Physics Letters A 373 (2009) 1826–1829

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

# Novel solutions to the combined dispersion equation

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ARTICLE INFO

*Article history:*
Received 18 February 2009
Accepted 16 March 2009
Available online 20 March 2009
Communicated by A.R. Bishop

*PACS:*
03.65.Ge

*Keywords:*
Jacobian elliptic function
Combined dispersion equation
Periodic solution

ABSTRACT

In this Letter, the combined dispersion equation was solved by the sub-equation method. It is shown that the combined dispersion equation with the special parameters can be solved and many novel solutions will derived in terms of Jacobi elliptic functions, where some known solutions will be recovered when the modulus arrives its limiting value.

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## 1. Introduction

The combined dispersion equation (CSE for short)

$$u_t + \alpha u^l u_x + \beta (u^2)_{xxx} + \gamma u_{xxx} = 0, \tag{1}$$

where $l$ is a positive integer and denotes the intensity of the convection term, $\alpha$ is the parameter of the convection term, $\beta$ is the parameter of the nonlinear dispersion term and $\gamma$ is the parameter of the linear dispersion terms.

Eq. (1) connects with two important nonlinear equation for solitary water waves. When $l = 1$ and $\gamma = 0$, Eq. (1) recovers the Rosenau–Hyman equation

$$u_t + \alpha u u_x + \beta (u^2)_{xxx} = 0, \tag{2}$$

which admits compactons [1]. When $l = 2$ and $\beta = 0$, Eq. (1) reduces to the famous mKdV equation

$$u_t + \alpha u^2 u_x + \gamma u_{xxx} = 0. \tag{3}$$

Actually, Eq. (1) can be taken as a special shallow water equation with linear and nonlinear dispersion [2], which is related to the famous Camassa–Holm equation [3]. Or, we can take Eq. (1) as a special general modified Camassa–Holm equation [4–6], since they take the similar forms when we solve them in the travelling wave frame $\xi = k(x - ct)$.

Since CSE is a current research interest in nonlinear water waves, in this Letter, we will show systematical results for the CSE (1) by using the knowledge of Jacobian elliptic functions [7–10] and elliptic equation [11–13], where many novel solutions will be found.

We will seek the travelling wave solution in the following frame

$$u = u(\xi), \quad \xi = k(x - ct), \tag{4}$$

here $c$ is wave speed and $k$ is wave number.

Substituting (4) into (1) yields

$$-c\frac{du}{d\xi} + \alpha u^l \frac{du}{d\xi} + \beta k^2 \frac{d^3 u^2}{d\xi^3} + \gamma k^2 \frac{d^3 u}{d\xi^3} = 0. \tag{5}$$

It is obvious that Eq. (5) cannot be solved directly for all $l$, only some special values of $l$ can make Eq. (5) solvable directly. In fact, $l = 2$ and $l = 3$ are two special cases. In the next sections, we will show the detailed results for these two cases.

## 2. Solutions to CSE for $l = 2$

When $l = 2$, Eq. (5) can be rewritten as

$$-c\frac{du}{d\xi} + \alpha u^2 \frac{du}{d\xi} + \beta k^2 \frac{d^3 u^2}{d\xi^3} + \gamma k^2 \frac{d^3 u}{d\xi^3} = 0. \tag{6}$$

In order to solve Eq. (6), we will map it to the following elliptic equation [7,8]

$$y'^2 = a_0 + a_2 y^2 + a_4 y^4, \tag{7}$$

i.e.

$$y'' = a_2 y + 2a_4 y^3, \tag{8}$$

where the prime denotes the derivatives in terms of its argument.

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doi:10.1016/j.physleta.2009.03.041

The connecting bridge between (6) and (7) is the following finite expansion

$$u = u(\xi) = \sum_{i=0}^{i=n} b_i y^i, \tag{9}$$

with $b_n \neq 0$.

Balancing the nonlinear term and the dispersion term in Eq. (6), we can derive the final expansion as

$$u = b_0 + b_1 y + b_2 y^2, \quad b_2 \neq 0. \tag{10}$$

Return (10) to (6) and then we have the following algebraic equations

$$-cb_1 + \alpha b_0^2 b_1 + 2\beta k^2(b_0 b_1 a_2 + 6 b_1 b_2 a_0) + \gamma k^2 b_1 a_2 = 0, \tag{11a}$$

$$-2cb_2 + 2\alpha\left(b_0 b_1^2 + b_0^2 b_2\right) + 8\beta k^2\left[a_2\left(b_1^2 + 2b_0 b_2\right) + 3b_2^2 a_0\right] + 8\gamma k^2 b_2 a_2 = 0, \tag{11b}$$

$$\alpha\left(b_1^3 + 6b_0 b_1 b_2\right) + \beta k^2(12 b_0 b_1 a_4 + 54 b_1 b_2 a_2) + 6\gamma k^2 b_1 a_4 = 0, \tag{11c}$$

$$4\alpha\left(b_1^2 b_2 + b_0 b_2^2\right) + \beta k^2\left[24 a_4\left(b_1^2 + 2b_0 b_2\right) + 64 b_2^2 a_2\right] + 24\gamma k^2 b_2 a_4 = 0, \tag{11d}$$

$$5\alpha b_1 b_2^2 + 120\beta k^2 b_1 b_2 a_4 = 0, \tag{11e}$$

$$2\alpha b_2^3 + 120\beta k^2 b_2^2 a_4 = 0, \tag{11f}$$

from which we have

$$b_2 = -\frac{60\beta k^2 a_4}{\alpha}, \qquad b_1 = 0, \qquad b_0 = \frac{\gamma}{8\beta} - \frac{20\beta k^2 a_2}{\alpha},$$

$$c = \alpha b_0^2 + 8\beta k^2 b_0 + 12\beta k^2 b_2 a_0 + 4\gamma k^2 a_2. \tag{12}$$

So the solution to (6) is

$$u = \frac{\gamma}{8\beta} - \frac{20\beta k^2 a_2}{\alpha} - \frac{60\beta k^2 a_4}{\alpha} y^2, \tag{13}$$

with the wave speed $c = \alpha b_0^2 + 8\beta k^2 b_0 + 12\beta k^2 b_2 a_0 + 4\gamma k^2 a_2$. Obviously, when $a_0$, $a_2$ and $a_4$ take different values, there will be many different kinds of solutions, we will show some next expressed in terms of different Jacobi elliptic functions [7,8].

(1) If $a_0 = 1$, $a_2 = -(1+m^2)$ and $a_4 = m^2$, then the solution is

$$u_1 = \frac{\gamma}{8\beta} + \frac{20\beta k^2(1+m^2)}{\alpha} - \frac{60\beta k^2 m^2}{\alpha} \operatorname{sn}^2(\xi, m), \tag{14}$$

where $0 \leqslant m \leqslant 1$, is called modulus of Jacobi elliptic functions, see [11,12], and $\operatorname{sn}(\xi, m)$ is Jacobi elliptic sine function, see [11–13].

(2) If $a_0 = 1 - m^2$, $a_2 = 2m^2 - 1$ and $a_4 = -m^2$, then the solution is

$$u_2 = \frac{\gamma}{8\beta} + \frac{20\beta k^2(1-2m^2)}{\alpha} + \frac{60\beta k^2 m^2}{\alpha} \operatorname{cn}^2(\xi, m), \tag{15}$$

where $\operatorname{cn}(\xi, m)$ is Jacobi elliptic cosine function, see [11–13].

(3) If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$ and $a_4 = -1$, then the solution is

$$u_3 = \frac{\gamma}{8\beta} - \frac{20\beta k^2(2-m^2)}{\alpha} + \frac{60\beta k^2}{\alpha} \operatorname{dn}^2(\xi, m), \tag{16}$$

where $\operatorname{dn}(\xi, m)$ is Jacobi elliptic function of the third kind, see [11–13].

(4) If $a_0 = m^2$, $a_2 = -(1+m^2)$ and $a_4 = 1$, then the solution is

$$u_4 = \frac{\gamma}{8\beta} + \frac{20\beta k^2(1+m^2)}{\alpha} - \frac{60\beta k^2}{\alpha} \operatorname{ns}^2(\xi, m), \tag{17}$$

with $\operatorname{ns}(\xi, m) \equiv \frac{1}{\operatorname{sn}(\xi,m)}$.

(5) If $a_0 = -m^2$, $a_2 = 2m^2 - 1$ and $a_4 = 1 - m^2$, then the solution is

$$u_5 = \frac{\gamma}{8\beta} + \frac{20\beta k^2(1-2m^2)}{\alpha} - \frac{60\beta k^2(1-m^2)}{\alpha} \operatorname{nc}^2(\xi, m), \tag{18}$$

with $\operatorname{nc}(\xi, m) \equiv \frac{1}{\operatorname{cn}(\xi,m)}$.

(6) If $a_0 = -1$, $a_2 = 2 - m^2$ and $a_4 = m^2 - 1$, then the solution is

$$u_6 = \frac{\gamma}{8\beta} - \frac{20\beta k^2(2-m^2)}{\alpha} + \frac{60\beta k^2(1-m^2)}{\alpha} \operatorname{nd}^2(\xi, m), \tag{19}$$

with $\operatorname{nd}(\xi, m) \equiv \frac{1}{\operatorname{dn}(\xi,m)}$.

(7) If $a_0 = 1$, $a_2 = 2 - m^2$ and $a_4 = 1 - m^2$, then the solution is

$$u_7 = \frac{\gamma}{8\beta} - \frac{20\beta k^2(2-m^2)}{\alpha} - \frac{60\beta k^2(1-m^2)}{\alpha} \operatorname{sc}^2(\xi, m), \tag{20}$$

with $\operatorname{sc}(\xi, m) \equiv \frac{\operatorname{sn}(\xi,m)}{\operatorname{cn}(\xi,m)}$.

(8) If $a_0 = 1$, $a_2 = 2m^2 - 1$ and $a_4 = (m^2 - 1)m^2$, then the solution is

$$u_8 = \frac{\gamma}{8\beta} - \frac{20\beta k^2(2m^2-1)}{\alpha} + \frac{60\beta k^2(1-m^2)m^2}{\alpha} \operatorname{sd}^2(\xi, m), \tag{21}$$

with $\operatorname{sd}(\xi, m) \equiv \frac{\operatorname{sn}(\xi,m)}{\operatorname{dn}(\xi,m)}$.

(9) If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$ and $a_4 = 1$, then the solution is

$$u_9 = \frac{\gamma}{8\beta} - \frac{20\beta k^2(2-m^2)}{\alpha} - \frac{60\beta k^2}{\alpha} \operatorname{cs}^2(\xi, m), \tag{22}$$

with $\operatorname{cs}(\xi, m) \equiv \frac{\operatorname{cn}(\xi,m)}{\operatorname{sn}(\xi,m)}$.

(10) If $a_0 = 1$, $a_2 = -(1+m^2)$ and $a_4 = m^2$, then the solution is

$$u_{10} = \frac{\gamma}{8\beta} + \frac{20\beta k^2(1+m^2)}{\alpha} - \frac{60\beta k^2 m^2}{\alpha} \operatorname{cd}^2(\xi, m), \tag{23}$$

with $\operatorname{cd}(\xi, m) \equiv \frac{\operatorname{cn}(\xi,m)}{\operatorname{dn}(\xi,m)}$.

(11) If $a_0 = m^2(m^2 - 1)$, $a_2 = 2m^2 - 1$ and $a_4 = 1$, then the solution is

$$u_{11} = \frac{\gamma}{8\beta} + \frac{20\beta k^2(1-2m^2)}{\alpha} - \frac{60\beta k^2}{\alpha} \operatorname{ds}^2(\xi, m), \tag{24}$$

with $\operatorname{ds}(\xi, m) \equiv \frac{\operatorname{dn}(\xi,m)}{\operatorname{sn}(\xi,m)}$.

(12) If $a_0 = m^2$, $a_2 = -(1+m^2)$ and $a_4 = 1$, then the solution is

$$u_{12} = \frac{\gamma}{8\beta} + \frac{20\beta k^2(1+m^2)}{\alpha} - \frac{60\beta k^2}{\alpha} \operatorname{dc}^2(\xi, m), \tag{25}$$

with $\operatorname{dc}(\xi, m) \equiv \frac{\operatorname{dn}(\xi,m)}{\operatorname{cn}(\xi,m)}$.

(13) If $a_0 = (1 - m^2)/4$, $a_2 = (1 + m^2)/2$ and $a_4 = (1 - m^2)/4$, then the solution is

$$u_{13} = \frac{\gamma}{8\beta} - \frac{10\beta k^2(1+m^2)}{\alpha} - \frac{15\beta k^2(1-m^2)}{\alpha}\left[\frac{\operatorname{cn}(\xi, m)}{1 \pm \operatorname{sn}(\xi, m)}\right]^2. \tag{26}$$

(14) If $a_0 = -(1 - m^2)/4$, $a_2 = (1 + m^2)/2$ and $a_4 = -(1 - m^2)/4$, then the solution is

$$u_{14} = \frac{\gamma}{8\beta} - \frac{10\beta k^2(1+m^2)}{\alpha} + \frac{15\beta k^2(1-m^2)}{\alpha}\left[\frac{\operatorname{dn}(\xi, m)}{1 \pm m \operatorname{sn}(\xi, m)}\right]^2. \tag{27}$$

(15) If $a_0 = m^2/4$, $a_2 = -(2 - m^2)/2$ and $a_4 = m^2/4$, then the solution is

$$u_{15} = \frac{\gamma}{8\beta} + \frac{10\beta k^2(2-m^2)}{\alpha} - \frac{15\beta k^2 m^2}{\alpha}\left[\frac{m\,\mathrm{sn}(\xi,m)}{1 \pm \mathrm{dn}(\xi,m)}\right]^2. \tag{28}$$

(16) If $a_0 = 1/4$, $a_2 = (1 - 2m^2)/2$ and $a_4 = 1/4$, then the solution is

$$u_{16} = \frac{\gamma}{8\beta} - \frac{10\beta k^2(1-2m^2)}{\alpha} - \frac{15\beta k^2}{\alpha}\left[\frac{\mathrm{sn}(\xi,m)}{1 \pm \mathrm{cn}(\xi,m)}\right]^2. \tag{29}$$

(17) If $a_0 = 1/4$, $a_2 = -(2 - m^2)/2$ and $a_4 = m^4/4$, then the solution is

$$u_{17} = \frac{\gamma}{8\beta} + \frac{10\beta k^2(2-m^2)}{\alpha} - \frac{15\beta k^2 m^4}{\alpha}\left[\frac{\mathrm{sn}(\xi,m)}{1 \pm \mathrm{dn}(\xi,m)}\right]^2. \tag{30}$$

There still exist many other kinds of Jacobi elliptic functions, we do not show here. It is known that when $m \to 1$, $\mathrm{sn}(\xi,m) \to \tanh\xi$, $\mathrm{cn}(\xi,m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}(\xi,m) \to \mathrm{sech}\,\xi$ and when $m \to 0$, $\mathrm{sn}(\xi,m) \to \sin\xi$, $\mathrm{cn}(\xi,m) \to \cos\xi$, so we also can derive solutions expressed in terms of hyperbolic functions and trigonometric functions. For example, when $m \to 1$, $u_{15}$ becomes

$$u_{15'} = \frac{\gamma}{8\beta} + \frac{10\beta k^2}{\alpha} - \frac{15\beta k^2}{\alpha}\tanh\left(\frac{\xi}{2}\right), \tag{31}$$

and

$$u_{15''} = \frac{\gamma}{8\beta} + \frac{10\beta k^2}{\alpha} - \frac{15\beta k^2}{\alpha}\coth\left(\frac{\xi}{2}\right). \tag{32}$$

**Remarks.** If the mapped elliptic equation is chosen as

$$y'^2 = a_0 + a_1 y + a_2 y^2 + a_4 y^4, \tag{33}$$

then if and only if $a_1$ is zero, Eq. (6) can be solved. So we have the same results with the two different mapped equations, the new mapped equation (33) will result in no new solutions to Eq. (6).

## 3. Solutions to CSE for $l = 3$

When $l = 3$, Eq. (5) can be rewritten as

$$-c\frac{du}{d\xi} + \alpha u^3\frac{du}{d\xi} + \beta k^2\frac{d^3u^2}{d\xi^3} + \gamma k^2\frac{d^3u}{d\xi^3} = 0. \tag{34}$$

Similarly, the connecting bridge between (34) and (7) will let us derive the final expansion as

$$u = b_0 + b_1 y, \quad b_1 \neq 0. \tag{35}$$

Return (35) to (34) and then we have the following algebraic equations

$$-cb_1 + \alpha b_0^3 b_1 + 2\beta k^2 b_0 b_1 a_2 + \gamma k^2 b_1 a_2 = 0, \tag{36a}$$

$$3\alpha b_0^2 b_1^2 + 8\beta k^2 b_1^2 a_2 = 0, \tag{36b}$$

$$3\alpha b_0 b_1^3 + 12\beta k^2 b_0 b_1 a_4 + 6\gamma k^2 b_1 a_4 = 0, \tag{36c}$$

$$\alpha b_1^4 + 24\beta k^2 b_1^2 a_4 = 0, \tag{36d}$$

from which we have

$$b_1 = \pm\frac{3\gamma}{10\beta}\sqrt{\frac{a_4}{a_2}}, \qquad b_0 = \frac{\gamma}{10\beta},$$
$$c = \alpha b_0^3 + 2\beta k^2 b_0 a_2 + \gamma k^2 a_2, \tag{37}$$

with the constraint

$$\frac{a_4}{a_2} > 0. \tag{38}$$

So the solution to (34) is

$$u = \frac{\gamma}{10\beta} \pm \frac{3\gamma}{10\beta}\sqrt{\frac{a_4}{a_2}}\,y, \tag{39}$$

with the wave speed $c = \alpha b_0^3 + 2\beta k^2 b_0 a_2 + \gamma k^2 a_2$. Obviously, when $a_0$, $a_2$ and $a_4$ take different values and $a_2$ and $a_4$ satisfy the constraint (38), there will be many different kinds of solutions, we will show some next expressed in terms of different Jacobi elliptic functions [7,8].

(1) If $a_0 = 1 - m^2$, $a_2 = 2m^2 - 1$ and $a_4 = -m^2$ with $m^2 < \frac{1}{2}$, then the solution is

$$u_1 = \frac{\gamma}{10\beta} \pm \frac{3\gamma}{10\beta}\sqrt{\frac{m^2}{1-2m^2}}\,\mathrm{cn}(\xi,m). \tag{40}$$

(2) If $a_0 = -m^2$, $a_2 = 2m^2 - 1$ and $a_4 = 1 - m^2$ with $m^2 > \frac{1}{2}$, then the solution is

$$u_2 = \frac{\gamma}{10\beta} \pm \frac{3\gamma}{10\beta}\sqrt{\frac{1-m^2}{2m^2-1}}\,\mathrm{nc}(\xi,m). \tag{41}$$

(3) If $a_0 = 1$, $a_2 = 2 - m^2$ and $a_4 = 1 - m^2$, then the solution is

$$u_3 = \frac{\gamma}{10\beta} \pm \frac{3\gamma}{10\beta}\sqrt{\frac{1-m^2}{2-m^2}}\,\mathrm{sc}(\xi,m). \tag{42}$$

(4) If $a_0 = 1$, $a_2 = 2m^2 - 1$ and $a_4 = m^2(m^2 - 1)$ with $m^2 < \frac{1}{2}$, then the solution is

$$u_4 = \frac{\gamma}{10\beta} \pm \frac{3\gamma}{10\beta}\sqrt{\frac{m^2(m^2-1)}{2m^2-1}}\,\mathrm{sd}(\xi,m). \tag{43}$$

(5) If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$ and $a_4 = 1$, then the solution is

$$u_5 = \frac{\gamma}{10\beta} \pm \frac{3\gamma}{10\beta}\sqrt{\frac{1}{2-m^2}}\,\mathrm{cs}(\xi,m). \tag{44}$$

(6) If $a_0 = m^2(m^2 - 1)$, $a_2 = 2m^2 - 1$ and $a_4 = 1$ with $m^2 > \frac{1}{2}$, then the solution is

$$u_6 = \frac{\gamma}{10\beta} \pm \frac{3\gamma}{10\beta}\sqrt{\frac{1}{2m^2-1}}\,\mathrm{ds}(\xi,m). \tag{45}$$

(7) If $a_0 = (1 - m^2)/4$, $a_2 = (1 + m^2)/2$ and $a_4 = (1 - m^2)/4$, then the solution is

$$u_7 = \frac{\gamma}{10\beta} \pm \frac{3\gamma}{10\beta}\sqrt{\frac{1-m^2}{2(1+m^2)}}\,\frac{\mathrm{cn}(\xi,m)}{1 \pm \mathrm{sn}(\xi,m)}. \tag{46}$$

(8) If $a_0 = 1/4$, $a_2 = (1 - 2m^2)/2$ and $a_4 = 1/4$ with $m^2 < \frac{1}{2}$, then the solution is

$$u_8 = \frac{\gamma}{10\beta} \pm \frac{3\gamma}{10\beta}\sqrt{\frac{1}{2(1-2m^2)}}\,\frac{\mathrm{sn}(\xi,m)}{1 \pm \mathrm{cn}(\xi,m)}. \tag{47}$$

**Remarks.** If the mapped elliptic equation is chosen as (33), Eq. (34) can still be solved. But the new mapped equation (33) will result in no changes to solutions of Eq. (34) except a new wave speed $c = \alpha b_0^3 + 2\beta k^2 b_0 a_2 + 3\beta k^2 b_1 a_1 + \gamma k^2 a_2$ with $b_0$ and $b_1$ still given by (37).

## 4. Conclusion

In this Letter, we presented the process to find exact solutions for the CSE with the help from the bridge connecting CSE to the elliptic equation and obtained some novel types of solutions, these solutions may be applied to describe and/or explain some phenomena found in the nonlinear water waves, since the model has been proposed to model nonlinear water waves. Another result given in this Letter is that choosing an appropriate mapping equation is of great importance in solving nonlinear equations, since the wrong chosen mapping equation may results in no solution to the solved nonlinear equations.

## Acknowledgement

Many thanks are due to support from National Natural Science Foundation of China (No. 90511009) and National Basic Research program of China (Grants 2006CB403600 and 2005CB42204).

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