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Periodic structures of oceanic Rossby wave under the influence of wind stress

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Abstract

A simple oceanic barotropic potential vorticity equation on β-plane with the influence of wind stress is applied to investigate the nonlinear Rossby wave in a shear flow. By the reductive perturbation method, we derived the rotational modified KdV (rmKdV for short) equation. And then with the help of Jacobi elliptic functions, we obtain various periodic structures for these equatorial Rossby waves. It is shown that the wind stress is very important for these periodic structures of rational form.

1. Introduction

Atmosphere and ocean construct an interactional coupled system. Ocean influences atmosphere by supplying heat and energy, and the atmosphere offers the ocean momentum by the wind stress on the interface between the ocean and the atmosphere, consequently driving the oceanic movement which called wind-driven current. So it is necessary to consider the wind stress in the studies for ocean surface motions, especially for the oceanic waves. The theory of Rossby wave and Kelvin wave is considered probably to be related with ENSO [1]. At present, there are different opinions about the development of ENSO, but the common acceptance is: in the tropical Pacific there prevails westward trade wind, when the westward trade wind weakens, even eastward wind appears, ENSO comes [1].

In this paper, a simple oceanic barotropic potential vorticity equation on β-plane with the influence of wind stress is applied to investigate the nonlinear Rossby wave in a shear flow, where the reductive perturbation method is used to derive rotational modified KdV equation, i.e. rmKdV equation. And then the basic structures of this rmKdV equation are obtained by using the knowledge of Jacobi elliptic functions and elliptic equation. We have taken elliptic equation as an intermediate transformation to solve nonlinear wave equations [2–4], and obtained many periodic solutions and solitary wave solutions. However, there are still more research needed to do in order to find more solutions of different forms. In Ref. [5], we derived periodic solutions of rational forms, which are due to external forcing. All these studies

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may help to learn about coherent structures such as atmospheric blocking events, long lived eddies in the ocean or coherent structures in the Jovian atmosphere such as the Great Red Spot.

2. Derivation of rmKdV equation with wind stress

The governing equation is an incompressible homogeneous fluid on an infinite β-plane with a free surface $h(x,y,t)$. First we consider the nonlinear horizontal momentum equations of motion as following:

$$u_t + uu_x + vu_y - \beta yv = -gh_x + \lambda\tau^x \tag{1a}$$

$$v_t + uv_x + vv_y + \beta yu = -gh_y + \lambda\tau^y \tag{1b}$$

where $\beta > 0$ is the planetary-vorticity gradient and τ^x and τ^y are the terms due to the wind stress in the x and y directions. The similar systems to Eq. (1) without nonlinear advection terms have been used to study the ENSO in Refs. [1,6]. From Eq. (1), the vorticity equation under the β-plane approximation can be easily obtained

$$\frac{\mathrm{d}(\zeta + \beta y)}{\mathrm{d}t} + (\zeta + \beta y)D = \lambda\left(\frac{\partial\tau^y}{\partial x} - \frac{\partial\tau^x}{\partial y}\right) \tag{2}$$

where ζ is the vertical vorticity and D is the horizontal divergence.

For the wind stress, the simplest assumption is that the wind stress is taken to be in proportion to the magnitude of the wind velocity above certain height, which is called Laminar flow [7]. As we know, the velocity of wind and ocean flow near their interface is continuous, so if we simply define the wind stress in proportion to the ocean flow for the surface oceanic motion, then Eq. (2) can be rewritten as

$$\frac{\mathrm{d}(\nabla^2\psi_* + \beta y)}{\mathrm{d}t} + (\nabla^2\psi_* + \beta y)D = \lambda_*\nabla^2\psi_* \tag{3}$$

where ψ_* is the stream function and ∇^2 is the horizontal Laplacian operator, λ_* is the re-scaled wind stress parameter.

If the whole layer is not divergent and the total stream function is written as

$$\psi_* = -\int^y [\bar{u}(r) - c]\,\mathrm{d}r + \psi' \tag{4}$$

then Eq. (3) can be rewritten as

$$\left[\frac{\partial}{\partial t} + (\bar{u} - c)\frac{\partial}{\partial x}\right]\nabla^2\psi' + J(\psi', \nabla^2\psi') + (\beta - \bar{u}'')\frac{\partial\psi'}{\partial x} = \lambda_*(\nabla^2\psi' - \bar{u}') \tag{5}$$

Next we take Gardner–Morikawa transformation [8], i.e.

$$X = \varepsilon x, \quad T = \varepsilon^3 t, \quad y = y, \tag{6}$$

and consider that the external forcing is weak, i.e. $\mathrm{O}(\lambda_*) \sim \mathrm{O}(\varepsilon^3)$, the perturbation stream function is expanded as

$$\psi' = \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \cdots \tag{7}$$

then from Eq. (5), we have

$$\mathrm{O}(\varepsilon) : \wp(\psi_1) = 0 \tag{8}$$

$$\mathrm{O}(\varepsilon^2) : \wp(\psi_2) = -\frac{\partial\psi_1}{\partial X}\frac{\partial^3\psi_1}{\partial y^3} + \frac{\partial\psi_1}{\partial y}\frac{\partial^3\psi_1}{\partial X\partial y^2} - \kappa\bar{u}' \tag{9}$$

$$\mathrm{O}(\varepsilon^3) : \wp(\psi_3) = -\frac{\partial\psi_1}{\partial X}\frac{\partial^3\psi_2}{\partial y^3} - \frac{\partial\psi_2}{\partial X}\frac{\partial^3\psi_1}{\partial y^3} + \frac{\partial\psi_1}{\partial y}\frac{\partial^3\psi_2}{\partial X\partial y^2} + \frac{\partial\psi_2}{\partial y}\frac{\partial^3\psi_1}{\partial X\partial y^2} - \frac{\partial^3\psi_1}{\partial T\partial y^2} - (\bar{u}' - c)\frac{\partial^3\psi_1}{\partial X^3} + \kappa\frac{\partial^2\psi_1}{\partial y^2} \tag{10}$$

with the operator $\wp$ is defined as

$$\wp() = (\bar{u}' - c)\frac{\partial^3}{\partial X\partial y^2} + (\beta - \bar{u}'')\frac{\partial}{\partial X} \tag{11}$$

ψ_1 satisfies Eq. (8), whose solution can be taken as

$$\psi_1 = A(X,T)G(y) \tag{12}$$

where $G(y)$ satisfies

$$\frac{\mathrm{d}^2G}{\mathrm{d}y^2}+QG=0 \tag{13}$$

with

$$Q\equiv\frac{\beta-\bar{u}''}{\bar{u}-c} \tag{14}$$

and boundary condition

$$G|_{y=y_1}=0,\quad G|_{y=y_2}=0 \tag{15}$$

Substituting the first-order solution (12) into the second order expansion Eq. (9), we have

$$\wp(\psi_2)=A\frac{\partial A}{\partial X}\left[\frac{\mathrm{d}G}{\mathrm{d}y}\frac{\mathrm{d}^2G}{\mathrm{d}y^2}-G\frac{\mathrm{d}^3G}{\mathrm{d}y^3}\right]-\kappa\bar{u}' \tag{16}$$

whose solution can be chosen as

$$\psi_2=A^2H(y)+\mathrm{d}X \tag{17}$$

where d is a constant.

Combining the results for ψ_1 and ψ_2 with the third order expansion Eq. (10) yields

$$\wp(\psi_3)=-\frac{\mathrm{d}^2G}{\mathrm{d}y^2}\frac{\partial A}{\partial T}-(\bar{u}-c)G\frac{\partial^3A}{\partial X^3}-\left[G\frac{\mathrm{d}^3H}{\mathrm{d}y^3}+2H\frac{\mathrm{d}^3G}{\mathrm{d}y^3}-2\frac{\mathrm{d}G}{\mathrm{d}y}\frac{\mathrm{d}^2H}{\mathrm{d}y^2}-\frac{\mathrm{d}H}{\mathrm{d}y}\frac{\mathrm{d}^2G}{\mathrm{d}y^2}\right]A^2\frac{\partial A}{\partial X}+\left(\kappa\frac{\mathrm{d}^2G}{\mathrm{d}y^2}-\mathrm{d}\frac{\mathrm{d}^3G}{\mathrm{d}y^3}\right)A \tag{18}$$

which can be multiplied by G and integrated with respect to y to reach the following rmKdV equation

$$\frac{\partial A}{\partial T}+\alpha A^2\frac{\partial A}{\partial X}+\mu\frac{\partial^3A}{\partial X^3}+\gamma A=0 \tag{19}$$

with

$$\begin{aligned}
&\alpha=\frac{I_1}{I_0},\quad \mu=\frac{I_2}{I_0},\quad \gamma=\frac{I_3}{I_0},\quad I_0=-\int_{y_1}^{y_2}\frac{G}{\bar{u}-c}\frac{\mathrm{d}^2G}{\mathrm{d}y^2}\mathrm{d}y\\
&I_1=\int_{y_1}^{y_2}\left[G\frac{\mathrm{d}^3H}{\mathrm{d}y^3}+2H\frac{\mathrm{d}^3G}{\mathrm{d}y^3}-2\frac{\mathrm{d}G}{\mathrm{d}y}\frac{\mathrm{d}^2H}{\mathrm{d}y^2}-\frac{\mathrm{d}H}{\mathrm{d}y}\frac{\mathrm{d}^2G}{\mathrm{d}y^2}\right]\frac{G}{\bar{u}-c}\mathrm{d}y\\
&I_2=-\int_{y_1}^{y_2}G^2\mathrm{d}y,\quad I_3=\int_{y_1}^{y_2}\left(\kappa\frac{\mathrm{d}^2G}{\mathrm{d}y^2}-\mathrm{d}\frac{\mathrm{d}^3G}{\mathrm{d}y^3}\right)\frac{G}{\bar{u}-c}\mathrm{d}y
\end{aligned} \tag{20}$$

3. Solutions to the rmKdV equation

Following the method mentioned in Ref. [9], we can decompose $A(X,T)$ as

$$A(X,T)=W(\sigma)V(\xi),\quad \sigma=T,\quad \xi=\delta(T)[X-\theta(T)] \tag{21}$$

where $W(\sigma)$, $V(\xi)$, $\delta(T)$ and $\theta(T)$ are certain functions for their argument to be determined. Here the decomposition (21) is equivalent to introducing Lagrangean variables [10] $\{\xi,\sigma\}$, which are related to the Eulerian ones $\{X,T\}$ through formula (21).

Substituting (21) into rmKdV Eq. (19) results in

$$\left[\frac{W_\sigma}{W}+\gamma\right]V+\frac{\delta_\sigma}{\delta}\xi V_\xi+\delta\left[-\theta_\sigma V_\xi+\alpha W^2V^2V_\xi+\mu\delta^2V_{\xi\xi\xi}\right]=0 \tag{22}$$

Here we put some restrictions on Eq. (22), i.e.

$$-\theta_\sigma V_\xi+\alpha W^2V^2V_\xi+\mu\delta^2V_{\xi\xi\xi}=0 \tag{23}$$

and

$$\left[\frac{W_\sigma}{W}+\gamma\right]V+\frac{\delta_\sigma}{\delta}\xi V_\xi=0 \tag{24}$$

Eq. (23) can be integrated once with respect to ξ to arrive at

$$-\theta_\sigma V + \frac{\alpha W^2}{3} V^3 + \mu\delta^2 V_{\xi\xi} = B(\sigma) \tag{25}$$

Eq. (25) can be solved by introducing the following fractional transformation [11]

$$V(\xi) = \frac{b_0(\sigma) + b_1(\sigma) z^2(\xi)}{1 + b_2(\sigma) z^2(\xi)} \tag{26}$$

where $z(\xi)$ satisfies elliptic equation

$$z_\xi^2 = a_0 + a_1 z^2 + a_2 z^4 \tag{27}$$

In order to obtain nontrivial solution, there is a constraint

$$b_0 b_2 - b_1 \neq 0 \tag{28}$$

so we consider two special cases, the first is

Case 1. $b_0 = 0$, $b_1 \neq 0$ and $b_2 \neq 0$

In this case, we have

$$b_1 = -\frac{B}{2a_0\mu\delta^2}, \quad b_2 = \frac{4a_1\mu\delta^2 - \theta_\sigma}{12a_0\mu\delta^2} \tag{29}$$

with constraint

$$\theta_\sigma^2 = 16\mu^2\delta^4\left(a_1^2 - 3a_0a_2\right) \tag{30}$$

and

$$\left(4a_1\mu\delta^2 - \theta_\sigma\right)^3 + 6\theta_\sigma\left(4a_1\mu\delta^2 - \theta_\sigma\right)^2 + 144a_0a_2\mu^2\delta^4\left(4a_1\mu\delta^2 - \theta_\sigma\right) - 72\alpha W^2 B = 0 \tag{31}$$

From Eq. (30), we have another constraint

$$a_1^2 - 3a_0a_2 \geqslant 0 \tag{32}$$

From Refs. [2–5,8,11], we know that there are many kinds of solutions satisfying the constraint (32) to elliptic Eq. (27). For example, when $a_0 = 1$, $a_1 = -(1 + m^2)$ and $a_2 = m^2$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and the solution is

$$z_1 = \mathrm{sn}(\xi, m) \tag{33}$$

where $0 \leqslant m \leqslant 1$, is called modulus of Jacobi elliptic functions and $\mathrm{sn}(\xi, m)$ is Jacobi elliptic sine function, see [8,12–15]. Thus $V(\xi)$ can be determined as

$$V_1 = -\frac{6B\,\mathrm{sn}^2(\xi, m)}{12\mu\delta^2 + \left[-4(1+m^2)\mu\delta^2 - \theta_\sigma\right]\mathrm{sn}^2(\xi, m)} \tag{34}$$

This result for $V(\xi)$ can be used to solve Eq. (24), which is firstly integrated with respect to ξ from $-l$ to l with l positive constant of a bounded value

$$\left[\frac{W_\sigma}{W} + \gamma - \frac{\delta_\sigma}{2\delta}\right]\int_{-l}^{l} V^2\,\mathrm{d}\xi + \frac{\delta_\sigma}{2\delta}\int_{-l}^{l}\mathrm{d}(V^2\xi) = 0 \tag{35}$$

Owing to the fact

$$\int_{-l}^{l} V_1^2\,\mathrm{d}\xi = \frac{B^2}{4\mu\delta^2}\left[\frac{(p^2+1)E(\xi)}{2p^2(p^2-1)(m^2-p^2)} + \frac{\xi}{2p^4(p^2-1)} + 2p^2(p^2-1)\pi(\xi, p^2)\right]\Bigg|_{-l}^{l}, \tag{36}$$

where $E(\xi)$ and $\pi(\xi, p^2)$ are Legendre's incomplete elliptic integrals of the second kind and the third kind, respectively, with

$$p = \frac{4(1+m^2)\mu\delta^2 + \theta_\sigma}{12\mu\delta^2} \tag{37}$$

is bounded (actually, if V takes other elliptic functions, similar results can be reached, too), so we take assumption that

$$\int_{-l}^{l} \mathrm{d}(V^2\xi) = s_1 \int_{-l}^{l} V_1^2 \,\mathrm{d}\xi \tag{38}$$

where s_1 is a constant, its value depending on the specific form of V. Then we have

$$\frac{W_\sigma}{W} + \gamma + (s_1 - 1)\frac{\delta_\sigma}{2\delta} = 0 \tag{39}$$

From Eqs. (30) and (31), one has

$$\frac{W_\sigma}{3W} = \frac{\delta_\sigma}{\delta} \tag{40}$$

Combining Eq. (39) with Eq. (40) leads to

$$W = W_0 \mathrm{e}^{-\frac{6\gamma}{5+s_1}T}, \quad \delta = \delta_0 \mathrm{e}^{-\frac{2\gamma}{5+s_1}T} \tag{41}$$

and

$$\theta = \pm\frac{\mu\delta_0^2(s_1+5)\sqrt{1-m^2+m^4}}{\gamma}\left(1 - \mathrm{e}^{-\frac{4\gamma}{5+s_1}T}\right) \tag{42}$$

So $A(X,T)$ in Eq. (21) can be written as

$$A_1 = -\frac{6W_0 B\,\mathrm{sn}^2(\xi,m)\mathrm{e}^{-\frac{6\gamma}{5+s_1}T}}{12\mu\delta^2 + [-4(1+m^2)\mu\delta^2 - \theta_\sigma]\mathrm{sn}^2(\xi,m)} \tag{43}$$

with δ and θ given by Eqs. (41) and (42), and $\xi = \delta(T)[X - \theta(T)]$.

Similarly, when a_0, a_1 and a_2 take other sets of values, there are different form solutions for V, which will results in different s_1, this indicates different W, δ, θ and $\xi = \delta(T)[X - \theta(T)]$. For example,

(1) If $a_0 = 1 - m^2$, $a_2 = 2m^2 - 1$ and $a_4 = -m^2$, then the solution is

$$z_2 = \mathrm{cn}(\xi,m), \quad V_2 = -\frac{6B\,\mathrm{cn}^2(\xi,m)}{12(1-m^2)\mu\delta^2 + \left[4(2m^2-1)\mu\delta^2 - \theta_\sigma\right]\mathrm{cn}^2(\xi,m)} \tag{44}$$

where $\mathrm{cn}(\xi,m)$ is Jacobi elliptic cosine function, see [8,12–15], and

$$A_2 = -\frac{6W_0 B\,\mathrm{cn}^2(\xi,m)\mathrm{e}^{-\frac{6\gamma}{5+s_2}T}}{12(1-m^2)\mu\delta^2 + \left[4(2m^2-1)\mu\delta^2 - \theta_\sigma\right]\mathrm{cn}^2(\xi,m)} \tag{45}$$

(2) If $a_0 = m^2 - 1$, $a_2 = 2 - m^2$ and $a_4 = -1$, then the solution is

$$z_3 = \mathrm{dn}(\xi,m), \quad V_3 = -\frac{6B\,\mathrm{dn}^2(\xi,m)}{12(m^2-1)\mu\delta^2 + \left[4(2-m^2)\mu\delta^2 - \theta_\sigma\right]\mathrm{dn}^2(\xi,m)} \tag{46}$$

where $\mathrm{dn}(\xi,m)$ is Jacobi elliptic function of the third kind, see [8,12–15], and

$$A_3 = -\frac{6W_0 B\,\mathrm{dn}^2(\xi,m)\mathrm{e}^{-\frac{6\gamma}{5+s_3}T}}{12(m^2-1)\mu\delta^2 + \left[4(2-m^2)\mu\delta^2 - \theta_\sigma\right]\mathrm{dn}^2(\xi,m)} \tag{47}$$

(3) If $a_0 = m^2$, $a_2 = -(1 + m^2)$ and $a_4 = 1$, then the solution is

$$z_4 = \mathrm{ns}(\xi,m) \equiv \frac{1}{\mathrm{sn}(\xi,m)}, \quad V_4 = -\frac{6B\,\mathrm{ns}^2(\xi,m)}{12m^2\mu\delta^2 + \left[-4(1+m^2)\mu\delta^2 - \theta_\sigma\right]\mathrm{ns}^2(\xi,m)} \tag{48}$$

and

$$A_4 = -\frac{6W_0 B\,\mathrm{ns}^2(\xi,m)\mathrm{e}^{-\frac{6\gamma}{5+s_4}T}}{12m^2\mu\delta^2 + \left[-4(1+m^2)\mu\delta^2 - \theta_\sigma\right]\mathrm{ns}^2(\xi,m)} \tag{49}$$

(4) If $a_0 = -m^2$, $a_2 = 2m^2 - 1$ and $a_4 = 1 - m^2$, then the solution is

$$z_5 = \mathrm{nc}(\xi, m) \equiv \frac{1}{\mathrm{cn}(\xi, m)}, \quad V_5 = -\frac{6B\mathrm{nc}^2(\xi, m)}{-12m^2\mu\delta^2 + [4(2m^2-1)\mu\delta^2 - \theta_\sigma]\mathrm{nc}^2(\xi, m)} \tag{50}$$

and

$$A_5 = -\frac{6W_0 B\mathrm{nc}^2(\xi, m)\mathrm{e}^{-\frac{6\gamma}{5+s_5}T}}{-12m^2\mu\delta^2 + [4(2m^2-1)\mu\delta^2 - \theta_\sigma]\mathrm{nc}^2(\xi, m)} \tag{51}$$

(5) If $a_0 = -1$, $a_2 = 2 - m^2$ and $a_4 = m^2 - 1$, then the solution is

$$z_6 = \mathrm{nd}(\xi, m) \equiv \frac{1}{\mathrm{dn}(\xi, m)}, \quad V_6 = -\frac{6B\mathrm{nd}^2(\xi, m)}{-12\mu\delta^2 + [4(2-m^2)\mu\delta^2 - \theta_\sigma]\mathrm{nd}^2(\xi, m)} \tag{52}$$

and

$$A_6 = -\frac{6W_0 B\mathrm{nd}^2(\xi, m)\mathrm{e}^{-\frac{6\gamma}{5+s_6}T}}{-12\mu\delta^2 + [4(2-m^2)\mu\delta^2 - \theta_\sigma]\mathrm{nd}^2(\xi, m)} \tag{53}$$

(6) If $a_0 = 1$, $a_2 = 2 - m^2$ and $a_4 = 1 - m^2$, then the solution is

$$z_7 = \mathrm{sc}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)}, \quad V_7 = -\frac{6B\mathrm{sc}^2(\xi, m)}{12\mu\delta^2 + [4(2-m^2)\mu\delta^2 - \theta_\sigma]\mathrm{sc}^2(\xi, m)} \tag{54}$$

and

$$A_7 = -\frac{6W_0 B\mathrm{sc}^2(\xi, m)\mathrm{e}^{-\frac{6\gamma}{5+s_7}T}}{12\mu\delta^2 + [4(2-m^2)\mu\delta^2 - \theta_\sigma]\mathrm{sc}^2(\xi, m)} \tag{55}$$

(7) If $a_0 = 1$, $a_2 = 2m^2 - 1$ and $a_4 = (m^2 - 1)m^2$, then the solution is

$$z_8 = \mathrm{sd}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}, \quad V_8 = -\frac{6B\mathrm{sd}^2(\xi, m)}{12\mu\delta^2 + [4(2m^2-1)\mu\delta^2 - \theta_\sigma]\mathrm{sd}^2(\xi, m)} \tag{56}$$

and

$$A_8 = -\frac{6W_0 B\mathrm{sd}^2(\xi, m)\mathrm{e}^{-\frac{6\gamma}{5+s_8}T}}{12\mu\delta^2 + [4(2m^2-1)\mu\delta^2 - \theta_\sigma]\mathrm{sd}^2(\xi, m)} \tag{57}$$

(8) If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$ and $a_4 = 1$, then the solution is

$$z_9 = \mathrm{cs}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)}, \quad V_9 = -\frac{6B\mathrm{cs}^2(\xi, m)}{12(1-m^2)\mu\delta^2 + [4(2-m^2)\mu\delta^2 - \theta_\sigma]\mathrm{cs}^2(\xi, m)} \tag{58}$$

and

$$A_9 = -\frac{6W_0 B\mathrm{cs}^2(\xi, m)\mathrm{e}^{-\frac{6\gamma}{5+s_9}T}}{12(1-m^2)\mu\delta^2 + [4(2-m^2)\mu\delta^2 - \theta_\sigma]\mathrm{cs}^2(\xi, m)} \tag{59}$$

(9) If $a_0 = 1$, $a_2 = -(1 + m^2)$ and $a_4 = m^2$, then the solution is

$$z_{10} = \mathrm{cd}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)}, \quad V_{10} = -\frac{6B\mathrm{cd}^2(\xi, m)}{12\mu\delta^2 + [-4(1+m^2)\mu\delta^2 - \theta_\sigma]\mathrm{cd}^2(\xi, m)} \tag{60}$$

and

$$A_{10} = -\frac{6W_0 B\mathrm{cd}^2(\xi, m)\mathrm{e}^{-\frac{6\gamma}{5+s_{10}}T}}{12\mu\delta^2 + [-4(1+m^2)\mu\delta^2 - \theta_\sigma]\mathrm{cd}^2(\xi, m)} \tag{61}$$

(10) If $a_0 = m^2(m^2 - 1)$, $a_2 = 2m^2 - 1$ and $a_4 = 1$, then the solution is

$$z_{11} = \mathrm{ds}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}, \quad V_{11} = -\frac{6B\mathrm{ds}^2(\xi, m)}{12m^2(m^2-1)\mu\delta^2 + [4(2m^2-1)\mu\delta^2 - \theta_\sigma]\mathrm{ds}^2(\xi, m)} \tag{62}$$

and

$$A_{11} = -\frac{6W_0 B \mathrm{ds}^2(\xi, m) \mathrm{e}^{-\frac{6\gamma}{5+s_{11}}T}}{12m^2(m^2-1)\mu\delta^2 + \left[4(2m^2-1)\mu\delta^2 - \theta_\sigma\right]\mathrm{ds}^2(\xi, m)} \tag{63}$$

(11) If $a_0 = m^2$, $a_2 = -(1 + m^2)$ and $a_4 = 1$, then the solution is

$$z_{12} = \mathrm{dc}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)}, \quad V_{12} = -\frac{6B\mathrm{dc}^2(\xi, m)}{12m^2\mu\delta^2 + \left[-4(1+m^2)\mu\delta^2 - \theta_\sigma\right]\mathrm{dc}^2(\xi, m)} \tag{64}$$

and

$$A_{12} = -\frac{6W_0 B \mathrm{dc}^2(\xi, m) \mathrm{e}^{-\frac{6\gamma}{5+s_{12}}T}}{12m^2\mu\delta^2 + \left[-4(1+m^2)\mu\delta^2 - \theta_\sigma\right]\mathrm{dc}^2(\xi, m)} \tag{65}$$

Case 2. $b_0 \neq 0$, $b_1 = 0$ and $b_2 \neq 0$

In this case, we have

$$b_0 = -\frac{B}{2a_2\mu\delta^2} b_2, \quad b_2 = \frac{12a_2\mu\delta^2}{4a_1\mu\delta^2 - \theta_\sigma} \tag{66}$$

with the same constraint as (30) and (31). Similarly we can obtain solutions just similar to solutions from A_1 to A_{12}, here we omit the details.

4. Conclusion

A simple oceanic barotropic potential vorticity equation on β-plane with the influence of wind stress is applied to investigate the nonlinear Rossby wave in a shear flow. By the reductive perturbation method, we derived the rmKdV equation. And then we obtain various periodic structures for these equational Rossby waves with the help of Jacobi elliptic functions. So we know that the wind stress is of great importance for these periodic structures of rational form. Of course, these periodic structures also contain solitons and solitary waves, which can be used in explaining different practical oceanic wave phenomena. This needs more further research.

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