xi\,. \tag{38}$$

Substituting Eq. (38) into Eq. (37) results in

$$\gamma a_0 + a_0'(T) = \gamma a_1 + a_1'(T) = \gamma a_2 + a_2'(T) = 0\,, \tag{39a}$$

$$a_1[f'X + g' + \alpha f a_0 - (1+m^2)\beta f^3 a_2] = 0\,, \tag{39b}$$

$$2a_2(f'X+g') + \alpha f(a_1^2 + 2a_0 a_2) - 8(1+m^2)\beta f^3 a_2 = 0\,, \tag{39c}$$

$$a_1 f[\alpha a_2 + 2m^2\beta f^2] = 0\,, \tag{39d}$$

$$a_2 f[\alpha a_2 + 12m^2\beta f^2] = 0\,, \tag{39e}$$

from which we have

$$f = k\,, \quad g = -kc_0\int^T \alpha\,\mathrm{e}^{-\gamma(\tau)}\,\mathrm{d}\tau + 4(1+m^2)k^3\int^T \beta\,\mathrm{d}\tau\,,$$

$$a_0 = c_0\,\mathrm{e}^{-\gamma(T)}\,, \qquad a_1 = 0\,, \qquad a_2 = -12m^2k^2\frac{\beta}{\alpha} \tag{40}$$

with the constraint

$$\frac{\beta}{\alpha}\,\mathrm{e}^{\gamma(T)} = c_1\,, \tag{41}$$

where k, c_0, and c_1 are all none-zero constants.

So the solution to Eq. (37) is

$$A = c_0\,\mathrm{e}^{-\gamma(T)} - 12m^2k^2\frac{\beta}{\alpha}\mathrm{sn}^2\xi\,, \tag{42}$$

which is another periodic-like solution.

4 Conclusion

A simple shallow-water model with influence of external forcing on a β-plane is applied to investigate the nonlinear equatorial Rossby waves in a shear flow. By the perturbation method, the extended KdV equation with an external forcing is derived for large amplitude equatorial Rossby wave in a shear flow. And then various periodic-like structures for these equatorial Rossby waves are obtained with the help of Jacobi elliptic functions. It is shown that the results are different for equatorial Rossby waves without a source and with an external forcing, and the external source plays an important role in forming periodic-like structures. Of course, these periodic-like structures contain solitons, solitary waves, and they also have their different practical applications in explaining atmospheric events. This needs more further research. Moreover, in this paper, we only consider two special cases of external forcing and find some exact results. For more various external sources, this effort provides a better starting point for the treatment of general external sources and their impacts on the equatorial Rossby waves and climate changes.

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