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Vortex of Fluid Field as Viewed from Curvature*

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Abstract *The vortex is a common phenomenon in fluid field. In this paper, vortex can be represented by curvature c, which varies with arc length s. The variance of point (x, y) with arc length in stream line satisfies a 2-order variable-coefficient linear ordinary differential equation. The type vortex can be analyzed qualitatively by this ordinary differential equation.*

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1 Introduction

The simple method analyzing complex fluid field is to divide fluid field into different geometry topological types. Topological fluid dynamics can be applied to fluid mechanics and magnetic fluid mechanics.[1,2] Geometrically, the center point of 2-D fluid field denotes the circular eddy fluid field, the focus point denotes spiral fluid field,[3] the saddle point denotes saddle fluid field, for example funnel structure of tornado under storm cloud layer in (x, z)-plane is a saddle fluid field,[4] the saddle-focus point in 3-D fluid field denotes updraft motion due to convergence in atmospheric lower level and divergence in atmospheric upper level.[5] This analysis methods reduce complex fluid field structure to simple geometry-topological representation. In the book "A new kind of science", the well-known specialist of cellular automata S. Wolfram[6] proposed that the simple curvature $c(s)$ which varies with arc length s can denote fairly complicated fluid field shapes. The curvature $c(s)$ can also denote the fold phenomena in nonlinear science. In this paper, from common circular vortex, spiral vortex, to Karman vortex street fluid field, we do not compute the stream line by curvature $c(s)$ but analyze qualitatively ordinary differential equation which is related to curvature $c(s)$. The vortex type can be analyzed from physical view.

2 Curvature and Common Vortex

Mathematically, the curvature c of a curve in a plane is defined as derivative of tangent direction with respect to arc length s

$$c \equiv \frac{\mathrm{d}\alpha}{\mathrm{d}s}\,, \tag{1}$$

where α is the angle between the tangent of the curve and x axis, while s is the arc length. Generally, if the curve is counter-clockwise rotation, α increases with s, $c > 0$. If the curve is clockwise rotation, α decreases with s, $c < 0$.

The reciprocal of curvature c is called curvature radius. The variation of curvature implies the variance of crooked degree for a curve. When the curve is much straight, the curvature radius is much larger and the curvature is smaller. The smaller the curvature radius, the larger the curvature, the larger crooked degree for a curve. If curvature $c(x)$ of a curve $y(x)$ is given, in order to determine the curve $y(x)$, it is necessary to seek a 2-order nonlinear ordinary differential equation, $y'' - c(x)(1 + y'^2)^{3/2} = 0$. It is very difficult. Similarly, if $c(s)$ is given, the curve $x(s)$ and $y(s)$ satisfy a similar ordinary differential equation. So, Wolfram draw out many complex curve with a computer.

However, from the point of physical view, the curvature in a plane can be taken as stream line, the structure of vortex can be obtained from variation of curvature $c(s)$. Figure 1 shows three common vortex stream lines.

From Fig. 1 it is obvious that the variation of curvature c with arc length s is very simple. In Fig. 1(a), the curvature radius is a constant (for example, equal to 1), so the curvature equates also to 1. The arrow in Fig. 1 denotes the increase of arc length. In Fig. 1(b), curvature radius decreases (or increases) with increase of arc length. For example, the form of curvature is $c(s) = s$ or $c(s) = 1/s$. In Fig. 1(c), the variation of curvature jumps between clockwise rotation and counter clockwise rotation. That is, the signs of curvature change at intervals. The transition points between clockwise rotation and counter clockwise rotation are turning points ($I_1, I_2, \ldots$ points in Fig. 1(c)). Hence, $c(s) = 0$ and $c'(s) \neq 0$ in turning points. So the

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curvature $c(s)$ of Fig. 1(c) takes oscillating form, such as $c(s) = \sin s, \cos s, \ldots$

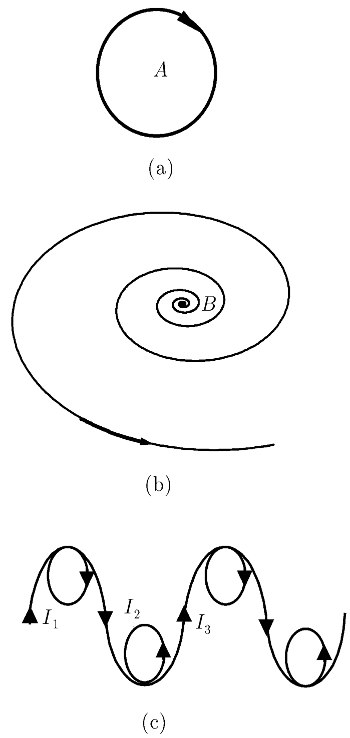

Fig. 1 Three typical curve shapes, (a) circular vortex, (b) spiral vortex, and (c) Karman vortex street.

3 The Differential Equation of $\mathrm{d}\boldsymbol{x}/\mathrm{d}\boldsymbol{s}$ and $\mathrm{d}\boldsymbol{y}/\mathrm{d}\boldsymbol{s}$

As stated in the former section, if $c(s)$ is given, it is very difficult to seek the analytical form of the curve (or stream line). We use the variation of point (x, y) with arc length s in the stream line,[7]

$$p = \frac{\mathrm{d}x}{\mathrm{d}s} = \cos\alpha \,, \qquad q = \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\alpha \,. \tag{2}$$

Obviously, the first and second derivatives of Eq. (2) with respect to arc length s yield

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}s} &= \frac{\mathrm{d}^2x}{\mathrm{d}s^2} = \sin\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s} = -c(s)q \,, \\ \frac{\mathrm{d}q}{\mathrm{d}s} &= \frac{\mathrm{d}^2y}{\mathrm{d}s^2} = \cos\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s} = c(s)p \,, \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{\mathrm{d}^2p}{\mathrm{d}s^2} &= -c'(s)q - c(s)\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{c'(s)}{c(s)}\frac{\mathrm{d}p}{\mathrm{d}s} - c^2(s)p \,, \\ \frac{\mathrm{d}^2q}{\mathrm{d}s^2} &= c'(s)p - c(s)\frac{\mathrm{d}p}{\mathrm{d}s} = \frac{c'(s)}{c(s)}\frac{\mathrm{d}q}{\mathrm{d}s} - c^2(s)q \,. \end{aligned} \tag{4}$$

Equation (4) shows that $p(s)$ and $q(s)$ satisfy the following 2-order ordinary differential equation with variable coefficients

$$\frac{\mathrm{d}^2z}{\mathrm{d}s^2} + a(s)\frac{\mathrm{d}z}{\mathrm{d}s} + b(s)z = 0 \,, \tag{5}$$

where

$$a(s) = -\frac{c'(s)}{c(s)} \,, \qquad b(s) = c^2(s) \,. \tag{6}$$

The physical meaning of Eq. (5) is that the three terms on the left-hand side denote acceleration, damping force, and restoring force,[8–11] respectively. $a(s)$ and $b(s)$ in Eq. (6) denote damping coefficient and restoring force coefficient. Though we do not find the representation of stream line $x(s)$ and $y(s)$, we can analyze Eq. (5) qualitatively to determine the type of stream line from geometry.[12]

4 Qualitative Analysis of Vortex Structure

For circular vortex in Fig. 1(a), curvature $c(s)$ is 1, from Eq. (6), we obtain $a = 0$ and $b = 1$, so equation (5) reduces to

$$[z(s)]'' + z(s) = 0 \,. \tag{7}$$

Equation (7) is an oscillation equation without damping force and positive restoring force. The equilibrium point $(0, 0)$ in phase plane $(\mathrm{d}z/\mathrm{d}s, \mathrm{d}^2z/\mathrm{d}s^2)$ is a center point,[8] as point A shown in Fig. 1(a). Hence, the stream line around point A is a circular vortex. That is to say, a vortex, whose curvature is constant is a circular vortex structure.

For the spiral vortex in Fig. 1(b), the curvature is

$$c(s) = s \,, \quad \text{and} \quad c(s) = \frac{1}{s} \,, \tag{8}$$

respectively.

Hence $a(s)$ and $b(s)$ are

$$a(s) = -\frac{1}{s}, \quad b(s) = s^2 \,; \quad a(s) = \frac{1}{s}, \quad b(s) = \frac{1}{s^2} \,, \tag{9}$$

respectively.

Equation (8) shows that the curvature $c(s)$ of point B_1 and point B_2 in Fig. 1(b) are very large, for point B_1 there is $s \to \infty$, $c(s) \to \infty$, and for point B_2 there is $s \to 0$, $c(s) \to \infty$. And the relative ratio between damping force and restoring force is

$$\begin{aligned} a^2 - 4b &= \frac{1 - 4s^4}{s^2} \quad \text{when} \quad s \to \infty \,, \\ a^2 - 4b &= -\frac{3}{s^2} \quad \text{when} \quad s \to 0 \,. \end{aligned} \tag{10}$$

According to qualitative theory of differential equation,[9] equation (10) shows that the equilibrium points B_1 and B_2 in Fig. 1(b) are of focus type. Hence the stream line of curvature (8) takes spiral structure.

For Karman vortex street in Fig. 1(c), the curvature is

$$c(s) = \sin s \,. \tag{11}$$

At the points $I_1, I_2, \ldots$ of Fig. 1(c), $s = 0, \pi, 2\pi, \ldots$, so, $c(s) = 0$ and $c'(s) = 1 \neq 0$. They are turning points. In Eq. (6) $a = 0$ and $b = 1$. At this moment, equation

(5) is an oscillating equation without damping force. The equilibrium states are center point. So the stream line around these points has a closed form such as circular vortex. When these points are approached, the damping coefficient jumps between positive and negative. Hence the stream line of Fig. 1(c) jumps between counter clockwise and clockwise.

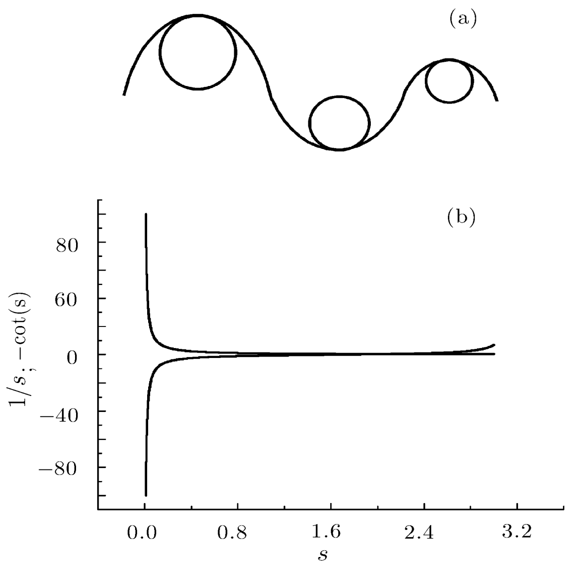

Fig. 2 (a) Another Karman vortex street; (b) The position of eddy center.

In Ref. [6], Wolfram draw another Karman vortex street, where the eddy continuously decreases, as shown in Fig. 2(a). The curvature of this eddy is

$$c(s) = s \sin s\,. \tag{12}$$

At this moment, the coefficients of Eq. (5) are

$$a(s) = -\frac{1}{s} - \cot s\ ; \qquad b(s) = s^2 \sin^2 s\,. \tag{13}$$

As stated above, the place where Karman vortex street exists is $a(s) = 0$, that is the crossover point for function $1/s$ and function $-\cot s$, as shown in Fig. 2(b).

From Fig. 2(b) it is obvious that the arc distance between two eddies decreases with the increase of arc length s.

5 Conclusion

The vortex is a common phenomenon in fluid field. In this paper, vortex is represented by curvature c, which varies with arc length s. The variance of point (x, y) with arc length in stream line satisfies a 2-order variable-coefficient linear ordinary differential equation. This ordinary differential equation is applied to analyze qualitatively the type vortex, from common circular vortex, spiral vortex to Karman vortex street etc. fluid field. We do not compute the stream line from curvature $c(s)$ but analyze qualitatively ordinary differential equation, which is related to curvature $c(s)$ to derive the vortex type from physical view.

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