Commun. Theor. Phys. (Beijing, China) **43** (2005) pp. 423–426
© International Academic Publishers
Vol. 43, No. 3, March 15, 2005

New Rational Form Solutions to mKdV Equation*

FU Zun-Tao,[1,2,†] LIU Shi-Kuo,[1] and LIU Shi-Da[1,2]

[1]School of Physics, Peking University, Beijing 100871, China[‡]

[2]State Key Laboratory for Turbulence and Complex System, Peking University, Beijing 100871, China

(Received April 19, 2004)

Abstract *In this paper, new basic functions, which are composed of three basic Jacobi elliptic functions, are chosen as components of finite expansion. This finite expansion can be taken as an ansatz and applied to solve nonlinear wave equations. As an example, mKdV equation is solved, and more new rational form solutions are derived, such as periodic solutions of rational form, solitary wave solutions of rational form, and so on.*

PACS numbers: 03.65.Ge
Key words: elliptic equation, Jacobi elliptic function, periodic wave solution, rational form

1 Introduction

We have taken elliptic equation as an intermediate transformation to solve nonlinear wave equations,[1−3] and obtained many periodic solutions and solitary wave solutions. However, there are still more research needed to do to find more solutions of different forms. In Ref. [4], we derived periodic solutions of rational forms, which are due to external forcing. It is an interesting issue to apply different methods to obtain this kind of solutions of rational forms.

In this paper, we will revisit the elliptic equation methods[1,2] and the elliptic function expansion method,[5,6] and show that we can construct this kind of solutions of rational forms just by the finite expansion methods, where new basic functions are chosen. So first of all, let us show results about this new finite expansion method.

2 Novel Application of Jacobi Elliptic Functions

In Refs. [1] and [2], nonlinear wave equations are supposed to take the following solution,

$$u = u(y) = \sum_{j=0}^{j=n} b_j y^j, \quad y = y(\xi)\,, \tag{1}$$

where y satisfies the elliptic equation,[7]

$$y'^2 = \sum_{i=0}^{4} a_i y^i, \quad a_4 \neq 0\,, \tag{2}$$

where $y' = \mathrm{d}y/\mathrm{d}\xi$, then

$$y'' = \frac{a_1}{2} + a_2 y + \frac{3a_3}{2} y^2 + 2a_4 y^3\,. \tag{3}$$

When parameters a_i $(i = 0, 1, 2, 3, 4)$ take different values, then elliptic equation (2) or (3) has different solutions, among which the interesting three are basic Jacobi elliptic functions,[7] i.e. Jacobi elliptic sine function $\mathrm{sn}(\xi, m)$, Jacobi elliptic cosine function $\mathrm{cn}(\xi, m)$, and Jacobi elliptic function of the third kind $\mathrm{dn}(\xi, m)$, where $0 \leq m \leq 1$ is called modulus of Jacobi elliptic functions, see Refs. [7] ∼ [10].

In Refs. [1] and [2], the components of the finite expansion are solutions to elliptic equation (2) or (3). Actually, there are more different basic functions can be taken as the components of the finite expansion, such as basic Jacobi elliptic functions in the elliptic function expansion method,[5,6] sine or cosine function in the sine-cosine method,[11] hyperbolic functions in the hyperbolic function expansion method,[12,13] and so on.

Here, we will introduce new basic functions, which are composed of the three basic Jacobi elliptic functions, taking the following forms

$$f(\xi) = \frac{\mathrm{sn}\,\xi}{1 + p\,\mathrm{sn}\,\xi}, \quad g(\xi) = \frac{\mathrm{cn}\,\xi}{1 + p\,\mathrm{sn}\,\xi}\,, \quad h(\xi) = \frac{\mathrm{dn}\,\xi}{1 + p\,\mathrm{sn}\,\xi}\,, \tag{4}$$

where p is a constant to be determined.

Obviously, there are the following relations between the three basic functions $f(\xi)$, $g(\xi)$, and $h(\xi)$,

$$g^2 = 1 - 2pf + (p^2 - 1)f^2\,, \quad h^2 = 1 - 2pf + (p^2 - m^2)f^2\,, \tag{5}$$

and

$$f' = gh, \quad g' = -ph + (p^2 - 1)fh\,, \quad h' = -pg + (p^2 - m^2)fg\,. \tag{6}$$

Then the finite expansion for these three basic functions can be written as

$$u = u(f, g, h) = r_0 + \sum_{j=1}^{j=n} (r_j f^j + s_j g f^{j-1} + t_j h f^{j-1})\,,$$

*The project supported by National Natural Science Foundation of China under Grant No. 40305006 and the Ministry of Science and Technology of China through Special Public Welfare Project under Grant No. 2002DIB20070

†Correspondence author, E-mail: fuzt@pku.edu.cn
‡Correspondence address

$$r_n^2 + s_n^2 + t_n^2 \neq 0\,. \tag{7}$$

If $p^2 \neq 1$ and $p^2 \neq m^2$, we suppose the rank of u is

$$O(u) = n\,, \tag{8}$$

then from Eqs. (4) and (5) one has

$$O(f) = O(g) = O(h)\,, \tag{9}$$

and from Eqs. (5) and (6) one has

$$O(u') = n+1, \quad O(u'') = n+2, \cdots,$$
$$O(u^{(j)}) = n + j\,. \tag{10}$$

Applying Eqs. (8), (9), and (10) to the finite expansion and specific nonlinear wave equation, partial balance between the highest degree nonlinear term and the highest order derivative term will let us determine the expansion rank n.

Of course, there are two special cases need consideration. The first one is $p^2 = 1$, then from Eqs. (4) $\sim$ (6) one has

$$f(\xi) = \frac{\operatorname{sn}\xi}{1 \pm \operatorname{sn}\xi}, \quad g(\xi) = \frac{\operatorname{cn}\xi}{1 \pm \operatorname{sn}\xi},$$
$$h(\xi) = \frac{\operatorname{dn}\xi}{1 \pm \operatorname{sn}\xi} \tag{11}$$

with

$$g^2 = 1 \mp 2f, \quad h^2 = 1 \mp 2f + (1-m^2)f^2\,, \tag{12}$$

and

$$f' = gh, \quad g' = \mp h, \quad h' = \mp g + (1-m^2)fg\,, \tag{13}$$

from which one can easily derive that g satisfies the elliptic equation (2) with $a_1 = a_3 = 0$, $a_0 = a_4 = (1-m^2)/4$, $a_2 = (1+m^2)/2$.

And the second case is $p^2 = m^2$, then from Eqs. (4), (5), and (6) one has

$$f(\xi) = \frac{\operatorname{sn}\xi}{1 \pm m\operatorname{sn}\xi}, \quad g(\xi) = \frac{\operatorname{cn}\xi}{1 \pm m\operatorname{sn}\xi},$$
$$h(\xi) = \frac{\operatorname{dn}\xi}{1 \pm m\operatorname{sn}\xi} \tag{14}$$

with

$$g^2 = 1 \mp 2mf + (m^2-1)f^2, \quad h^2 = 1 \mp 2mf\,, \tag{15}$$

and

$$f' = gh, \quad g' = \mp mh + (m^2-1)fh, \quad h' = \mp mg\,, \tag{16}$$

from which one can easily derive that h satisfies the elliptic equation (2) with $a_1 = a_3 = 0$, $a_0 = a_4 = -(1-m^2)/4$, $a_2 = (1+m^2)/2$.

Obviously, for the two special cases, there is no relation (9) between f, g, and h. However, we still can apply these two special cases to solve nonlinear wave equations, where the finite expansion is replaced by Eq. (1). In the next section, we will show the details.

3 Rational Form Solutions to mKdV Equation

mKdV equation reads[7]

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0\,. \tag{17}$$

We seek its travelling wave solution in the following frame:

$$u = u(\xi), \quad \xi = x - ct\,, \tag{18}$$

then equation (17) can be rewritten as

$$-cu' + \alpha u^2 u' + \beta u''' = 0\,. \tag{19}$$

First of all, we consider the general case $p^2 \neq 1$ and $p^2 \neq m^2$. Substituting Eqs. (7), (8), (9), and (10) into Eq. (19), we can derive $n = 1$, i.e., the finite expansion can be written as

$$u = r_0 + r_1 f + s_1 g + t_1 h, \quad r_1^2 + s_1^2 + t_1^2 \neq 0\,. \tag{20}$$

For the ansatz solution (20), there are some cases to be considered. We firstly show some simple cases to illustrate the applications of new basic functions and finite expansion.

Case 1 $s_1^2 + t_1^2 = 0$, then $r_1^2 \neq 0$, the ansatz solution (20) reduces to

$$u = r_0 + r_1 f, \quad r_1^2 \neq 0\,. \tag{21}$$

Combining Eq. (21) with Eq. (19) leads to

$$[-cr_1 + \alpha r_0^2 r_1 + \beta r_1(6p^2 - m^2 - 1)]gh$$
$$+[2\alpha r_0 r_1^2 - 6\beta r_1 p(2p^2 - m^2 - 1)]fgh$$
$$+[\alpha r_1^3 + 6\beta r_1(p^2-1)(p^2-m^2)]f^2gh = 0\,, \tag{22}$$

i.e.

$$-cr_1 + \alpha r_0^2 r_1 + \beta r_1(6p^2 - m^2 - 1) = 0\,, \tag{23a}$$
$$2\alpha r_0 r_1^2 - 6\beta r_1 p(2p^2 - m^2 - 1) = 0\,, \tag{23b}$$
$$\alpha r_1^3 + 6\beta r_1(p^2-1)(p^2-m^2) = 0\,, \tag{23c}$$

from which the parameters can be determined,

$$r_1 = \pm\sqrt{-\frac{6\beta}{\alpha}(p^2-1)(p^2-m^2)}\,,$$
$$c = -\frac{3\beta p^2(2p^2-m^2-1)^2}{2(p^2-1)(p^2-m^2)} + \beta(6p^2-m^2-1)\,,$$
$$r_0 = \pm\frac{3\beta p(2p^2-m^2-1)}{\alpha} \times \sqrt{-\frac{\alpha}{6\beta(p^2-1)(p^2-m^2)}}\,. \tag{24}$$

So the solution to Eq. (17)

$$u_1 = r_0 + r_1 f = \pm\frac{3\beta p(2p^2-m^2-1)}{\alpha} \times \sqrt{-\frac{\alpha}{6\beta(p^2-1)(p^2-m^2)}} \pm \sqrt{-\frac{6\beta}{\alpha}(p^2-1)(p^2-m^2)}\,\frac{\operatorname{sn}\xi}{1+p\operatorname{sn}\xi}\,, \tag{25}$$

with

$$c = -\frac{3\beta p^2(2p^2-m^2-1)^2}{2(p^2-1)(p^2-m^2)} + \beta(6p^2-m^2-1)\,. \tag{26}$$

Moreover, it is known that when $m \to 1$, $\operatorname{sn}(\xi,m) \to \tanh\xi$, $\operatorname{cn}(\xi,m) \to \operatorname{sech}\xi$, $\operatorname{dn}(\xi,m) \to \operatorname{sech}\xi$ and when $m \to 0$, $\operatorname{sn}(\xi,m) \to \sin\xi$, $\operatorname{cn}(\xi,m) \to \cos\xi$. So we can get more kinds of solutions of rational form expressed in terms of hyperbolic functions or trigonometric functions,

$$u_2 = r_0 + r_1 f = \pm\frac{3\beta p(2p^2-2)}{\alpha}\sqrt{-\frac{\alpha}{6\beta(p^2-1)^2}} \pm\sqrt{-\frac{6\beta}{\alpha}(p^2-1)^2}\frac{\tanh\xi}{1+p\tanh\xi} \tag{27}$$

with

$$c = -2\beta\,, \tag{28}$$

$$u_3 = r_0 + r_1 f = \pm\frac{3\beta p(2p^2-1)}{\alpha}\sqrt{-\frac{\alpha}{6\beta(p^2-1)p^2}} \pm\sqrt{-\frac{6\beta}{\alpha}(p^2-1)p^2}\frac{\sin\xi}{1+p\sin\xi} \tag{29}$$

with

$$c = -\frac{3\beta(2p^2-1)^2}{2(p^2-1)} + \beta(6p^2-1)\,. \tag{30}$$

Obviously, u_1, u_2, and u_3 are three novel rational form solutions to mKdV equation (17).

When $p=0$, equstions (25) and (26) reduce to

$$r_0 = 0, \quad r_1 = \pm m\sqrt{-\frac{6\beta}{\alpha}}, \quad c = -\beta(1+m^2)\,, \tag{31}$$

then

$$u_4 = r_0 + r_1 f = \pm m\sqrt{-\frac{6\beta}{\alpha}}\,\mathrm{sn}\,\xi, \quad c = -\beta(1+m^2)\,, \tag{32}$$

which is just what we obtained by Jacobi elliptic function expansion method.[5,6]

Case 2 $t_1 = 0$, and $r_1^2 + s_1^2 \neq 0$, then the ansatz solution (20) reduces to

$$u = r_0 + r_1 f + s_1 g, \quad r_1^2 + s_1^2 \neq 0\,. \tag{33}$$

Combining Eq. (33) with Eq. (19) results in

$$cps_1 + \alpha[2r_0r_1s_1 - (r_0^2 - s_1^2)s_1p] - \beta(6p^2 - m^2 - 4)s_1p = 0\,, \tag{34a}$$

$$\begin{aligned}&-cs_1(p^2-1) + \alpha[2r_1^2s_1 - 2(3r_0r_1 - ps_1^2)s_1p + (r_0^2 - s_1^2)(p^2-1)s_1] \\ &\quad + \beta s_1(p^2-1)(6p^2-4m^2-1) + 3\beta s_1p^2(4p^2-m^2-3) = 0\,,\end{aligned} \tag{34b}$$

$$-cr_1 + \alpha[(r_0^2 - s_1^2)r_1 - 2r_0s_1^2p] + \beta r_1(6p^2 - m^2 - 1) = 0\,, \tag{34c}$$

$$\alpha[2(r_0r_1 - ps_1^2)r_1 - 2r_1s_1^2p + 2r_0s_1^2(p^2-1)] - 6\beta r_1p(2p^2 - m^2 - 1) = 0\,, \tag{34d}$$

$$\begin{aligned}&\alpha\{-[r_1^2 + (p^2-1)s_1^2]s_1p + 2(r_0r_1 - ps_1^2)(p^2-1)s_1 - 4r_1^2s_1p + 2r_0r_1s_1(p^2-1)\} \\ &\quad - 2\beta s_1p(p^2-1)(4p^2-m^2-3) - 10\beta s_1p(p^2-1)(p^2-m^2) = 0\,,\end{aligned} \tag{34e}$$

$$\alpha[r_1^2 + (p^2-1)s_1^2]r_1 + 2\alpha r_1s_1^2(p^2-1) + 6\beta r_1(p^2-1)(p^2-m^2) = 0\,, \tag{34f}$$

$$\alpha[r_1^2 + (p^2-1)s_1^2]r_1(p^2-1) + 2\alpha r_1^2s_1(p^2-1) + 6\beta s_1(p^2-1)^2(p^2-m^2) = 0\,, \tag{34g}$$

from which the following results can be derived,

$$s_1 = \pm\sqrt{-\frac{3\beta}{2\alpha}(p^2-m^2)}\,, \tag{35}$$

$$r_1 = \pm\sqrt{-\frac{3\beta}{2\alpha}(p^2-1)(p^2-m^2)}\,, \tag{36}$$

$$r_0 = \mp\frac{p}{p^2-m^2}\sqrt{-\frac{3\beta}{2\alpha}(p^2-1)(p^2-m^2)}\,, \tag{37}$$

and

$$c = \beta(6p^2 - m^2 - 1) + \frac{3\beta}{2(p^2-m^2)}(m^4 + p^2 - 2p^4) \tag{38}$$

with the constraint $p^2 > 1$ for real solutions.

So the solutions to Eq. (17) are

$$u_5 = r_0 + r_1 f + s_1 g = \mp\sqrt{-\frac{3\beta}{2\alpha}(p^2-m^2)}\Big[\sqrt{p^2-1}\Big(-\frac{p}{p^2-m^2} + \frac{\mathrm{sn}\,\xi}{1+p\,\mathrm{sn}\,\xi}\Big) + \frac{\mathrm{cn}\,\xi}{1+p\,\mathrm{sn}\,\xi}\Big] \tag{39}$$

and

$$u_6 = r_0 + r_1 f + s_1 g = \mp\sqrt{-\frac{3\beta}{2\alpha}(p^2-m^2)}\Big[\sqrt{p^2-1}\Big(-\frac{p}{p^2-m^2} + \frac{\mathrm{sn}\,\xi}{1+p\,\mathrm{sn}\,\xi}\Big) - \frac{\mathrm{cn}\,\xi}{1+p\,\mathrm{sn}\,\xi}\Big]\,. \tag{40}$$

Their corresponding limited solutions are

$$u_7 = \mp\sqrt{-\frac{3\beta}{2\alpha}(p^2-1)}\Big[\sqrt{p^2-1}\Big(-\frac{p}{p^2-1} + \frac{\tanh\xi}{1+p\tanh\xi}\Big) + \frac{\mathrm{sech}\,\xi}{1+p\tanh\xi}\Big] \tag{41}$$

and

$$u_8 = \mp\sqrt{-\frac{3\beta}{2\alpha}(p^2-1)}\Big[\sqrt{p^2-1}\Big(-\frac{p}{p^2-1} + \frac{\tanh\xi}{1+p\tanh\xi}\Big) - \frac{\mathrm{sech}\,\xi}{1+p\tanh\xi}\Big] \tag{42}$$

for $m = 1$ and

$$u_9 = \mp\sqrt{-\frac{3\beta}{2\alpha}p^2}\Big[\sqrt{p^2-1}\Big(-\frac{1}{p}+\frac{\sin\xi}{1+p\sin\xi}\Big)+\frac{\cos\xi}{1+p\sin\xi}\Big] \tag{43}$$

and

$$u_{10} = \mp\sqrt{-\frac{3\beta}{2\alpha}p^2}\Big[\sqrt{p^2-1}\Big(-\frac{1}{p}+\frac{\sin\xi}{1+p\sin\xi}\Big)-\frac{\cos\xi}{1+p\sin\xi}\Big] \tag{44}$$

for $m = 0$.

Similarly, u_5, u_6, u_7, u_8, u_9, and u_{10} are six pairs of novel rational form solutions to mKdV equation (17).

Secondly, we consider the two special cases $p^2 = 1$ or $p^2 = m^2$, where g and h satisfy the elliptic equation (2) with $a_1 = a_3 = 0$, and the finite expansion is Eq. (1). Then applying the elliptic equations (2) and (3) to mKdV equation (17) yields $n = 1$, i.e. the ansatz takes the form of

$$u = b_0 + b_1 y, \quad b_1 \neq 0\,. \tag{45}$$

Similarly, combining ansatz solution (45) with Eq. (19) leads to

$$b_0 = 0, \quad b_1 = \pm\sqrt{-\frac{6\beta}{\alpha}a_4}, \quad c = \frac{\beta a_2}{b_1}\,. \tag{46}$$

If $a_0 = a_4 = (1-m^2)/4$, $a_2 = (1+m^2)/2$, then $y = g$ with $p^2 = 1$, so the solutions to mKdV equation are

$$u_{11} = b_0 + b_1 y = \pm\sqrt{-\frac{3\beta(1-m^2)}{2\alpha}}\frac{\mathrm{cn}\,\xi}{1\pm\mathrm{sn}\,\xi}\,. \tag{47}$$

If $a_0 = a_4 = -(1-m^2)/4$, $a_2 = (1+m^2)/2$, then $y = h$ with $p^2 = m^2$, so the solutions to mKdV equation are

$$u_{12} = b_0 + b_1 y = \pm\sqrt{\frac{3\beta(1-m^2)}{2\alpha}}\frac{\mathrm{dn}\,\xi}{1\pm m\,\mathrm{sn}\,\xi}\,. \tag{48}$$

u_{11} and u_{12} are another two pairs of rational form solutions to mKdV equation (17).

4 Conclusion

In this paper, we introduce three basic functions, composed of three basic Jacobi elliptic functions, in the finite expasion method and apply it to solve nonlinear wave equations. More kinds of solutions are obtained, including periodic solutions of rational forms, solitary wave solutions constructed in terms of hyperbolic functions of rational forms. As application to mKdV equation, some of the obtained solutions have not been obtained by the Jacobi elliptic function expansion method,[5,6] the sine-cosine method,[11] the hyperbolic function expansion method,[12,13] the homogeneous balance method,[14,15] the nonlinear transformation method,[16−18] the trial function method[19,20] and others. So more applications of novel basic functions to solve other nonlinear systems are also applicable and deserved, due to the available symbolic computation softwares, such as *Maple*, *Matlab*, *Mathematica*, and so on, and the complete expansion (7) can be easily applied to more nonlinear systems.

References

[1] Z.T. Fu, S.D. Liu, and S.K. Liu, Commun. Theor. Phys. (Beijing, China) **39** (2003) 531.

[2] Z.T. Fu, S.K. Liu, and S.D. Liu, Commun. Theor. Phys. (Beijing, China) **40** (2003) 285.

[3] Z.T. Fu, Z. Chen, S.K. Liu, and S.D. Liu, Commun. Theor. Phys. (Beijing, China) **41** (2004) 675.

[4] Z.T. Fu, Z. Chen, S.D. Liu, and S.K. Liu, Commun. Theor. Phys. (Being, China) **42** (2003) 43.

[5] Z.T. Fu, S.K. Liu, S.D. Liu, and Q. Zhao, Phys. Lett. **A290** (2001) 72.

[6] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Phys. Lett. **A289** (2001) 69.

[7] S.K. Liu and S.D. Liu, *Nonlinear Equations in Physics*, Peking University Press, Beijing (2000).

[8] F. Bowman, *Introduction to Elliptic Functions with Applications*, Universities, London (1959).

[9] V. Prasolov and Y. Solovyev, *Elliptic Functions and Elliptic Integrals*, American Mathematical Society, Providence R.I. (1997).

[10] Z.X. Wang and D.R. Guo, *Special Functions*, World Scietific, Singapore (1989).

[11] C.T. Yan, Phys. Lett. **A224** (1996) 77.

[12] E.G. Fan, Phys. Lett. **A277** (2000) 212.

[13] E.J. Parkes and B.R. Duffy, Phys. Lett. **A229** (1997) 217.

[14] M.L. Wang, Phys. Lett. **A199** (1995) 169.

[15] M.L. Wang, Y.B. Zhou, and Z.B. Li, Phys. Lett. **A216** (1996) 67.

[16] R. Hirota, J. Math. Phys. **14** (1973) 810.

[17] N.A. Kudryashov, Phys. Lett. **A147** (1990) 287.

[18] Z.T. Fu, S.K. Liu, and S.D. Liu, Phys. Lett. **A299** (2002) 507.

[19] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Appl. Math. Mech. **22** (2001) 326.

[20] M. Otwinowski, R. Paul, and W.G. Laidlaw, Phys. Lett. **A128** (1988) 483.