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New Rational Form Solutions to Coupled Nonlinear Wave Equations*

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Abstract *The new rational form solutions to the elliptic equation are shown, and then these solutions to the elliptic equation are taken as a transformation and applied to solve nonlinear coupled wave equations. It is shown that more novel kinds of solutions are derived, such as periodic solutions of rational form, solitary wave solutions of rational form, and so on.*

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1 Introduction

We have taken elliptic equation as an intermediate transformation to solve nonlinear wave equations,[1−3] and obtained many periodic solutions and solitary wave solutions. However, there are still more researches to do to find more solutions of different forms. In Ref. [4], we derived periodic solutions of rational forms, which are due to external forcing. It is an interesting issue to apply different methods to obtain this kind of solutions of rational forms. In this paper, we will revisit the elliptic equation methods,[1,2] and show that we can construct this kind of solutions of rational forms just by the elliptic equation methods.[1,2] All we can do is due to that there are more kinds of solutions to elliptic equation. So first of all, let us show more solutions to elliptic equation and constrain that resulting in these solutions.

2 Novel Solutions to Elliptic Equation

In this paper, elliptic equation[5] takes the following form:

$$y'^2 = \sum_{i=0}^{i=4} a_i y^i \,, \tag{1}$$

where $y' = \mathrm{d}y/\mathrm{d}\xi$.

In Ref. [5], equation (1) is classified into four types, of which the first one is

$$y'^2 = a_0 + a_2 y^2 + a_4 y^4 \,, \quad a_4 \neq 0 \,, \tag{2}$$

which has twelve basic solutions.[5] These twelve solutions have been applied as an intermediate transformation in Refs. [1] ∼ [3] to solve nonlinear systems directly or indirectly. Many solutions have also been derived. Actually, there are many other kinds of solutions to elliptic equation (2), which are helpful to constructing more kinds of solutions to nonlinear systems. Here we show ten additional solutions of rational form to elliptic equation (2).

(i) If $a_0 = (1-m^2)/4$, $a_2 = (1+m^2)/2$, and $a_4 = (1-m^2)/4$, (where $0 \le m \le 1$ is called modulus of Jacobi elliptic functions, see Refs. [5] ∼ [8]), then the solutions to Eq. (2) are

$$y_{1a} = \frac{\mathrm{cn}\,(\xi,m)}{1+\mathrm{sn}\,(\xi,m)} = \frac{1-\mathrm{sn}(\xi,m)}{\mathrm{cn}(\xi,m)} \,, \tag{3}$$

$$y_{1b} = \frac{\mathrm{cn}(\xi,m)}{1-\mathrm{sn}\,(\xi,m)} = \frac{1+\mathrm{sn}\,(\xi,m)}{\mathrm{cn}(\xi,m)} \,, \tag{4}$$

where $\mathrm{sn}(\xi,m)$ and $\mathrm{cn}(\xi,m)$ are Jacobi elliptic sine function and cosine function, respectively (see Refs. [5] ∼ [8]).

(ii) If $a_0 = -(1-m^2)/4$, $a_2 = (1+m^2)/2$, and $a_4 = -(1-m^2)/4$, then the solutions to Eq. (2) are

$$y_{2a} = \frac{\mathrm{dn}(\xi,m)}{1+m\,\mathrm{sn}(\xi,m)} = \frac{1-m\,\mathrm{sn}(\xi,m)}{\mathrm{dn}(\xi,m)} \,, \tag{5}$$

and

$$y_{2b} = \frac{\mathrm{dn}(\xi,m)}{1-m\,\mathrm{sn}(\xi,m)} = \frac{1+m\,\mathrm{sn}(\xi,m)}{\mathrm{dn}(\xi,m)} \,, \tag{6}$$

where $\mathrm{dn}(\xi,m)$ is Jacobi elliptic function of the third kind (see Refs. [5] ∼ [8]).

(iii) If $a_0 = m^2/4$, $a_2 = -(2-m^2)/2$, and $a_4 = m^2/4$, then the solutions to Eq. (2) are

$$y_{3a} = \frac{m\,\mathrm{sn}(\xi,m)}{1+\mathrm{dn}\,(\xi,m)} = \frac{1-\mathrm{dn}\,(\xi,m)}{m\,\mathrm{sn}(\xi,m)} \,, \tag{7}$$

and

$$y_{3b} = \frac{m\,\mathrm{sn}(\xi,m)}{1-\mathrm{dn}\,(\xi,m)} = \frac{1+\mathrm{dn}\,(\xi,m)}{m\,\mathrm{sn}(\xi,m)} \,. \tag{8}$$

(iv) If $a_0 = 1/4$, $a_2 = (1-2m^2)/2$, and $a_4 = 1/4$, then the solutions to Eq. (2) are

$$y_{4a} = \frac{\mathrm{sn}(\xi,m)}{1+\mathrm{cn}(\xi,m)} = \frac{1-\mathrm{cn}(\xi,m)}{\mathrm{sn}(\xi,m)} \,, \tag{9}$$

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and

$$y_{4b} = \frac{\mathrm{sn}(\xi, m)}{1 - \mathrm{cn}(\xi, m)} = \frac{1 + \mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)} . \tag{10}$$

(v) If $a_0 = 1/4$, $a_2 = -(2 - m^2)/2$, and $a_4 = m^4/4$, then the solutions to Eq. (2) are

$$y_{5a} = \frac{\mathrm{sn}(\xi, m)}{1 + \mathrm{dn}(\xi, m)} , \tag{11}$$

and

$$y_{5b} = \frac{\mathrm{sn}(\xi, m)}{1 - \mathrm{dn}(\xi, m)} . \tag{12}$$

Moreover, it is known that when $m \to 1$, $\mathrm{sn}(\xi, m) \to \tanh\xi$, $\mathrm{cn}(\xi, m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}(\xi, m) \to \mathrm{sech}\,\xi$; and when $m \to 0$, $\mathrm{sn}(\xi, m) \to \sin\xi$, $\mathrm{cn}(\xi, m) \to \cos\xi$. So we also can get more kinds of solutions of rational form expressed in terms of hyperbolic functions and trigonometric functions.

(vi) If $a_0 = 1/4$, $a_2 = 1/2$, and $a_4 = 1/4$ with $m = 0$, then the solutions to Eq. (2) are

$$y_{6a} = \frac{\sin(\xi)}{1 + \cos(\xi)} , \tag{13}$$

and

$$y_{6b} = \frac{\sin(\xi)}{1 - \cos(\xi)} . \tag{14}$$

(vii) If $a_0 = 1/4$, $a_2 = -1/2$, and $a_4 = 1/4$ with $m = 1$, then the solutions to Eq. (2) are

$$y_{7a} = \frac{\tanh(\xi)}{1 + \mathrm{sech}\,(\xi)} , \tag{15}$$

and

$$y_{7b} = \frac{\tanh(\xi)}{1 - \mathrm{sech}\,(\xi)} . \tag{16}$$

These fourteen solutions are novel to Eq. (2), which are not shown in Refs. [1] $\sim$ [3], and [5]. So based on the above results, we can derive new rational solutions to nonlinear systems. In the next sections, we will show their applications to some coupled nonlinear wave equations.

3 Coupled mKdV Equations

We here consider coupled mKdV equations of the following form,[2]

$$u_t + \alpha u^2 u_x + \beta u_{xxx} + c_0 v_x = 0 , \tag{17a}$$

$$v_t + \gamma v v_x + \delta(uv)_x = 0 . \tag{17b}$$

Seeking their solution in the following frame,

$$u = u(\xi) , \quad v = v(\xi) , \quad \xi = x - ct , \tag{18}$$

then we can get

$$-cu' + \alpha u^2 u' + \beta u''' + c_0 v' = 0 , \tag{19a}$$

$$-cv' + \gamma v v' + \delta(uv)' = 0 . \tag{19b}$$

And then we suppose that equations (17) have the following solution:

$$u = u(y) = \sum_{j_1=0}^{j_1=n_1} b_{j_1} y^{j_1} ,$$

$$v = v(y) = \sum_{j_2=0}^{j_2=n_2} d_{j_2} y^{j_2} , \tag{20}$$

where y satisfies the elliptic equation (1), then

$$y'' = \frac{a_1}{2} + a_2 y + \frac{3a_3}{2} y^2 + 2a_4 y^3 , \tag{21a}$$

$$y''' = (a_2 + 3a_3 y + 6a_4 y^2) y' . \tag{21b}$$

There n in Eq. (20) can be determined by the partial balance between the highest order derivative terms and the highest degree nonlinear term in Eqs. (17). Here we know that the degrees of u and v are

$$O(u) = O(y^{n_1}) = n_1 , \quad O(v) = O(y^{n_2}) = n_2 , \tag{22}$$

and from Eqs. (1) and (21), one has

$$O(y'^2) = O(y^4) = 4 , \quad O(y'') = O(y^3) = 3 , \tag{23}$$

and actually one can have

$$O(y^{(l)}) = l + 1 . \tag{24}$$

So one has

$$O(u) = n_1 , \quad O(v) = n_2 , \quad O(u') = n_1 + 1 ,$$
$$O(u'') = n_1 + 2 , \quad O(u^{(l)}) = n_1 + l . \tag{25}$$

For coupled mKdV equations (17), we have $n_1 = 1$ and $n_2 = 1$, so the ansatz solution of Eq. (20) can be rewritten as

$$u = b_0 + b_1 y , \quad v = d_0 + d_1 y , \quad b_1 \neq 0 , \quad d_1 \neq 0 . \tag{26}$$

Substituting Eq. (26) into Eq. (19) leads to

$$b_0 = \mp \frac{3\beta a_3}{2\alpha} \sqrt{-\frac{\alpha}{6\beta a_4}} , \quad b_1 = \pm \sqrt{-\frac{6\beta a_4}{\alpha}} ,$$
$$d_1 = \mp \frac{2\delta}{\gamma} \sqrt{-\frac{6\beta a_4}{\alpha}} ,$$
$$d_0 = -\frac{3\beta a_3^2}{4\gamma a_4} + \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \pm \frac{3\delta\beta a_3}{\gamma\alpha} \sqrt{-\frac{\alpha}{6\beta a_4}} . \tag{27}$$

So if $a_3 = 0$, then

$$b_0 = 0 , \quad b_1 = \pm \sqrt{-\frac{6\beta a_4}{\alpha}} ,$$
$$d_1 = \mp \frac{2\delta}{\gamma} \sqrt{-\frac{6\beta a_4}{\alpha}} , \quad d_0 = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} , \tag{28}$$

and if taking the arbitrary constant $a_1 = 0$ then the transformation (1) takes the form of Eq. (2). These results combined with the rational solutions given in the above section can lead to more novel solutions to coupled mKdV equations.

(i) If $a_0 = (1 - m^2)/4$, $a_2 = (1 + m^2)/2$, and $a_4 = (1 - m^2)/4$, then the solutions to Eq. (17) are

$$u_1 = \pm\sqrt{-\frac{6\beta a_4}{\alpha}}y = \pm\sqrt{-\frac{3\beta(1-m^2)}{2\alpha}}\frac{\mathrm{cn}(\xi,m)}{1+\mathrm{sn}(\xi,m)}\,, \tag{29a}$$

$$v_1 = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y = \frac{\beta(1+m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{3\beta(1-m^2)}{2\alpha}}\frac{\mathrm{cn}(\xi,m)}{1+\mathrm{sn}(\xi,m)}\,; \tag{29b}$$

and

$$u_2 = \pm\sqrt{-\frac{6\beta a_4}{\alpha}}y = \pm\sqrt{-\frac{3\beta(1-m^2)}{2\alpha}}\frac{\mathrm{cn}(\xi,m)}{1-\mathrm{sn}(\xi,m)}\,, \tag{30a}$$

$$v_2 = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y = \frac{\beta(1+m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{3\beta(1-m^2)}{2\alpha}}\frac{\mathrm{cn}(\xi,m)}{1-\mathrm{sn}(\xi,m)}\,. \tag{30b}$$

(ii) If $a_0 = -(1-m^2)/4$, $a_2 = (1+m^2)/2$, and $a_4 = -(1-m^2)/4$, then the solutions to Eq. (17) are

$$u_3 = \pm\sqrt{-\frac{6\beta a_4}{\alpha}}y = \pm\sqrt{\frac{3\beta(1-m^2)}{2\alpha}}\frac{\mathrm{dn}(\xi,m)}{1+m\,\mathrm{sn}\,(\xi,m)}\,, \tag{31a}$$

$$v_3 = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y = \frac{\beta(1+m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{\frac{3\beta(1-m^2)}{2\alpha}}\frac{\mathrm{dn}(\xi,m)}{1+m\,\mathrm{sn}\,(\xi,m)}\,; \tag{31b}$$

and

$$u_4 = \pm\sqrt{-\frac{6\beta a_4}{\alpha}}y = \pm\sqrt{\frac{3\beta(1-m^2)}{2\alpha}}\frac{\mathrm{dn}(\xi,m)}{1-m\,\mathrm{sn}\,(\xi,m)}\,, \tag{32a}$$

$$v_4 = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y = \frac{\beta(1+m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{\frac{3\beta(1-m^2)}{2\alpha}}\frac{\mathrm{dn}(\xi,m)}{1-m\,\mathrm{sn}\,(\xi,m)}\,. \tag{32b}$$

(iii) If $a_0 = m^2/4$, $a_2 = -(2-m^2)/2$, and $a_4 = m^2/4$, then the solutions to Eq. (17) are

$$u_5 = \pm\sqrt{-\frac{6\beta a_4}{\alpha}}y = \pm\sqrt{-\frac{3\beta m^2}{2\alpha}}\frac{m\,\mathrm{sn}(\xi,m)}{1+\mathrm{dn}(\xi,m)}\,, \tag{33a}$$

$$v_5 = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y = -\frac{\beta(2-m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{3\beta m^2}{2\alpha}}\frac{m\,\mathrm{sn}\,(\xi,m)}{1+\mathrm{dn}(\xi,m)}\,; \tag{33b}$$

and

$$u_6 = \pm\sqrt{-\frac{6\beta a_4}{\alpha}}y = \pm\sqrt{-\frac{3\beta m^2}{2\alpha}}\frac{m\,\mathrm{sn}\,(\xi,m)}{1-\mathrm{dn}(\xi,m)}\,, \tag{34a}$$

$$v_6 = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y = -\frac{\beta(2-m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{3\beta m^2}{2\alpha}}\frac{m\,\mathrm{sn}\,(\xi,m)}{1-\mathrm{dn}(\xi,m)}\,. \tag{34b}$$

(iv) If $a_0 = 1/4$, $a_2 = (1-2m^2)/2$, and $a_4 = 1/4$, then the solutions to Eq. (17) are

$$u_7 = \pm\sqrt{-\frac{6\beta a_4}{\alpha}}y = \pm\sqrt{-\frac{3\beta}{2\alpha}}\frac{\mathrm{sn}(\xi,m)}{1+\mathrm{cn}(\xi,m)}\,, \tag{35a}$$

$$v_7 = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y = \frac{\beta(1-2m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{3\beta}{2\alpha}}\frac{\mathrm{sn}(\xi,m)}{1+\mathrm{cn}(\xi,m)}\,; \tag{35b}$$

and

$$u_8 = \pm\sqrt{-\frac{6\beta a_4}{\alpha}}y = \pm\sqrt{-\frac{3\beta}{2\alpha}}\frac{\mathrm{sn}(\xi,m)}{1-\mathrm{cn}(\xi,m)}\,, \tag{36a}$$

$$v_8 = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y = \frac{\beta(1-2m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{3\beta}{2\alpha}}\frac{\mathrm{sn}(\xi,m)}{1-\mathrm{cn}(\xi,m)}\,. \tag{36b}$$

(v) If $a_0 = 1/4$, $a_2 = -(2-m^2)/2$, and $a_4 = m^4/4$, then the solutions to Eq. (17) are just the same as Eqs. (33) and (34).

(vi) If $a_0 = 1/4$, $a_2 = 1/2$, and $a_4 = 1/4$ with $m = 0$, then the solutions to Eq. (17) are

$$u_9 = \pm\sqrt{-\frac{6\beta a_4}{\alpha}}y = \pm\sqrt{-\frac{3\beta}{2\alpha}}\frac{\sin(\xi)}{1+\cos(\xi)}\,, \tag{37a}$$

$$v_9=\frac{2\beta a_2}{\gamma}-\frac{4\delta c_0}{\gamma^2}\mp\frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y=\frac{\beta}{\gamma}-\frac{4\delta c_0}{\gamma^2}\mp\frac{2\delta}{\gamma}\sqrt{-\frac{3\beta}{2\alpha}}\frac{\sin(\xi)}{1+\cos(\xi)}\,; \tag{37b}$$

and

$$u_{10}=\pm\sqrt{-\frac{6\beta a_4}{\alpha}}y=\pm\sqrt{-\frac{3\beta}{2\alpha}}\frac{\sin(\xi)}{1-\cos(\xi)}\,, \tag{38a}$$

$$v_{10}=\frac{2\beta a_2}{\gamma}-\frac{4\delta c_0}{\gamma^2}\mp\frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y=\frac{\beta}{\gamma}-\frac{4\delta c_0}{\gamma^2}\mp\frac{2\delta}{\gamma}\sqrt{-\frac{3\beta}{2\alpha}}\frac{\sin(\xi)}{1-\cos(\xi)}\,. \tag{38b}$$

(vii) If $a_0=1/4$, $a_2=-1/2$, and $a_4=1/4$ with $m=1$, then the solutions to Eq. (17) are

$$u_{11}=\pm\sqrt{-\frac{6\beta a_4}{\alpha}}y=\pm\sqrt{-\frac{3\beta}{2\alpha}}\frac{\tanh(\xi)}{1+\mathrm{sech}\,(\xi)}\,, \tag{39a}$$

$$v_{11}=\frac{2\beta a_2}{\gamma}-\frac{4\delta c_0}{\gamma^2}\mp\frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y=-\frac{\beta}{\gamma}-\frac{4\delta c_0}{\gamma^2}\mp\frac{2\delta}{\gamma}\sqrt{-\frac{3\beta}{2\alpha}}\frac{\tanh(\xi)}{1+\mathrm{sech}\,(\xi)}\,; \tag{39b}$$

and

$$u_{12}=\pm\sqrt{-\frac{6\beta a_4}{\alpha}}y=\pm\sqrt{-\frac{3\beta}{2\alpha}}\frac{\tanh(\xi)}{1-\mathrm{sech}\,(\xi)}\,, \tag{40a}$$

$$v_{12}=\frac{2\beta a_2}{\gamma}-\frac{4\delta c_0}{\gamma^2}\mp\frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}y=-\frac{\beta}{\gamma}-\frac{4\delta c_0}{\gamma^2}\mp\frac{2\delta}{\gamma}\sqrt{-\frac{3\beta}{2\alpha}}\frac{\tanh(\xi)}{1-\mathrm{sech}\,(\xi)}\,. \tag{40b}$$

Actually, the above twelve sets of solutions have not been reported in literature.

4 System of Variant Boussinesq Equations

The system of variant Boussinesq equations reads[9]

$$H_t+(Hu)_x+u_{xxx}=0\,, \tag{41a}$$

$$u_t+H_x+uu_x=0\,, \tag{41b}$$

which is a model for water waves, where $u(x,t)$ is the velocity and $H(x,t)$ is the total depth.

We seek its travelling wave solutions in the following frame:

$$u=u(\xi)\,,\quad H=H(\xi)\,,\quad \xi=x-ct\,, \tag{42}$$

where c is the wave velocity.

And then we suppose that equation (41) have the following solution:

$$H=H(y)=\sum_{j_1=0}^{j_1=n_1}b_{j_1}y^{j_1}\,,\quad u=u(y)=\sum_{j_2=0}^{j_2=n_2}d_{j_2}y^{j_2}\,, \tag{43}$$

where y satisfies the elliptic Eq. (1). There n in Eq. (43) can be determined by the partial balance between the highest order derivative terms and the highest degree nonlinear term in Eqs. (41). For the system of variant Boussinesq equations (41), we have $n_1=2$ and $n_2=1$, so the ansatz solution of Eq. (43) can be rewritten as

$$H=b_0+b_1y+b_2y^2\,,\quad u=d_0+d_1y\,,$$
$$b_2\neq 0\,,\quad d_1\neq 0\,. \tag{44}$$

Then substituting Eq. (44) into Eq. (42) and collecting the each order of y yield the algebraic equations about coefficients $b_j(j=0,1,2)$, $d_j(j=0,1)$, and $a_i(i=0,1,2,3,4)$, we have

$$b_2=-2a_4\,,\quad b_1=-a_3\,,\quad b_0=-\frac{1}{2}\Big[a_2\pm\frac{ca_3}{2\sqrt{a_4}}\Big]\,,$$

$$d_1=\pm2\sqrt{a_4}\,,\quad d_0=c\pm\frac{a_3}{2\sqrt{a_4}}\,. \tag{45}$$

So if $a_3=0$, then

$$b_2=-2a_4\,,\quad b_1=0\,,\quad b_0=-\frac{a_2}{2}\,,$$
$$d_0=c\,,\quad d_1=\pm2\sqrt{a_4}\,, \tag{46}$$

and if taking the arbitrary constant $a_1=0$ then the transformation (1) takes the form of Eq. (2). Obviously, the above solutions require constraint $a_4>0$, so the solutions have the following ten sets.

(i) If $a_0=(1-m^2)/4$, $a_2=(1+m^2)/2$, and $a_4=(1-m^2)/4$, then the solutions to Eq. (41) are

$$u_1=c\pm2\sqrt{a_4}y=c\pm\sqrt{1-m^2}\frac{\mathrm{cn}(\xi,m)}{1+\mathrm{sn}(\xi,m)}\,, \tag{47a}$$

$$H_1=-\frac{a_2}{2}-2a_4y^2=-\frac{1+m^2}{4}-\frac{(1-m^2)\mathrm{cn}^2(\xi,m)}{2[1+\mathrm{sn}(\xi,m)]^2}\,; \tag{47b}$$

and

$$u_2=c\pm2\sqrt{a_4}y=c\pm\sqrt{1-m^2}\frac{\mathrm{cn}(\xi,m)}{1-\mathrm{sn}(\xi,m)}\,, \tag{48a}$$

$$H_2=-\frac{a_2}{2}-2a_4y^2=-\frac{1+m^2}{4}-\frac{(1-m^2)\mathrm{cn}^2(\xi,m)}{2[1-\mathrm{sn}(\xi,m)]^2}\,. \tag{48b}$$

(ii) If $a_0=m^2/4$, $a_2=-(2-m^2)/2$, and $a_4=m^2/4$, then the solutions to Eq. (41) are

$$u_3=c\pm2\sqrt{a_4}y=c\pm\frac{m^2\mathrm{sn}(\xi,m)}{1+\mathrm{dn}(\xi,m)}\,, \tag{49a}$$

$$H_3=-\frac{a_2}{2}-2a_4y^2=\frac{2-m^2}{4}-\frac{m^4\mathrm{sn}^2(\xi,m)}{2[1+\mathrm{dn}(\xi,m)]^2}\,; \tag{49b}$$

and

$$u_4=c\pm2\sqrt{a_4}y=c\pm\frac{m^2\mathrm{sn}(\xi,m)}{1-\mathrm{dn}(\xi,m)}\,, \tag{50a}$$

$$H_4 = -\frac{a_2}{2} - 2a_4y^2 = \frac{2-m^2}{4} - \frac{m^4\mathrm{sn}^2(\xi,m)}{2[1-\mathrm{dn}(\xi,m)]^2}\,. \quad (50b)$$

(iii) If $a_0 = 1/4$, $a_2 = (1-2m^2)/2$, and $a_4 = 1/4$, then the solutions to Eq. (41) are

$$u_5 = c \pm 2\sqrt{a_4}y = c \pm \frac{\mathrm{sn}(\xi,m)}{1+\mathrm{cn}(\xi,m)}\,, \quad (51a)$$

$$H_5 = -\frac{a_2}{2} - 2a_4y^2 = -\frac{1-2m^2}{4} - \frac{\mathrm{sn}^2(\xi,m)}{2[1+\mathrm{cn}(\xi,m)]^2}\,; \quad (51b)$$

and

$$u_6 = c \pm 2\sqrt{a_4}y = c \pm \frac{\mathrm{sn}(\xi,m)}{1-\mathrm{cn}(\xi,m)}\,, \quad (52a)$$

$$H_6 = -\frac{a_2}{2} - 2a_4y^2 = -\frac{1-2m^2}{4} - \frac{\mathrm{sn}^2(\xi,m)}{2[1-\mathrm{cn}(\xi,m)]^2}\,. \quad (52b)$$

(iv) If $a_0 = 1/4$, $a_2 = -(2-m^2)/2$, and $a_4 = m^4/4$, then the solutions to Eq. (41) are just the same as Eqs. (49) and (50).

(v) If $a_0 = 1/4$, $a_2 = 1/2$, and $a_4 = 1/4$ with $m = 0$, then the solutions to Eq. (41) are

$$u_7 = c \pm 2\sqrt{a_4}y = c \pm \frac{\sin(\xi)}{1+\cos(\xi)}\,, \quad (53a)$$

$$H_7 = -\frac{a_2}{2} - 2a_4y^2 = -\frac{1}{4} - \frac{\sin^2(\xi)}{2[1+\cos(\xi)]^2}\,; \quad (53b)$$

and

$$u_8 = c \pm 2\sqrt{a_4}y = c \pm \frac{\sin(\xi)}{1-\cos(\xi)}\,, \quad (54a)$$

$$H_8 = -\frac{a_2}{2} - 2a_4y^2 = -\frac{1}{4} - \frac{\sin^2(\xi)}{2[1-\cos(\xi)]^2}\,; \quad (54b)$$

(vi) If $a_0 = 1/4$, $a_2 = -1/2$, and $a_4 = 1/4$ with $m = 1$, then the solutions to Eq. (41) are

$$u_9 = c \pm 2\sqrt{a_4}y = c \pm \frac{\tanh(\xi)}{1+\mathrm{sech}\,(\xi)}\,, \quad (55a)$$

$$H_9 = -\frac{a_2}{2} - 2a_4y^2 = \frac{1}{4} - \frac{\tanh^2(\xi)}{2[1+\mathrm{sech}\,(\xi)]^2}\,; \quad (55b)$$

and

$$u_{10} = c \pm 2\sqrt{a_4}y = c \pm \frac{\tanh(\xi)}{1-\mathrm{sech}\,(\xi)}\,, \quad (56a)$$

$$H_{10} = -\frac{a_2}{2} - 2a_4y^2 = \frac{1}{4} - \frac{\tanh^2(\xi)}{2[1-\mathrm{sech}\,(\xi)]^2}\,. \quad (56b)$$

5 Coupled Nonlinear Klein–Gordon Schrödinger Equations

Coupled nonlinear Klein–Gordon Schrödinger equations[2] reads

$$u_{tt} - c_0^2u_{xx} + f_0^2u - \gamma|v|^2 = 0\,, \quad (57a)$$

$$\mathrm{i}v_t + \alpha v_{xx} + \beta uv = 0\,. \quad (57b)$$

We solve Eqs. (57) in the following frame:

$$u = u(\xi)\,, \quad v = \phi(\xi)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \quad \xi = p(x - c_gt)\,. \quad (58)$$

Substituting Eq. (58) into Eqs. (57) leads to

$$p^2(c_g^2 - c_0^2)u'' + f_0^2u - \gamma\phi^2 = 0\,, \quad (59a)$$

$$\alpha p^2\phi'' + \mathrm{i}p(2\alpha k - c_g)\phi' + (\omega - \alpha k^2)\phi + \beta u\phi = 0\,. \quad (59b)$$

Setting $c_g = 2\alpha k$, $\omega - \alpha k^2 = -\delta$, then one has

$$u'' + f_1u - \gamma_1\phi^2 = 0\,, \quad (60a)$$

$$\phi'' - \delta_1\phi + \beta_1u\phi = 0\,, \quad (60b)$$

where

$$f_1 = \frac{f_0^2}{p^2(c_g^2 - c_0^2)}\,, \quad \gamma_1 = \frac{\gamma}{p^2(c_g^2 - c_0^2)}\,,$$

$$\delta_1 = \frac{\delta}{\alpha p^2}\,, \quad \beta_1 = \frac{\beta}{\alpha p^2}\,. \quad (61)$$

Similarly, we assume that the solutions of Eqs. (60) take the form of Eq. (43), and then get $n_1 = n_2 = 2$ for Eqs. (60), i.e.

$$u = b_0 + b_1y + b_2y^2\,, \quad \phi = d_0 + d_1y + d_2y^2\,,$$

$$b_2 \neq 0\,, \quad d_2 \neq 0\,, \quad (62)$$

where y satisfies elliptic equation (1). Then substituting Eq. (62) into Eqs. (60) leads to

$$b_2 = -\frac{6a_4}{\beta_1}\,, \quad b_1 = -\frac{3a_3}{\beta_1}\,,$$

$$b_0 = \frac{1}{\beta_1}\Big[\delta_1 + \frac{f_1}{2} + \frac{3a_3^2}{8a_4} - 2a_2\Big]\,, \quad (63)$$

and

$$d_2 = \pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}\,, \quad d_1 = \pm\frac{3a_3}{\sqrt{-\beta_1\gamma_1}}\,,$$

$$d_0 = \pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big[2a_2 + \frac{f_1}{2} - \frac{3a_3^2}{8a_4}\Big]\,. \quad (64)$$

If $a_3 = 0$, then $b_1 = d_1 = a_1 = 0$ and

$$b_0 = \frac{1}{\beta_1}\Big[\delta_1 + \frac{f_1}{2} - 2a_2\Big]\,,$$

$$d_0 = \pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big[2a_2 + \frac{f_1}{2}\Big]\,, \quad (65)$$

then the transformation takes the form of Eq. (2). Similarly, we can derive some rational solutions to Eqs. (57), too.

(i) If $a_0 = (1-m^2)/4$, $a_2 = (1+m^2)/2$, and $a_4 = (1-m^2)/4$, then the solutions to Eq. (57) are

$$u_1 = \frac{1}{\beta_1}\Big(\delta_1 + \frac{f_1}{2} - 2a_2\Big) - \frac{6a_4}{\beta_1}y^2 = \frac{1}{\beta_1}\Big(\delta_1 + \frac{f_1}{2} - 1 - m^2\Big) - \frac{3(1-m^2)}{2\beta_1}\frac{\mathrm{cn}^2(\xi,m)}{[1+\mathrm{sn}(\xi,m)]^2}\,, \quad (66a)$$

$$v_1 = \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}(2a_2 + \frac{f_1}{2}) \pm \frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$

$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(1+m^2+\frac{f_1}{2}\Big)\pm\frac{3(1-m^2)}{2\sqrt{-\beta_1\gamma_1}}\frac{\mathrm{cn}^2(\xi,m)}{[1+\mathrm{sn}(\xi,m)]^2}\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,; \tag{66b}$$

and

$$u_2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-2a_2\Big)-\frac{6a_4}{\beta_1}y^2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-1-m^2\Big)-\frac{3(1-m^2)}{2\beta_1}\frac{\mathrm{cn}^2(\xi,m)}{[1-\mathrm{sn}(\xi,m)]^2}\,, \tag{67a}$$

$$v_2=\Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(2a_2+\frac{f_1}{2}\Big)\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$

$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(1+m^2+\frac{f_1}{2}\Big)\pm\frac{3(1-m^2)}{2\sqrt{-\beta_1\gamma_1}}\frac{\mathrm{cn}^2(\xi,m)}{[1-\mathrm{sn}(\xi,m)]^2}\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{67b}$$

(ii) If $a_0=-(1-m^2)/4$, $a_2=(1+m^2)/2$, and $a_4=-(1-m^2)/4$, then the solutions to Eq. (57) are

$$u_3=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-2a_2\Big)-\frac{6a_4}{\beta_1}y^2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-1-m^2\Big)+\frac{3(1-m^2)}{2\beta_1}\frac{\mathrm{dn}^2(\xi,m)}{[1+m\,\mathrm{sn}(\xi,m)]^2}\,, \tag{68a}$$

$$v_3=\Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(2a_2+\frac{f_1}{2}\Big)\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$

$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(1+m^2+\frac{f_1}{2}\Big)\mp\frac{3(1-m^2)}{2\sqrt{-\beta_1\gamma_1}}\frac{\mathrm{dn}^2(\xi,m)}{[1+m\,\mathrm{sn}(\xi,m)]^2}\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,; \tag{68b}$$

and

$$u_4=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-2a_2\Big)-\frac{6a_4}{\beta_1}y^2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-1-m^2\Big)+\frac{3(1-m^2)}{2\beta_1}\frac{\mathrm{dn}^2(\xi,m)}{[1-m\,\mathrm{sn}(\xi,m)]^2}\,, \tag{69a}$$

$$v_4=\Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(2a_2+\frac{f_1}{2}\Big)\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$

$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(1+m^2+\frac{f_1}{2}\Big)\mp\frac{3(1-m^2)}{2\sqrt{-\beta_1\gamma_1}}\frac{\mathrm{dn}^2(\xi,m)}{[1-m\,\mathrm{sn}(\xi,m)]^2}\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{69b}$$

(iii) If $a_0=m^2/4$, $a_2=-(2-m^2)/2$, and $a_4=m^2/4$, then the solutions to Eq. (57) are

$$u_5=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-2a_2\Big)-\frac{6a_4}{\beta_1}y^2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}+2-m^2\Big)-\frac{3m^2}{2\beta_1}\frac{m^2\mathrm{sn}^2(\xi,m)}{[1+\mathrm{dn}(\xi,m)]^2}\,, \tag{70a}$$

$$v_5=\Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(2a_2+\frac{f_1}{2}\Big)\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$

$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(-2+m^2+\frac{f_1}{2}\Big)\pm\frac{3m^2}{2\sqrt{-\beta_1\gamma_1}}\frac{m^2\mathrm{sn}^2(\xi,m)}{[1+\mathrm{dn}(\xi,m)]^2}\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,; \tag{70b}$$

and

$$u_6=\frac{1}{\beta_1}(\delta_1+\frac{f_1}{2}-2a_2)-\frac{6a_4}{\beta_1}y^2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}+2-m^2\Big)-\frac{3m^2}{2\beta_1}\frac{m^2\mathrm{sn}^2(\xi,m)}{[1-\mathrm{dn}(\xi,m)]^2}\,, \tag{71a}$$

$$v_6=\Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(2a_2+\frac{f_1}{2}\Big)\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$

$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(-2+m^2+\frac{f_1}{2}\Big)\pm\frac{3m^2}{2\sqrt{-\beta_1\gamma_1}}\frac{m^2\mathrm{sn}^2(\xi,m)}{[1-\mathrm{dn}(\xi,m)]^2}\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{71b}$$

(iv) If $a_0=1/4$, $a_2=(1-2m^2)/2$, and $a_4=1/4$, then the solutions to Eq. (57) are

$$u_7=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-2a_2\Big)-\frac{6a_4}{\beta_1}y^2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-1+2m^2\Big)-\frac{3}{2\beta_1}\frac{\mathrm{sn}^2(\xi,m)}{[1+\mathrm{cn}(\xi,m)]^2}\,, \tag{72a}$$

$$v_7=\Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(2a_2+\frac{f_1}{2}\Big)\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$

$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(1-2m^2+\frac{f_1}{2}\Big)\pm\frac{3}{2\sqrt{-\beta_1\gamma_1}}\frac{\mathrm{sn}^2(\xi,m)}{[1+\mathrm{cn}(\xi,m)]^2}\Big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,; \tag{72b}$$

and

$$u_8=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-2a_2\Big)-\frac{a_4}{\beta_1}y^2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-1+2m^2\Big)-\frac{3}{2\beta_1}\frac{\mathrm{sn}^2(\xi,m)}{[1-\mathrm{cn}(\xi,m)]^2}\,, \tag{73a}$$

$$v_8 = \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(2a_2+\frac{f_1}{2}\Big)\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]e^{i(kx-\omega t)}$$
$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(1-2m^2+\frac{f_1}{2}\Big)\pm\frac{3}{2\sqrt{-\beta_1\gamma_1}}\frac{\mathrm{sn}^2(\xi,m)}{[1-\mathrm{cn}(\xi,m)]^2}\Big]e^{i(kx-\omega t)}\,. \tag{73b}$$

(v) If $a_0=1/4$, $a_2=-(2-m^2)/2$, and $a_4=m^4/4$, then the solutions to Eq. (57) are just the same as Eqs. (70) and (71).

(vi) If $a_0=1/4$, $a_2=1/2$, and $a_4=1/4$ with $m=0$, then the solutions to Eq. (57) are

$$u_9=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-2a_2\Big)-\frac{6a_4}{\beta_1}y^2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-1\Big)-\frac{3}{2\beta_1}\frac{\sin^2(\xi)}{[1+\cos(\xi)]^2}\,, \tag{74a}$$
$$v_9 = \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(2a_2+\frac{f_1}{2}\Big)\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]e^{i(kx-\omega t)}$$
$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(1+\frac{f_1}{2}\Big)\pm\frac{3}{2\sqrt{-\beta_1\gamma_1}}\frac{\sin^2(\xi)}{[1+\cos(\xi)]^2}\Big]e^{i(kx-\omega t)}\,; \tag{74b}$$

and

$$u_{10}=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-2a_2\Big)-\frac{6a_4}{\beta_1}y^2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-1\Big)-\frac{3}{2\beta_1}\frac{\sin^2(\xi)}{[1-\cos(\xi)]^2}\,, \tag{75a}$$
$$v_{10} = \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(2a_2+\frac{f_1}{2}\Big)\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]e^{i(kx-\omega t)}$$
$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(1+\frac{f_1}{2}\Big)\pm\frac{3}{2\sqrt{-\beta_1\gamma_1}}\frac{\sin^2(\xi)}{[1-\cos(\xi)]^2}\Big]e^{i(kx-\omega t)}\,. \tag{75b}$$

(vii) If $a_0=1/4$, $a_2=-1/2$, and $a_4=1/4$ with $m=1$, then the solutions to Eq. (57) are

$$u_{11}=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-2a_2\Big)-\frac{6a_4}{\beta_1}y^2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}+1\Big)-\frac{3}{2\beta_1}\frac{\tanh^2(\xi)}{[1+\mathrm{sech}\,(\xi)]^2}\,, \tag{76a}$$
$$v_{11} = \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(2a_2+\frac{f_1}{2}\Big)\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]e^{i(kx-\omega t)}$$
$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(-1+\frac{f_1}{2}\Big)\pm\frac{3}{2\sqrt{-\beta_1\gamma_1}}\frac{\tanh^2(\xi)}{[1+\mathrm{sech}\,(\xi)]^2}\Big]e^{i(kx-\omega t)}\,; \tag{76b}$$

and

$$u_{12}=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}-2a_2\Big)-\frac{6a_4}{\beta_1}y^2=\frac{1}{\beta_1}\Big(\delta_1+\frac{f_1}{2}+1\Big)-\frac{3}{2\beta_1}\frac{\tanh^2(\xi)}{[1-\mathrm{sech}\,(\xi)]^2}\,, \tag{77a}$$
$$v_{12} = \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(2a_2+\frac{f_1}{2}\Big)\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]e^{i(kx-\omega t)}$$
$$= \Big[\pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big(-1+\frac{f_1}{2}\Big)\pm\frac{3}{2\sqrt{-\beta_1\gamma_1}}\frac{\tanh^2(\xi)}{[1-\mathrm{sech}\,(\xi)]^2}\Big]e^{i(kx-\omega t)}\,. \tag{77b}$$

6 Conclusion

In this paper, we consider elliptic equation as a transformation to solve nonlinear wave equations. More kinds of solutions can be obtained since there are more new solutions to elliptic equation, including periodic solutions of rational forms and solitary wave solutions constructed in terms of hyperbolic functions of rational forms. Applying transformation (1) to some coupled nonlinear wave equations, we obtained solutions that have not been obtained by the sine-cosine method,[10] the homogeneous balance method,[9,11] the hyperbolic function expansion method,[12,13] the Jacobi elliptic function expansion method,[14,15] the nonlinear transformation method,[16,17] the trial function method,[18,19] or others.[20] So more applications of novel solutions of elliptic equation to solve other nonlinear systems are also applicable and deserved.

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