Commun. Theor. Phys. (Beijing, China) **45** (2006) pp. 55–60
© International Academic Publishers Vol. 45, No. 1, January 15, 2006

Exact Jacobian Elliptic Function Solutions to sinh-Gordon Equation*

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(Received May 27, 2005)

Abstract *In this paper, two transformations are introduced to solve sinh-Gordon equation by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different transformations are required in order to obtain more kinds of solutions to the sinh-Gordon equation.*

PACS numbers: 03.65.Ge
Key words: Jacobian elliptic function, transformation, sinh-Gordon equation

1 Introduction

The sinh-Gordon (ShG for short) equation[1−8]

$$u_{xt} = \alpha \,\mathrm{sinh} u \tag{1}$$

is widely applied in physics and engineering, for example, integrable quantum field theories,[1] noncommutative field theories,[2] fluid dynamics,[3] and so on. Due to the wide applications of sinh-Gordon equation, many achievements have been obtained in different aspects.[3−9] For instance, ShG equation is known to be completely integrable[4] because it possesses similarity reductions to the third Painlevé equation.[5]

Due to the special form of the sinh-Gordon equation, it is very difficult to solve them directly, so one needs some transformations. In this paper, based on the introduced transformations, we will show systematical results about solutions for ShG equation (1) by using the knowledge of elliptic equation and Jacobian elliptic functions.[10−12]

2 The First Kind of Transformation and Solutions to ShG Equation

The first transformation is introduced in the form

$$u = 2\sinh^{-1} v \quad or \quad v = \sinh\frac{u}{2}\,, \tag{2}$$

and then

$$\mathrm{sinh} u = 2\sinh\frac{u}{2}\cosh\frac{u}{2} = 2v\sqrt{1+v^2}\,, \tag{3}$$

and

$$u_{tx} = \frac{2}{\sqrt{1+v^2}}v_{tx} - \frac{2v}{(1+v^2)\sqrt{1+v^2}}v_t v_x\,. \tag{4}$$

Combining Eqs. (3) and (4) with Eq. (1), the ShG equation can be rewritten as

$$(1+v^2)v_{tx} - v v_t v_x - \alpha v(1+v^2)^2 = 0\,. \tag{5}$$

Equation (5) can be solved in the frame

$$v = v(\xi), \quad \xi = k(x-ct)\,, \tag{6}$$

where k and c are wave number and wave speed, respectively.

Substituting Eqs. (6) into Eqs. (5), we have

$$(1+v^2)\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} - v\Big(\frac{\mathrm{d}v}{\mathrm{d}\xi}\Big)^2 + \alpha_1 v(1+v^2)^2 = 0\,,$$
$$\alpha_1 \equiv \frac{\alpha}{k^2 c}\,. \tag{7}$$

And then we suppose equation (7) has the following solution:

$$v = v(y) = \sum_{j=0}^{j=n} b_j y^j, \quad b_n \neq 0, \quad y = y(\xi)\,, \tag{8}$$

where y satisfies elliptic equation[10,13]

$$y'^2 = a_0 + a_2 y^2 + a_4 y^4, \quad a_4 \neq 0, \quad y' \equiv \frac{\mathrm{d}y}{\mathrm{d}\xi}\,. \tag{9}$$

There n in Eq. (8) can be determined by the partial balance between the highest order derivative terms and the highest degree nonlinear term in Eq. (7). Here we know that the degree of v is

$$O(v) = O(y^n) = n\,, \tag{10}$$

and from Eq. (9), one has

$$O(y'^2) = O(y^4) = 4\,,$$
$$O(y'') = O(y^3) = 3\,,$$
$$O(y^{(l)}) = l+1\,. \tag{11}$$

So one has

$$O(v) = n, \quad O(v') = n+1\,,$$

*The project supported by National Natural Science Foundation of China under Grant No. 40305006
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$$O(v'') = n + 2, \quad O(v^{(l)}) = n + l\,. \tag{12}$$

For ShG equation (1), we have $n = 1$, so the ansatz solution of Eq. (7) can be rewritten as

$$v = b_0 + b_1 y, \quad b_1 \neq 0\,. \tag{13}$$

Substituting Eq. (13) into Eq. (7) results in an algebraic equation for y, which can be used to determine expansion coefficients in Eq. (13) and some constraints can also be obtained. Here we have

$$b_0 = 0, \quad b_1 = \pm\sqrt{-\frac{a_4}{\alpha_1}}\,,$$
$$\alpha_1 = \frac{\alpha}{k^2 c} = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_0 a_4}}{2}\,. \tag{14}$$

In order to derive real solutions, there are two constraints,

$$-\frac{a_4}{\alpha_1} > 0, \quad a_2^2 - 4a_0 a_4 \geq 0\,. \tag{15}$$

There are some cases to be considered. For example,

Case 1 If $a_0 = 1 - m^2$, $a_2 = 2m^2 - 1$, $a_4 = -m^2$, where $0 \leq m \leq 1$ is called modulus of Jacobian elliptic functions,[13−15] then

$$y = \operatorname{cn}\xi, \quad b_0 = 0, \quad b_1 = \pm\frac{m}{\sqrt{1 - m^2}}\,,$$
$$c = \frac{\alpha}{k^2(1 - m^2)}, \quad 0 < m < 1\,, \tag{16}$$

where k is an arbitrary constant, and $\operatorname{cn}\xi$ is Jacobian elliptic cosine function.[13−15] So the solution to ShG equation (1) is

$$u_1 = 2\sinh^{-1}\Big(\pm\frac{m}{\sqrt{1 - m^2}}\operatorname{cn}\xi\Big)\,. \tag{17}$$

Case 2 If $a_0 = -m^2$, $a_2 = 2m^2 - 1$, $a_4 = 1 - m^2$, then

$$y = \operatorname{nc}\xi \equiv \frac{1}{\operatorname{cn}\xi}, \quad b_0 = 0\,,$$
$$b_1 = \pm\frac{\sqrt{1 - m^2}}{m}\,,$$
$$c = -\frac{\alpha}{m^2 k^2}, \quad 0 < m < 1\,, \tag{18}$$

where k is an arbitrary constant, and then the solution to ShG equation (1) is

$$u_2 = 2\sinh^{-1}\Big(\pm\frac{\sqrt{1 - m^2}}{m}\operatorname{nc}\xi\Big)\,. \tag{19}$$

Case 3 If $a_0 = 1$, $a_2 = 2 - m^2$, $a_4 = 1 - m^2$, then

$$y = \operatorname{sc}\xi \equiv \frac{\operatorname{sn}\xi}{\operatorname{cn}\xi}, \quad b_0 = 0, \quad b_1 = \pm 1\,,$$
$$c = -\frac{\alpha}{(1 - m^2)k^2}, \quad 0 < m < 1\,, \tag{20}$$

where k is an arbitrary constant, and $\operatorname{sn}\xi$ is Jacobian elliptic sine function.[13−15] So the solution to ShG equation (1) is

$$u_3 = 2\sinh^{-1}(\pm\operatorname{sc}\xi)\,. \tag{21}$$

Case 4 If $a_0 = 1$, $a_2 = 2 - m^2$, $a_4 = 1 - m^2$, then

$$y = \operatorname{sc}\xi, \quad b_0 = 0, \quad b_1 = \pm\sqrt{1 - m^2}\,,$$
$$c = -\frac{\alpha}{k^2}, \quad 0 < m < 1\,, \tag{22}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_4 = 2\sinh^{-1}(\pm\sqrt{1 - m^2}\operatorname{sc}\xi)\,. \tag{23}$$

Case 5 If $a_0 = 1$, $a_2 = 2m^2 - 1$, $a_4 = (m^2 - 1)m^2$, then

$$y = \operatorname{sd}\xi \equiv \frac{\operatorname{sn}\xi}{\operatorname{dn}\xi}, \quad b_0 = 0, \quad b_1 = \pm m\,,$$
$$c = \frac{\alpha}{(1 - m^2)k^2}, \quad 0 < m < 1\,, \tag{24}$$

where k is an arbitrary constant, and $\operatorname{dn}\xi$ is Jacobian elliptic function of the third kind.[13−15] So the solution to ShG equation (1) is

$$u_5 = 2\sinh^{-1}(\pm m\operatorname{sd}\xi)\,. \tag{25}$$

Case 6 If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$, $a_4 = 1$, then

$$y = \operatorname{cs}\xi \equiv \frac{\operatorname{cn}\xi}{\operatorname{sn}\xi}, \quad b_0 = 0, \quad b_1 = \pm 1\,,$$
$$c = -\frac{\alpha}{k^2}, \quad 0 < m \leq 1\,, \tag{26}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_6 = 2\sinh^{-1}(\pm\operatorname{cs}\xi)\,. \tag{27}$$

When $m \to 1$, u_6 recovers

$$u_7 = 2\sinh^{-1}(\pm\operatorname{csch}\xi)\,. \tag{28}$$

Case 7 If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$, $a_4 = 1$, then

$$y = \operatorname{cs}\xi, \quad b_0 = 0, \quad b_1 = \pm\frac{1}{\sqrt{1 - m^2}}\,,$$
$$c = -\frac{\alpha}{(1 - m^2)k^2}, \quad 0 < m < 1\,, \tag{29}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_8 = 2\sinh^{-1}\Big(\pm\frac{1}{\sqrt{1 - m^2}}\operatorname{cs}\xi\Big)\,. \tag{30}$$

Case 8 If $a_0 = m^2(m^2 - 1)$, $a_2 = 2m^2 - 1$, $a_4 = 1$, then

$$y = \operatorname{ds}\xi \equiv \frac{\operatorname{dn}\xi}{\operatorname{sn}\xi}, \quad b_0 = 0, \quad b_1 = \pm\frac{1}{m}\,,$$
$$c = -\frac{\alpha}{m^2 k^2}, \quad 0 < m \leq 1\,, \tag{31}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_9 = 2\sinh^{-1}\Big(\pm\frac{1}{m}\operatorname{ds}\xi\Big)\,. \tag{32}$$

When $m \to 1$, u_9 recovers u_7.

Apart from these Jacobian elliptic solutions, y also has some rational solutions in terms of Jacobian elliptic functions, such as the following.

Case 9 If $a_0 = (1-m^2)/4$, $a_2 = (1+m^2)/2$, $a_4 = (1-m^2)/4$, then

$$y = \frac{\mathrm{cn}\,\xi}{1 \pm \mathrm{sn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm \frac{\sqrt{1-m^2}}{1+m}, \\ c = -\frac{4\alpha}{(1+m)^2 k^2}, \quad 0 < m < 1, \tag{33}$$

where k is an arbitrary constant. So the solution to ShG equation (1) is

$$u_{10} = 2\sinh^{-1}\Big(\pm \frac{\sqrt{1-m^2}}{1+m}\frac{\mathrm{cn}\,\xi}{1 \pm \mathrm{sn}\,\xi}\Big). \tag{34}$$

Case 10 If $a_0 = (1-m^2)/4$, $a_2 = (1+m^2)/2$, $a_4 = (1-m^2)/4$, then

$$y = \frac{\mathrm{cn}\,\xi}{1 \pm \mathrm{sn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm \frac{\sqrt{1-m^2}}{1-m}, \\ c = -\frac{4\alpha}{(1-m)^2 k^2}, \quad 0 < m < 1, \tag{35}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{11} = 2\sinh^{-1}\Big(\pm \frac{\sqrt{1-m^2}}{1-m}\frac{\mathrm{cn}\,\xi}{1 \pm \mathrm{sn}\,\xi}\Big). \tag{36}$$

Remark Transformation (2) and the solutions from u_1 to u_{11} in terms of Jacobian elliptic functions have not been reported in the literature.

3 The Second Kind of Transformation and Solutions to ShG Equation

The second transformation is introduced in the form

$$u = 2\cosh^{-1} v \quad \text{or} \quad v = \cosh\frac{u}{2}, \tag{37}$$

and then

$$\sinh u = 2\sinh\frac{u}{2}\cosh\frac{u}{2} = 2v\sqrt{v^2-1}, \tag{38}$$

and

$$u_{tx} = \frac{2}{\sqrt{v^2-1}} v_{tx} - \frac{2v}{(v^2-1)\sqrt{v^2-1}} v_t v_x. \tag{39}$$

Combining Eqs. (38) and (39) with Eq. (1), the ShG equation can be rewritten as

$$(v^2-1)v_{tx} - v v_t v_x - \alpha v (v^2-1)^2 = 0. \tag{40}$$

We can see that equation (40) is similar to Eq. (5), so it can be easily solved just like what we have done to Eq. (5). Here we have

$$b_0 = 0, \quad b_1 = \pm\sqrt{-\frac{a_4 k^2 c}{\alpha}},$$

$$\frac{\alpha}{k^2 c} = \frac{a_2 \pm \sqrt{a_2^2 - 4a_0 a_4}}{2}, \tag{41}$$

with constraints

$$-\frac{a_4 k^2 c}{\alpha} > 0, \quad a_2^2 - 4a_0 a_4 \geq 0. \tag{42}$$

Similarly, there are some cases to be considered. For example,

Case 1 If $a_0 = 1$, $a_2 = -(1+m^2)$, $a_4 = m^2$, then

$$y = \mathrm{sn}\,\xi, \quad b_0 = 0, \quad b_1 = \pm m, \\ c = -\frac{\alpha}{k^2}, \quad 0 < m \leq 1, \tag{43}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{12} = 2\cosh^{-1}(\pm m\mathrm{sn}\,\xi). \tag{44}$$

Case 2 If $a_0 = 1$, $a_2 = -(1+m^2)$, $a_4 = m^2$, then

$$y = \mathrm{sn}\,\xi, \quad b_0 = 0, \quad b_1 = \pm 1, \\ c = -\frac{\alpha}{k^2 m^2}, \quad 0 < m \leq 1, \tag{45}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{13} = 2\cosh^{-1}(\pm \mathrm{sn}\,\xi). \tag{46}$$

Actually, when $m \to 1$, u_{12} and u_{13} both recover to

$$u_{14} = 2\cosh^{-1}(\pm\tanh\xi), \quad c = -\frac{\alpha}{k^2}. \tag{47}$$

Case 3 If $a_0 = 1-m^2$, $a_2 = 2m^2-1$, $a_4 = -m^2$, then

$$y = \mathrm{cn}\,\xi, \quad b_0 = 0, \quad b_1 = \pm 1, \\ c = \frac{\alpha}{k^2 m^2}, \quad 0 < m \leq 1, \tag{48}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{15} = 2\cosh^{-1}(\pm \mathrm{cn}\,\xi). \tag{49}$$

Actually, when $m \to 1$, u_{15} recovers to

$$u_{16} = 2\cosh^{-1}(\pm \mathrm{sech}\,\xi), \quad c = \frac{\alpha}{k^2}. \tag{50}$$

Case 4 If $a_0 = m^2-1$, $a_2 = 2-m^2$, $a_4 = -1$, then

$$y = \mathrm{dn}\,\xi, \quad b_0 = 0, \quad b_1 = \pm 1, \\ c = \frac{\alpha}{k^2}, \quad 0 < m \leq 1, \tag{51}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{17} = 2\cosh^{-1}(\pm \mathrm{dn}\,\xi). \tag{52}$$

Actually, when $m \to 1$, u_{17} recovers u_{16}.

Case 5 If $a_0 = m^2-1$, $a_2 = 2-m^2$, $a_4 = -1$, then

$$y = \mathrm{dn}\,\xi, \quad b_0 = 0, \quad b_1 = \pm \frac{1}{\sqrt{1-m^2}},$$

$$c=\frac{\alpha}{(1-m^2)k^2},\quad 0<m<1\,, \tag{53}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{18}=2\cosh^{-1}\Big(\pm\frac{1}{\sqrt{1-m^2}}\operatorname{dn}\xi\Big)\,. \tag{54}$$

Case 6 If $a_0=m^2$, $a_2=-(1+m^2)$, $a_4=1$, then

$$y=\operatorname{ns}\xi,\quad b_0=0,\quad b_1=\pm\frac{1}{m}\,,$$
$$c=-\frac{\alpha}{k^2m^2},\quad 0<m\le 1\,, \tag{55}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{19}=2\cosh^{-1}\Big(\pm\frac{1}{m}\operatorname{ns}\xi\Big)\,. \tag{56}$$

Case 7 If $a_0=m^2$, $a_2=-(1+m^2)$, $a_4=1$, then

$$y=\operatorname{ns}\xi,\quad b_0=0,\quad b_1=\pm 1\,,$$
$$c=-\frac{\alpha}{k^2},\quad 0<m\le 1\,, \tag{57}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{20}=2\cosh^{-1}(\pm\operatorname{ns}\xi)\,. \tag{58}$$

Actually, when $m\to 1$, u_{19} and u_{20} both recover to

$$u_{21}=2\cosh^{-1}(\pm\coth\xi),\quad c=-\frac{\alpha}{k^2}\,. \tag{59}$$

Case 8 If $a_0=-m^2$, $a_2=2m^2-1$, $a_4=1-m^2$, then

$$y=\operatorname{nc}\xi,\quad b_0=0,\quad b_1=\pm 1\,,$$
$$c=-\frac{\alpha}{(1-m^2)k^2},\quad 0<m<1\,, \tag{60}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{22}=2\cosh^{-1}(\pm\operatorname{nc}\xi)\,. \tag{61}$$

Case 9 If $a_0=-1$, $a_2=2-m^2$, $a_4=m^2-1$, then

$$y=\operatorname{nd}\xi,\quad b_0=0,\quad b_1=\pm 1\,,$$
$$c=-\frac{\alpha}{(m^2-1)k^2},\quad 0<m<1\,, \tag{62}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{23}=2\cosh^{-1}(\pm\operatorname{nd}\xi)\,. \tag{63}$$

Case 10 If $a_0=-1$, $a_2=2-m^2$, $a_4=m^2-1$, then

$$y=\operatorname{nd}\xi,\quad b_0=0,\quad b_1=\pm\sqrt{1-m^2}\,,$$
$$c=\frac{\alpha}{k^2},\quad 0<m<1\,, \tag{64}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{24}=2\cosh^{-1}(\pm\sqrt{1-m^2}\operatorname{nd}\xi)\,. \tag{65}$$

Case 11 If $a_0=1$, $a_2=2m^2-1$, $a_4=(m^2-1)m^2$, then

$$y=\operatorname{sd}\xi,\quad b_0=0,\quad b_1=\pm\sqrt{1-m^2}\,,$$
$$c=\frac{\alpha}{m^2k^2},\quad 0<m<1\,, \tag{66}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{25}=2\cosh^{-1}(\pm\sqrt{1-m^2}\operatorname{sd}\xi)\,. \tag{67}$$

Case 12 If $a_0=1$, $a_2=-(1+m^2)$, $a_4=m^2$, then

$$y=\operatorname{cd}\xi,\quad b_0=0,\quad b_1=\pm m\,,$$
$$c=-\frac{\alpha}{k^2},\quad 0<m\le 1\,, \tag{68}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{26}=2\cosh^{-1}(\pm m\operatorname{cd}\xi)\,. \tag{69}$$

Case 13 If $a_0=1$, $a_2=-(1+m^2)$, $a_4=m^2$, then

$$y=\operatorname{cd}\xi,\quad b_0=0,\quad b_1=\pm 1\,,$$
$$c=-\frac{\alpha}{m^2k^2},\quad 0<m\le 1\,, \tag{70}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{27}=2\cosh^{-1}(\pm\operatorname{cd}\xi)\,. \tag{71}$$

Case 14 If $a_0=m^2(m^2-1)$, $a_2=2m^2-1$, $a_4=1$, then

$$y=\operatorname{ds}\xi,\quad b_0=0,\quad b_1=\pm\frac{1}{\sqrt{1-m^2}}\,,$$
$$c=-\frac{\alpha}{(1-m^2)k^2},\quad 0<m<1\,, \tag{72}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{28}=2\cosh^{-1}(\pm\frac{1}{\sqrt{1-m^2}}\operatorname{ds}\xi)\,. \tag{73}$$

Case 15 If $a_0=m^2$, $a_2=-(1+m^2)$, $a_4=1$, then

$$y=\operatorname{dc}\xi,\quad b_0=0,\quad b_1=\pm\frac{1}{m}\,,$$
$$c=-\frac{\alpha}{m^2k^2},\quad 0<m\le 1\,, \tag{74}$$

where k is an arbitrary constant, and the solution to ShG equation (1) is

$$u_{29}=2\cosh^{-1}\Big(\pm\frac{1}{m}\operatorname{dc}\xi\Big)\,. \tag{75}$$

Case 16 If $a_0=m^2$, $a_2=-(1+m^2)$, $a_4=1$, then

$$y=\operatorname{dc}\xi,\quad b_0=0,\quad b_1=\pm 1\,,$$
$$c=-\frac{\alpha}{k^2},\quad 0<m\le 1\,, \tag{76}$$

where k is an arbitrary constant, and then the solution to ShG equation (1) is

$$u_{30}=2\cosh^{-1}(\pm\mathrm{dc}\,\xi)\,. \tag{77}$$

Case 17 If $a_0=-(1-m^2)/4$, $a_2=(1+m^2)/2$, $a_4=-(1-m^2)/4$, then

$$y=\frac{\mathrm{dn}\,\xi}{1\pm m\,\mathrm{sn}\,\xi},\quad b_0=0,\quad b_1=\pm\frac{\sqrt{1-m^2}}{1+m},$$
$$c=\frac{4\alpha}{(1+m)^2k^2},\quad 0<m<1\,, \tag{78}$$

where k is an arbitrary constant, and then the solution to ShG equation (1) is

$$u_{31}=2\cosh^{-1}\Big(\pm\frac{\sqrt{1-m^2}}{1+m}\frac{\mathrm{dn}\,\xi}{1\pm m\,\mathrm{sn}\,\xi}\Big)\,. \tag{79}$$

Case 18 If $a_0=-(1-m^2)/4$, $a_2=(1+m^2)/2$, $a_4=-(1-m^2)/4$, then

$$y=\frac{\mathrm{dn}\,\xi}{1\pm m\,\mathrm{sn}\,\xi},\quad b_0=0,\quad b_1=\pm\frac{\sqrt{1-m^2}}{1-m},$$
$$c=\frac{4\alpha}{(1-m)^2k^2},\quad 0<m<1\,, \tag{80}$$

where k is an arbitrary constant, then the solution to ShG equation (1) is

$$u_{32}=2\cosh^{-1}\Big(\pm\frac{\sqrt{1-m^2}}{1-m}\frac{\mathrm{dn}\,\xi}{1\pm m\,\mathrm{sn}\,\xi}\Big)\,. \tag{81}$$

Case 19 If $a_0=m^2/4$, $a_2=-(2-m^2)/2$, $a_4=m^2/4$, then

$$y=\frac{m\,\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi},\quad b_0=0\,,$$
$$b_1=\pm\frac{m}{\sqrt{2-m^2+2\sqrt{1-m^2}}}\,,$$
$$c=-\frac{4\alpha}{(\sqrt{2-m^2+2\sqrt{1-m^2}})k^2},\quad 0<m\le 1\,, \tag{82}$$

where k is an arbitrary constant, and then the solution to ShG equation (1) is

$$u_{33}=2\cosh^{-1}\Big(\pm\frac{m}{\sqrt{2-m^2+2\sqrt{1-m^2}}}\frac{m\,\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi}\Big)\,. \tag{83}$$

Case 20 If $a_0=m^2/4$, $a_2=-(2-m^2)/2$, $a_4=m^2/4$, then

$$y=\frac{m\,\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi},\quad b_0=0\,,$$
$$b_1=\pm\frac{m}{\sqrt{2-m^2-2\sqrt{1-m^2}}}\,,$$
$$c=-\frac{4\alpha}{(\sqrt{2-m^2-2\sqrt{1-m^2}})k^2},\quad 0<m\le 1\,, \tag{84}$$

where k is an arbitrary constant, and then the solution to ShG equation (1) is

$$u_{34}=2\cosh^{-1}\Big(\pm\frac{m}{\sqrt{2-m^2-2\sqrt{1-m^2}}}\frac{m\,\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi}\Big)\,. \tag{85}$$

When $m\to 1$, u_{33} and u_{34} both recover to

$$u_{35}=2\cosh^{-1}\Big(\pm\frac{\tanh\xi}{1\pm\mathrm{sech}\,\xi}\Big),\quad c=-\frac{4\alpha}{k^2}\,. \tag{86}$$

Case 21 If $a_0=1/4$, $a_2=-(2-m^2)/2$, $a_4=m^4/4$, then

$$y=\frac{\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi},\quad b_0=0\,,$$
$$b_1=\pm\frac{m^2}{\sqrt{2-m^2+2\sqrt{1-m^2}}}\,,$$
$$c=-\frac{4\alpha}{(\sqrt{2-m^2+2\sqrt{1-m^2}})k^2},\quad 0<m\le 1\,, \tag{87}$$

where k is an arbitrary constant.

So the solution to ShG equation (1) is

$$u_{36}=2\cosh^{-1}\Big(\pm\frac{m^2}{\sqrt{2-m^2+2\sqrt{1-m^2}}}\frac{\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi}\Big)\,. \tag{88}$$

Case 22 If $a_0=1/4$, $a_2=-(2-m^2)/2$, $a_4=m^4/4$, then

$$y=\frac{\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi},\quad b_0=0\,,$$
$$b_1=\pm\frac{m^2}{\sqrt{2-m^2-2\sqrt{1-m^2}}}\,,$$
$$c=-\frac{4\alpha}{(\sqrt{2-m^2-2\sqrt{1-m^2}})k^2},\quad 0<m\le 1\,, \tag{89}$$

where k is an arbitrary constant, and then the solution to ShG equation (1) is

$$u_{37}=2\cosh^{-1}\Big(\pm\frac{m^2}{\sqrt{2-m^2-2\sqrt{1-m^2}}}\frac{\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi}\Big)\,. \tag{90}$$

When $m\to 1$, u_{36} and u_{37} both recover to u_{35}.

Remark Transformation (37) and the solutions from u_{12} to u_{37} in terms of Jacobian elliptic functions have not been given in the literature.

4 Conclusion

In this paper, two transformations are introduced to solve sinh-Gordon equation by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different transformations are required in order to obtain more kinds of solutions to the sinh-Gordon equation. Here some new solutions have not been reported in the literature. It is shown that different transformations play different roles in obtaining exact solutions, some transformations may not work for a specific parameter of ShG equation. Of course, still more efforts are needed to explore what kinds of transformations are more suitable to solving sinh-Gordon equation, because different transformations result in different partial balances for sinh-Gordon equation, which will lead to different expansion truncations in the elliptic equation expansion method. Finally, these will result in different solutions of the sine-Gordon equation.

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