Available online at www.sciencedirect.com
ScienceDirect
Physics Letters A 368 (2007) 238–244
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Envelope breather solution and envelope breather lattice solutions to the NLS equation

Zuntao Fu [a,b,*], Shida Liu [a,b], Shikuo Liu [a]

[a] School of Physics & Laboratory for Severe Storm and Flood Disaster, Peking University, Beijing 100871, China
[b] State Key Laboratory for Turbulence and Complex Systems, Peking University, Beijing 100871, China

Received 25 September 2006; received in revised form 21 March 2007; accepted 2 April 2007

Available online 6 April 2007

Communicated by A.R. Bishop

Abstract

In this Letter, dependent and independent variable transformations are introduced to solve the nonlinear Schrödinger (NLS) equation systematically by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different kinds of solutions can be obtained to the NLS equation, including many kinds of envelope breather solutions and envelope breather lattice solutions.

PACS: 03.65.Ge

Keywords: Jacobian elliptic function; The NLS equation; Envelope breather solution; Envelope breather lattice solution

1. Introduction

Among the soliton bearing nonlinear equations, the modified Korteweg–de Vries (mKdV) equation is of special interest [1,2]. For it possesses rich solutions, such as solitary solutions [1–4], periodic solutions [3–6], breather solution [1,2,7,8], breather lattice solutions [7,8]. A particularly interesting type of solution is the so-called breather kind of solution, usually this kind of solutions is unavailable and such solutions have to be solved numerically [7]. In some cases, however, the analytical expressions in closed form can be found, such as the breather lattice solution for the sine-Gordon equation [9] and for the mKdV equation [7,8].

In Refs. [7–9], Kevrekidis's research group has applied some ansatzes to obtain the breather lattice solutions to the mKdV equation and the sine-Gordon equation. The aim of present Letter is to present the envelope breather solutions and envelope breather lattice solutions of the NLS equation in a systematical way. Based on the introduced transformations, we will show systematical results about these breather-type solutions for the NLS equation by using the knowledge of elliptic equation and Jacobian elliptic functions [3,4,10–12].

The cubic nonlinear Schrödinger (NLS) equation reads [1,2]

$$iu_{t'}+\alpha' u_{x'x'}+\beta'|u|^2u=0, \tag{1}$$

where $i=\sqrt{-1}$.

Eq. (1) can be transformed as

$$\alpha'\frac{\partial^2 E}{\partial x'^2}+\left(kc-k^2\alpha'\right)E+\beta' E^3+i\left(\frac{\partial E}{\partial t'}+2k\alpha'\frac{\partial E}{\partial x'}\right)=0, \tag{2}$$

with the transformation

$$u(x',t')=E(x',t')e^{ik(x'-ct')}, \tag{3}$$

where the amplitude $E(x',t')$ is a real function of its arguments. Next, we will try to find the analytical expression of $E(x',t')$.

First of all, the real part and imaginary part of Eq. (2) can be separated into a set of equations, i.e.

$$\frac{\partial E}{\partial t'}+2k\alpha'\frac{\partial E}{\partial x'}=0, \tag{4a}$$

* Corresponding author at: School of Physics & Laboratory for Severe Storm and Flood Disaster, Peking University, Beijing 100871, China.
E-mail address: fuzt@pku.edu.cn (Z. Fu).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2007.04.011

$$\alpha' \frac{\partial^2 E}{\partial x'^2} + \left(kc - k^2\alpha'\right)E + \beta' E^3 = 0. \tag{4b}$$

Substituting Eq. (4a) into Eq. (4b) yields the modified KdV (mKdV) equation

$$\frac{\partial E}{\partial t'} + \alpha E^2 \frac{\partial E}{\partial x'} + \beta \frac{\partial^3 E}{\partial x'^3} = 0, \tag{5}$$

with

$$\alpha \equiv \frac{6\alpha'\beta'}{c - k\alpha'}, \qquad \beta \equiv \frac{2\alpha'^2}{c - k\alpha'}. \tag{6}$$

Set $t = t'$, $x = \beta^{-1/3}x'$ and $v = \pm\sqrt{\frac{\alpha}{6}}\beta^{-1/6}E$, Eq. (5) can be rewritten as

$$v_t + 6v^2 v_x + v_{xxx} = 0, \tag{7}$$

which is called the positive mKdV (pmKdV) equation [7,8].

If we set $t = t'$, $x = \beta^{-1/3}x'$ and $v = \pm\sqrt{-\frac{\alpha}{6}}\beta^{-1/6}E$, Eq. (5) can be rewritten as

$$v_t - 6v^2 v_x + v_{xxx} = 0, \tag{8}$$

which is called the negative mKdV (nmKdV) equation [7,8].

From the above relation between (7), (8) and (5), it is obvious that if one derives the solutions to (7) or (8), then the solutions to (5) can be obtained directly by the rescaled independent variables and dependent variable. Next, we will show the details to derive many kinds of solutions to (5), especially the breather solutions, the solutions describing the interaction of two-soliton and the shelf-shaped solutions, which have not been reported in the literature in a systematical way.

2. Envelope breather lattice solutions and envelope breather solutions: Case for the pmKdV equation

In order to obtain the breather solution and breather lattice solution to Eq. (5) satisfied by the amplitude E, the following transformation

$$v = 2\frac{\partial}{\partial x}\tan^{-1}\phi \tag{9}$$

must be introduced, and then ϕ satisfies

$$\left(1+\phi^2\right)(\phi_t + \phi_{xxx}) + 6\phi_x\left(\phi_x^2 - \phi\phi_{xx}\right) = 0. \tag{10}$$

Eq. (10) can be solved by introducing the following independent variable and dependent variable transformations

$$\xi = ax + bt + \xi_0, \qquad \eta = cx + dt + \eta_0, \tag{11}$$

and

$$\phi = AU(\xi)V(\eta), \tag{12}$$

where ξ_0 and η_0 are two constants, A is a constant to be determined, U and V satisfy the following elliptic equation

$$U_\xi^2 = s_1U^4 + p_1U^2 + q_1,$$
$$V_\eta^2 = s_2V^4 + p_2V^2 + q_2, \tag{13}$$

where s_1, s_2, p_1, p_2, q_1 and q_2 are determined constants.

Substituting (11) and (12) into (10) yields the following algebraic equations

$$b + p_1a^3 + 3p_2ac^2 = 0, \tag{14a}$$
$$q_1A^2a^2 + s_2c^2 = 0, \tag{14b}$$
$$q_2A^2c^2 + s_1a^2 = 0, \tag{14c}$$
$$d + 3p_1a^2c + p_2c^3 = 0. \tag{14d}$$

For the algebraic equations (14), some cases can be addressed, first of all, we will address some special cases.

Case 1. If $s_1 = s_2 = q_1 = q_2 = 0$ and $p_1 > 0$, $p_2 > 0$, then from Eq. (14), we have

$$U = \gamma_1 e^{\pm\sqrt{p_1}\xi}, \qquad V = \gamma_2 e^{\pm\sqrt{p_2}\eta}, \tag{15}$$

where γ_1 and γ_2 are two constants. The solution (12) can be written as

$$\phi_{2-1} = Ae^{\pm(\sqrt{p_1}\xi \pm \sqrt{p_2}\eta)}, \tag{16}$$

where A is a constant, and the solution (16) is a kind of shelf-shaped solution, whose graphical presentation is shown in Fig. 1, an obvious shelf shape.

Case 2. If only $s_2 = q_1 = 0$, then from Eq. (13), we have

$$U_\xi^2 = s_1U^4 + p_1U^2, \qquad V_\eta^2 = p_2V^2 + q_2. \tag{17}$$

Actually, if we set $U = \frac{1}{W}$, then we have

$$W_\xi^2 = p_1W^2 + s_1. \tag{18}$$

It is obvious that V and W satisfy the same equation, this equation has three subcases to be considered.

Case 2a. If $p_1 > 0$, $s_1 > 0$ or $p_2 > 0$, $q_2 > 0$, the solution to U or V is

$$\frac{1}{U} = \pm\sqrt{\frac{s_1}{p_1}}\sinh(\sqrt{p_1}\xi + c_1),$$

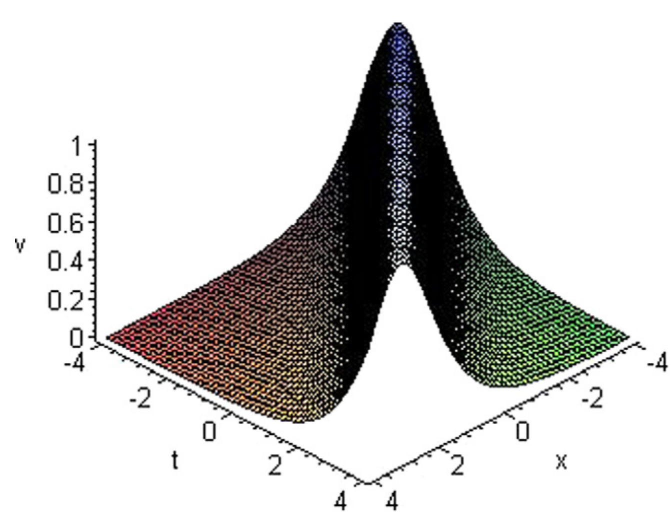

Fig. 1. The graphical presentation shows the space–time evolution of the shelf-shaped solution of Eqs. (15) and (16), where the parameters are chosen as $p_1 = p_2 = A = 1$, $a = b = \xi_0 = 2$, $c = d = \eta_0 = 1$.

$$V = \pm\sqrt{\frac{q_2}{p_2}}\sinh(\sqrt{p_2}\eta + c_2), \tag{19}$$

where c_1 and c_2 are two constants.

Case 2b. If $p_1 > 0$, $s_1 < 0$ or $p_2 > 0$, $q_2 < 0$, the solution to U or V is

$$\frac{1}{U} = \pm\sqrt{-\frac{s_1}{p_1}}\cosh(\sqrt{p_1}\xi + c_1),$$
$$V = \pm\sqrt{-\frac{q_2}{p_2}}\cosh(\sqrt{p_2}\eta + c_2), \tag{20}$$

where c_1 and c_2 are two constants.

Case 2c. If $p_1 < 0$, $s_1 > 0$ or $p_2 < 0$, $q_2 > 0$, the solution to U or V is

$$\frac{1}{U} = \pm\sqrt{-\frac{s_1}{p_1}}\sin(\sqrt{-p_1}\xi + c_1),$$
$$V = \pm\sqrt{-\frac{q_2}{p_2}}\sin(\sqrt{-p_2}\eta + c_2), \tag{21}$$

where c_1 and c_2 are two constants.

Similarly, the case for only $s_1 = q_2 = 0$ can be discussed. The solution (12) can be obtained by the combinations of above six pairs of U and V. For example, for $p_1 > 0$, $p_2 > 0$, $s_1 > 0$ and $q_2 < 0$, the solution is

$$\phi_{2-2} = \pm\sqrt{-\frac{p_1 q_2}{p_2 s_1}}\frac{\cosh(\sqrt{p_2}\eta + c_2)}{\sinh(\sqrt{p_1}\xi + c_1)}, \tag{22}$$

this is a solution describing the interaction of two solitons, whose graphical presentation is shown in Fig. 2, an obvious figure of the interaction of two solitons.

For $p_1 > 0$, $p_2 < 0$, $s_1 < 0$ and $q_2 > 0$, the solution is

$$\phi_{2-3} = \pm\sqrt{\frac{p_1 q_2}{p_2 s_1}}\frac{\sin(\sqrt{-p_2}\eta + c_2)}{\cosh(\sqrt{p_1}\xi + c_1)}, \tag{23}$$

obviously, this is a kind of breather solution, whose graphical presentation is shown in Fig. 3.

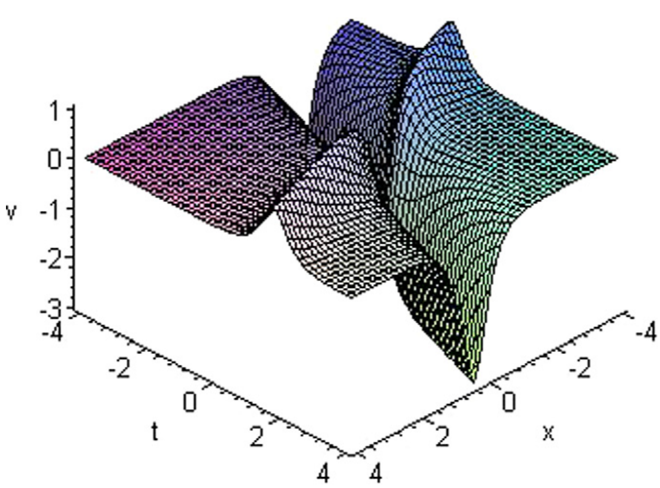

Fig. 2. The graphical presentation shows the space–time evolution of the interaction of two-soliton solution of Eq. (22), where the parameters are chosen as $p_1 = p_2 = s_1 = 1$, $q_2 = -1$, $a = b = c_1 = 1$, $c = d = c_2 = 2$.

Case 3. If q_1, q_2, s_1 and s_2 are all nonzero values, from (14), we can determine

$$A^4 = \frac{s_1 s_2}{q_1 q_2}, \qquad -\frac{a^2}{c^2} = \frac{s_2}{q_1 A^2} = \frac{q_2 A^2}{s_1},$$
$$b = -a(p_1 a^2 + 3p_2 c^2), \qquad d = -c(3p_1 a^2 + p_2 c^2). \tag{24}$$

From (24), it is obvious that the determined constants in (14) must satisfy the following constraints

$$\frac{s_1 s_2}{q_1 q_2} > 0, \qquad \frac{s_2}{q_1} < 0, \qquad \frac{q_2}{s_1} < 0, \tag{25}$$

this implies that not all combinations of Jacobi elliptic functions are solutions to the positive mKdV equation (10) under the above mentioned transformations, only the combination of a couple of the Jacobi elliptic functions satisfies the constraint (25), it can be a solution to the positive mKdV equation (10). Actually, there exist only 18 of this kind of combinations [13], we will address some of them in details.

Case 3-1. When $U = \mathrm{sn}(\xi, k)$ and $V = \mathrm{dn}(\eta, m)$, where $\mathrm{sn}(\xi, k)$ and $\mathrm{dn}(\eta, m)$ are the Jacobi sine elliptic function and the Jacobi elliptic function of the third kind, respectively, and k and m are their modulus [10–12]. Then from (13), we have

$$s_1 = k^2, \qquad p_1 = -(1 + k^2), \qquad q_1 = 1,$$
$$s_2 = -1, \qquad p_2 = 2 - m^2, \qquad q_2 = -(1 - m^2). \tag{26}$$

Substituting (26) into (24), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1 - m^2}{k^2}}, \qquad \frac{b}{a} = \left[a^2(1 + k^2) - 3c^2(2 - m^2)\right],$$
$$\frac{d}{c} = \left[3a^2(1 + k^2) - c^2(2 - m^2)\right],$$
$$A = \pm\left[\frac{k^2}{1 - m^2}\right]^{\frac{1}{4}}, \tag{27}$$

then the solution (12) is

$$\phi_{2-4} = \pm\left[\frac{k^2}{1 - m^2}\right]^{\frac{1}{4}}\left[\mathrm{sn}(\xi, k)\,\mathrm{dn}(\eta, m)\right], \tag{28}$$

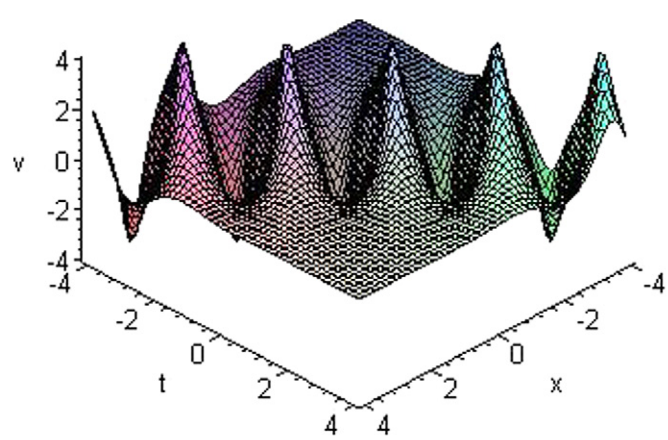

Fig. 3. The graphical presentation shows the space–time evolution of the breather solution of Eq. (23), where the parameters are chosen as $p_1 = q_2 = 1$, $p_2 = s_1 = -1$, $a = b = c_1 = 1$, $c = d = c_2 = 2$.

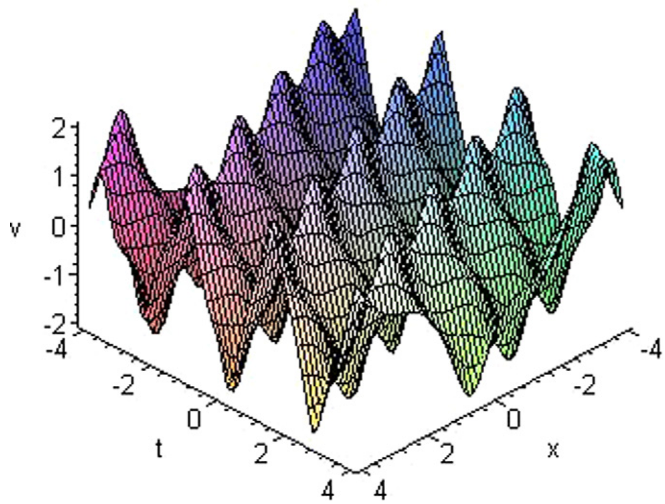

Fig. 4. The graphical presentation shows the space–time evolution of the breather lattice solution of Eqs. (27), (28), where the parameters are chosen as $a=1$, $c=1$, $m=0.8$, $\xi_0=\eta_0=0$, from which the other parameters can be determined as $b=-2.72$, $d=2.72$, $k=0.6$ and $A=1$.

which is a kind of breather lattice solution given in Refs. [7,8,13], and when $m\to 0$, $\mathrm{dn}(\eta,m)\to 1$, it turns to be a periodic wave solution

$$\phi_{2-4'}=\pm\sqrt{k}\,\mathrm{sn}(\xi,k). \tag{29}$$

Fig. 4 shows the evolution of the breather lattice solution with the periodic characteristics in both spatial and temporal directions, while for the normal breather solution which has the periodic characteristics just in a specific direction.

Case 3-2. When $U=\mathrm{cn}(\xi,k)$ and $V=\mathrm{sd}(\eta,m)=\frac{\mathrm{sn}(\eta,m)}{\mathrm{dn}(\eta,m)}$, where $\mathrm{cn}(\xi,k)$ is the Jacobi cosine elliptic function [10–12]. Then from (13), we have

$$s_1=-k^2,\qquad p_1=2k^2-1,\qquad q_1=1-k^2,$$
$$s_2=-m^2\bigl(1-m^2\bigr),\qquad p_2=2m^2-1,\qquad q_2=1, \tag{30}$$

then the solution (12) is

$$\phi_{2-5}=\pm\left[\frac{k^2m^2(1-m^2)}{1-k^2}\right]^{\frac14}\bigl[\mathrm{cn}(\xi,k)\,\mathrm{sd}(\eta,m)\bigr], \tag{31}$$

with

$$\frac{a^2}{c^2}=\frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}},\qquad \frac{b}{a}=\bigl[a^2\bigl(1-2k^2\bigr)+3c^2\bigl(1-2m^2\bigr)\bigr],$$
$$\frac{d}{c}=\bigl[3a^2\bigl(1-2k^2\bigr)+c^2\bigl(1-2m^2\bigr)\bigr],$$
$$A=\pm\left[\frac{k^2m^2(1-m^2)}{1-k^2}\right]^{\frac14}. \tag{32}$$

The periodic characteristics in Fig. 5 is different from what we see in Fig. 4, here the periodic characteristics only in a specific direction is obvious, in other directions it is much weaker.

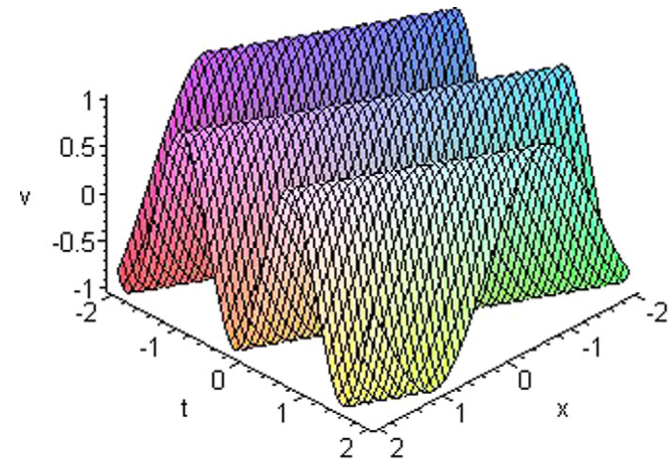

Fig. 5. The graphical presentation shows the space–time evolution of the breather lattice solution of Eqs. (31), (32), where the parameters are chosen as $a=1$, $c=1$, $m=0.5$, $\xi_0=\eta_0=0$, from which the other parameters can be determined as $b=2$, $d=2$, $k=0.5$ and $A=\frac12$.

3. Envelope breather lattice solutions and envelope breather solutions: Case for the nmKdV equation

For (8), the transformation is

$$v=2\frac{\partial}{\partial x}\tanh^{-1}\phi, \tag{33}$$

then ϕ satisfies

$$\bigl(1-\phi^2\bigr)(\phi_t+\phi_{xxx})-6\phi_x\bigl(\phi_x^2-\phi\phi_{xx}\bigr)=0. \tag{34}$$

Substituting (11) and (12) into (34) yields the following algebraic equations

$$b+p_1a^3+3p_2ac^2=0, \tag{35a}$$
$$q_1A^2a^2-s_2c^2=0, \tag{35b}$$
$$q_2A^2c^2-s_1a^2=0, \tag{35c}$$
$$d+3p_1a^2c+p_2c^3=0. \tag{35d}$$

Similar to (14), there are also some cases that can be addressed, first of all, we will address some special cases.

Case 1. If $s_1=s_2=q_1=q_2=0$ and $p_1>0$, $p_2>0$, then from Eq. (35), the solution (12) can be written as

$$\phi_{3-1}=Ae^{\pm(\sqrt{p_1}\xi\pm\sqrt{p_2}\eta)}, \tag{36}$$

where A is a constant, and the solution (36) is a kind of shelf-shaped solution, whose graphical presentation is shown in Fig. 6, an obvious shelf shape, but different from Fig. 1. Here the shelf-wave will blowup within finite region along two lines.

Case 2. If only $s_1=q_2=0$, then from Eq. (13), we have

$$U_\xi^2=p_1U^2+q_1,\qquad V_\eta^2=s_2V^4+p_2V^2. \tag{37}$$

Actually, if we set $V=\frac{1}{W}$, then

$$W_\eta^2=p_2W^2+s_2. \tag{38}$$

It is obvious that U and W satisfy the same equation, this equation has three subcases to be considered.

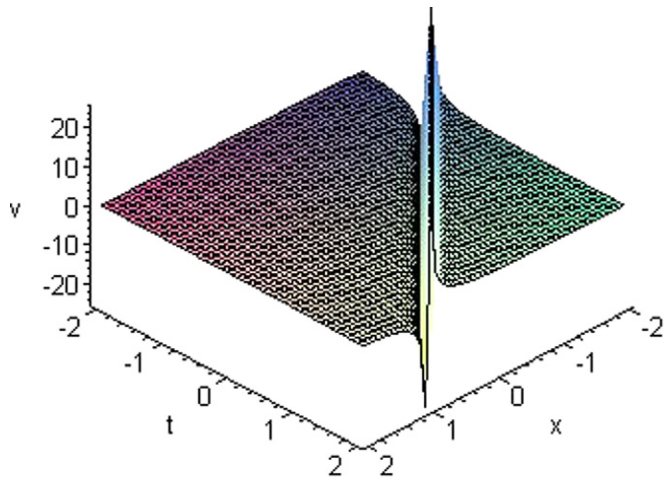

Fig. 6. The graphical presentation shows the space–time evolution of the shelf-shaped solution of Eq. (36), where the parameters are chosen as $p_1 = p_2 = A = 1$, $a = b = \xi_0 = 2$, $c = d = \eta_0 = 1$.

Case 2a. If $p_2 > 0$, $s_2 > 0$ or $p_1 > 0$, $q_1 > 0$, the solution to U or V is

$$\frac{1}{V} = \pm\sqrt{\frac{s_2}{p_2}}\sinh(\sqrt{p_2}\eta + c_2),$$
$$U = \pm\sqrt{\frac{q_1}{p_1}}\sinh(\sqrt{p_1}\xi + c_1), \tag{39}$$

where c_1 and c_2 are two constants.

Case 2b. If $p_2 > 0$, $s_2 < 0$ or $p_1 > 0$, $q_1 < 0$, the solution to U or V is

$$\frac{1}{V} = \pm\sqrt{-\frac{s_2}{p_2}}\cosh(\sqrt{p_2}\eta + c_2),$$
$$U = \pm\sqrt{-\frac{q_1}{p_1}}\cosh(\sqrt{p_1}\xi + c_1), \tag{40}$$

where c_1 and c_2 are two constants.

Case 2c. If $p_2 < 0$, $s_2 > 0$ or $p_1 < 0$, $q_1 > 0$, the solution to U or V is

$$\frac{1}{V} = \pm\sqrt{-\frac{s_2}{p_2}}\sin(\sqrt{-p_2}\eta + c_2),$$
$$U = \pm\sqrt{-\frac{q_1}{p_1}}\sin(\sqrt{-p_1}\xi + c_1), \tag{41}$$

where c_1 and c_2 are two constants.

Similarly, the case for only $s_2 = q_1 = 0$ can be discussed. The solution (12) can be obtained by the combinations of above six pairs of U and V. For example, for $p_2 > 0$, $p_1 > 0$, $s_2 > 0$ and $q_1 < 0$, the solution is

$$\phi_{3-2} = \pm\sqrt{-\frac{p_2 q_1}{p_1 s_2}}\frac{\cosh(\sqrt{p_1}\xi + c_1)}{\sinh(\sqrt{p_2}\eta + c_2)}, \tag{42}$$

this is another solution describing the interaction of two solitons, whose graphical presentation is shown in Fig. 7, an obvious different figure of the interaction of two solitons from Fig. 2.

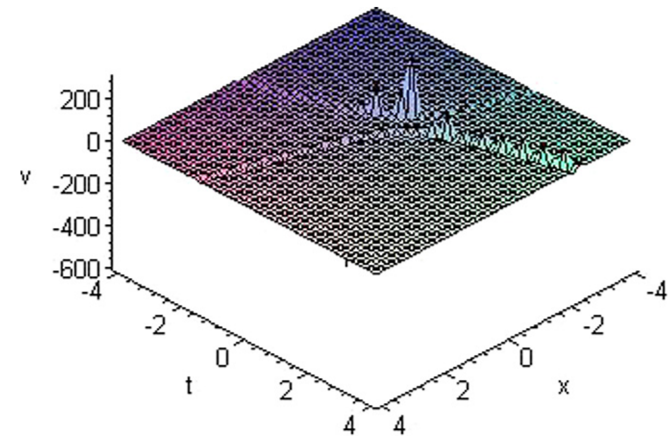

Fig. 7. The graphical presentation shows the space–time evolution of the interaction of two-soliton solution of Eq. (42), where the parameters are chosen as $p_1 = p_2 = s_2 = 1$, $q_1 = -1$, $a = b = c_1 = 1$, $c = d = c_2 = 2$.

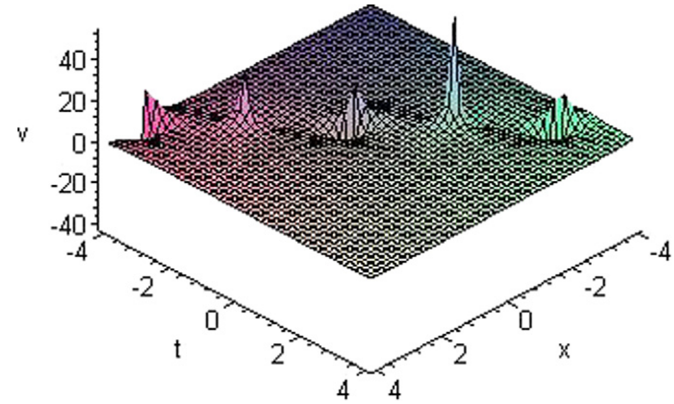

Fig. 8. The graphical presentation shows the space–time evolution of the breather solution of Eq. (43), where the parameters are chosen as $p_1 = s_2 = -1$, $p_2 = q_1 = 1$, $a = b = c_1 = 1$, $c = d = c_2 = 2$.

For $p_2 > 0$, $p_1 < 0$, $s_2 < 0$ and $q_1 > 0$, the solution is

$$\phi_{3-3} = \pm\sqrt{\frac{p_2 q_1}{p_1 s_2}}\frac{\sin(\sqrt{-p_1}\xi + c_1)}{\cosh(\sqrt{p_2}\eta + c_2)}, \tag{43}$$

obviously, this is another kind of breather solution, whose graphical presentation is shown in Fig. 8, which is with different figure from Fig. 3.

Case 3. If q_1, q_2, s_1 and s_2 are all nonzero values, from (35), we can determine

$$A^4 = \frac{s_1 s_2}{q_1 q_2}, \qquad \frac{a^2}{c^2} = \frac{s_2}{q_1 A^2} = \frac{q_2 A^2}{s_1},$$
$$b = -a\left(p_1 a^2 + 3p_2 c^2\right), \qquad d = -c\left(3p_1 a^2 + p_2 c^2\right). \tag{44}$$

From (44), it is obvious that the determined constants in (35) must satisfy the following constraints

$$\frac{s_1 s_2}{q_1 q_2} > 0, \qquad \frac{s_2}{q_1} > 0, \qquad \frac{q_2}{s_1} > 0, \tag{45}$$

this implies that not all combinations of Jacobi elliptic functions are solutions to the negative mKdV equation (34) under the above mentioned transformations, only the combination

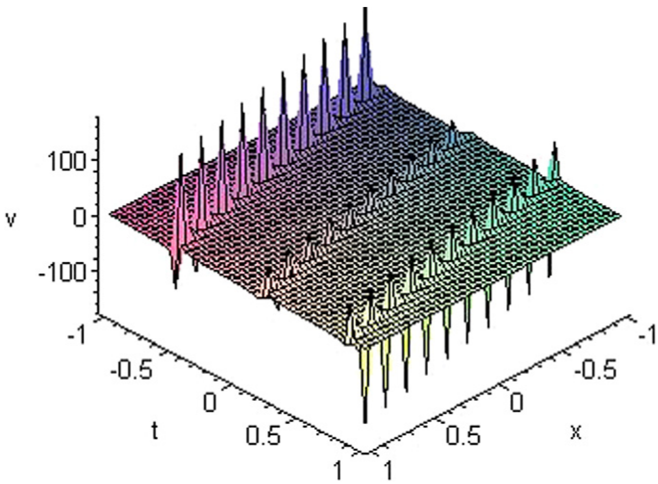

Fig. 9. The graphical presentation shows the space–time evolution of the breather lattice solution of Eqs. (46), (47), where the parameters are chosen as $a=1$, $c=1$, $m=\frac{\sqrt{3}}{2}$, $\xi_0=\eta_0=0$, from which the other parameters can be determined as $b=-5$, $d=-5$, $k=\frac{\sqrt{3}}{2}$ and $A=\frac{1}{2}$.

of a couple of the Jacobi elliptic functions satisfies the constraint (45), it can be a solution to the negative mKdV equation (34). Actually, there exist only 28 of this kind of combinations [14], we will address some of them in details.

Case 3-1. When $U=\mathrm{nd}(\xi,k)$ and $V=\mathrm{nd}(\eta,m)$, then from (13) and (44), the parameters can be determined as

$$\frac{a^2}{c^2}=\sqrt{\frac{1-m^2}{1-k^2}},\qquad \frac{b}{a}=-\big[a^2\big(2-k^2\big)+3c^2\big(2-m^2\big)\big],$$
$$\frac{d}{c}=-\big[3a^2\big(2-k^2\big)+c^2\big(2-m^2\big)\big],$$
$$A=\pm\big[\big(1-m^2\big)\big(1-k^2\big)\big]^{\frac{1}{4}}, \tag{46}$$

then the breather lattice solution is

$$\phi_{3-4}=\pm\big[\big(1-m^2\big)\big(1-k^2\big)\big]^{\frac{1}{4}}\big[\mathrm{nd}(\xi,k)\,\mathrm{nd}(\eta,m)\big]. \tag{47}$$

It is obvious that Fig. 9 describes a different kind of breather lattice solution from that given in Fig. 4. Compared to Fig. 4, the profiles in Fig. 9 for both spatial and temporal directions are intermittent. At the same time, the periodic characteristics of Fig. 9 in the spatial direction is quite different from that in the temporal direction.

Case 3-2. When $U=\mathrm{sn}(\xi,k)$ and $V=\mathrm{cs}(\eta,m)$, then from (13) and (44), the parameters can be determined as

$$\frac{a^2}{c^2}=\sqrt{\frac{1-m^2}{k^2}},\qquad \frac{b}{a}=\big[a^2\big(1+k^2\big)-3c^2\big(2-m^2\big)\big],$$
$$\frac{d}{c}=\big[3a^2\big(1+k^2\big)-c^2\big(2-m^2\big)\big],$$
$$A=\pm\left[\frac{k^2}{1-m^2}\right]^{\frac{1}{4}}, \tag{48}$$

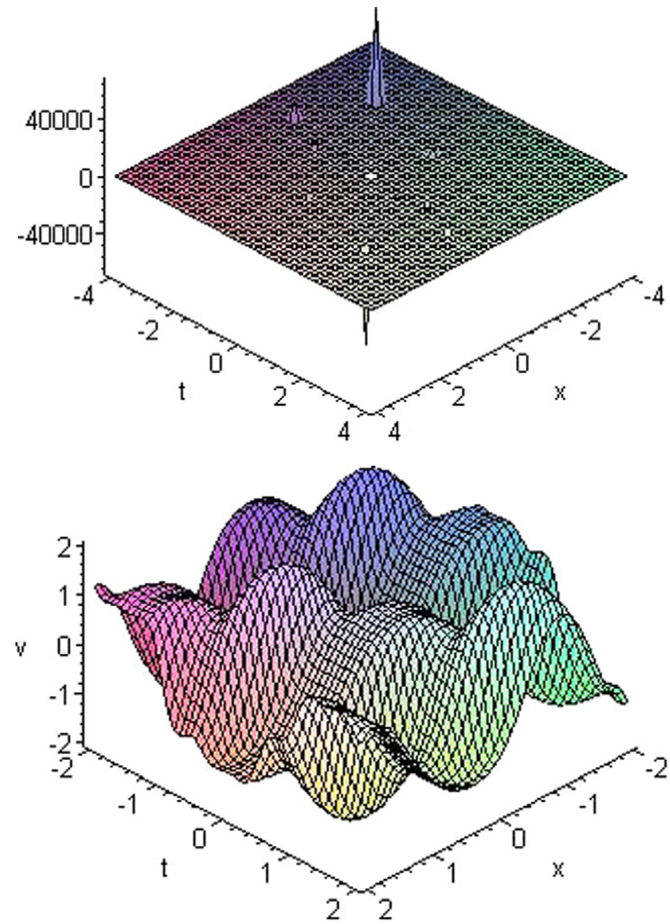

Fig. 10. The upper panel shows the space–time evolution of the breather lattice solution of Eqs. (48), (49), where the parameters are chosen as $a=1$, $c=1$, $m=0.8$, $\xi_0=\eta_0=0$, from which the other parameters can be determined as $b=-2.72$, $d=2.72$, $k=0.6$ and $\alpha=1$. While for the bottom panel, the parameters are chosen as $a=1$, $c=1$, $m=0$, $\xi_0=\eta_0=0$, from which the other parameters can be determined as $b=-4$, $d=4$, $k=1$ and $A=1$.

then the breather lattice solution is

$$\phi_{3-5}=\pm\left[\frac{k^2}{1-m^2}\right]^{\frac{1}{4}}\big[\mathrm{sn}(\xi,k)\,\mathrm{cs}(\eta,m)\big]. \tag{49}$$

When $k\to 1$ and $m\to 0$, the breather lattice solution (49) turns to be a solution

$$\phi_{3-6}=\pm\big[\tanh(\xi)\cot(\eta)\big], \tag{50}$$

with

$$a^2=c^2,\qquad b=-\frac{\gamma}{4a},\qquad d=\pm\frac{\gamma}{4a}. \tag{51}$$

From Fig. 10, it is obvious that for different values of m and k, the same breather lattice solution will also show different characteristics, small or large. Especially, when m and k take their limiting values, the behavior will be quite different from what given in Fig. 10. Actually, the characteristics of the breather lattice solution shown in the upper panel of Fig. 10 is sporadic, the magnitude of v has an antisymmetric characteristics along a specific direction. But for the bottom panel of Fig. 10, the profiles in any directions are smooth.

4. Conclusion

In this Letter, dependent and independent variable transformations are introduced to solve the nonlinear Schrödinger (NLS) equation systematically by using the knowledge of elliptic equation and Jacobian elliptic functions. Here first of all, the variable satisfying the nonlinear Schrödinger (NLS) equation is separated into real part and imaginary part, where the amplitude, i.e. the envelope, is related to the mKdV equation. The rescaled independent variable and dependent variable satisfy the positive mKdV equation or the negative mKdV equation. It is shown that different kinds of solutions can be obtained to the NLS equation, including many kinds of envelope breather solution and envelope breather lattice solutions. Furthermore, when different independent variable transformations are adopted, there will be different results. For example, when we choose the independent variable transformation

$$\xi = ax + \frac{1}{a}t + \xi_0, \qquad \eta = ax - \frac{1}{a}t + \eta_0, \tag{52}$$

which is given in Ref. [2], some breather lattice solutions expressed in terms of Jacobi elliptic functions will be omitted.

From the view point of constructing the connection between the solutions of different family of nonlinear equations, there is still much work needed to do, since a wide variety of different physical systems can be described by a relatively small set of universal equations [15]. In this Letter, we have chosen a simple envelope representation of the wave function (a real amplitude and a phase linearly dependent on both space and time coordinates) for NLS equation, so the governed equation for the amplitude is turned to be the positive mKdV equation or the negative mKdV equation by suitable combinations. Actually, this approach can extended in a more general framework, as it has been done in Refs. [16,17], where a correspondence between envelope solutions of generalized NLS equations and solutions of generalized KdV equations is constructed. In their framework, the envelope representation allows to split a generalized NLS equation into a pair of the Madelung's fluid equations, when the connection between phase and amplitude is assumed, an equation for the only amplitude can be established to be the generalized KdV equation, Eq. (5) is just a special case. The generalized KdV equation can be solved to obtain its breather-typed solutions by means of suitable transformations (including Miura transformation, since the Miura transformation is an exact transformation between the solution of the mKdV equation and that of the KdV equation), just as we have done in Sections 2 and 3.

Due to wide applications of the NLS equation, the analytical solutions given in this Letter will be helpful in related research.

Acknowledgements

Many thanks are due to supports from National Natural Science Foundation of China (Nos. 40305006 and 90511009).

References

[1] P.G. Drazin, R.S. Johnson, Solitons: An Introduction, Cambridge Univ. Press, New York, 1989.
[2] G.L. Lamb Jr., Elements of Soliton Theory, John Wiley and Sons, New York, 1980.
[3] Z.T. Fu, S.K. Liu, S.D. Liu, Q. Zhao, Phys. Lett. A 290 (2001) 72.
[4] S.K. Liu, Z.T. Fu, S.D. Liu, Q. Zhao, Phys. Lett. A 289 (2001) 69.
[5] Z.T. Fu, L. Zhang, S.K. Liu, S.D. Liu, Phys. Lett. A 325 (2004) 364.
[6] Z.T. Fu, S.D. Liu, S.K. Liu, Phys. Lett. A 326 (2004) 364.
[7] P.G. Kevrekidis, A. Khare, A. Saxena, Phys. Rev. E 68 (2003) 047701.
[8] P.G. Kevrekidis, A. Khare, A. Saxena, G. Herring, J. Phys. A 37 (2004) 10959.
[9] P.G. Kevrekidis, A. Saxena, A.R. Bishop, Phys. Rev. E 64 (2001) 026613.
[10] S.K. Liu, S.D. Liu, Nonlinear Equations in Physics, Peking Univ. Press, Beijing, 2000.
[11] Z.X. Wang, D.R. Guo, Special Functions, World Scientific, Singapore, 1989.
[12] P.F. Byrd, M.D. Friedman, Handbook of Elliptic Integrals for Engineers and Physics, Springer-Verlag, Berlin, 1954.
[13] Z.T. Fu, S.D. Liu, S.K. Liu, J. Phys. A, in press.
[14] Z.T. Fu, S.D. Liu, S.K. Liu, submitted for publication.
[15] J.M. Christian, G.S. McDonald, P. Chamorro-Posada, J. Phys. A 39 (2006) 15355.
[16] R. Fedele, H. Schamel, Eur. Phys. J. B 27 (2002) 313.
[17] R. Fedele, Phys. Scripta 65 (2002) 502.