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Bose-Einstein 凝聚态中的一维 Gross-Pitaevskii 方程的包络周期解和孤立波解

高斌[1,2]　刘式适[1,†]　付遵涛[1]　刘式达[1]

1. 北京大学物理学院, 北京 100871; 2. 玉溪师范学院物理系, 玉溪 653100; †通讯作者, E-mail: liusk@pku.edu.cn

摘要　非线性 Schrödinger 方程的精确解对于理解 Bose-Einstein 凝聚态动力学演化有重要的作用。应用 Jacobi 椭圆函数展开法,求得描述 Bose-Einstein 凝聚态的一维 Gross-Pitaevskii 方程的包络周期解。在极限情况下,包络周期解可导出包络孤立波解。

关键词　Gross-Pitaevskii 方程; Jacobi 椭圆函数; 包络周期解; 包络孤立波解
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Envelope Periodic and Solitary Solutions to One Dimensional Gross-Pitaevskii Equation in Bose-Einstein Condensates

GAO Bin[1,2], LIU Shikuo[1,†], FU Zuntao[1], LIU Shida[1]

1. School of Physics, Peking University, Beijing 100871; 2. Department of Physics, Yuxi Normal University, Yuxi 653100;
† Corresponding Author, E-mail: liusk@pku.edu.cn

Abstract　The construction of exact solutions of nonlinear Schrödinger equations plays an important role in understanding the dynamic evolution of Bose-Einstein condensates. The Jacobi elliptic function expansion method is applied to construct the envelope periodic solutions to one dimensional Gross-Pitaevskii equation in Bose-Einstein condensates. The envelope periodic solutions can degenerate to the envelope solitary solutions under the limited condition.

Key words　Gross-Pitaevskii equation; Jacobi elliptic function; envelope periodic solution; envelope solitary solution

描写 Bose-Einstein 凝聚态(简称 BEC)的动力学演化方程通常被称为 Gross-Pitaevskii (简称 GP) 方程[1-2],它可写为

$$\mathrm{i}\frac{\partial u}{\partial t} + \alpha\nabla^2 u + \beta|u|^2 u - V(\mathrm{r})u = 0, \qquad (1)$$

其中 u 是波函数, α 是色散系数,是常数, β 可称为 Landau 系数,它正比于原子的散射长度,可以是常数,也可是时间的函数, $V(\mathrm{r})$ 是外力势, ∇^2 为三维 Laplace 算子。

对于一般的外力势 $V(\mathrm{r})$, GP 方程很难求解,但对于磁陷阱的约束势, $V(\mathrm{r})$ 可表示为二次的形式,即

$$V(\mathrm{r}) = a\left(x^2 + y^2 + \lambda z^2\right), \qquad (2)$$

其中 a 和 λ 是常数,特别是在柱对称的情况下(如雪茄型凝聚态), 三维 GP 方程(1)能够化为下列一维 GP 方程[3-5]:

$$\mathrm{i}\frac{\partial u}{\partial t} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta|u|^2 u + \gamma x^2 u = 0, \qquad (3)$$

其中 γ 是常数。一维 GP 方程(3)也可以称为是带有一个喷射式外力势的一维非线性 Schrödinger 方程。利用 Darboux 变换[6-7]、双曲函数展开法[8-9]和其他变换[10]可以求得一维 GP 方程(3)的孤立子解。

本文应用 Jacobi 椭圆函数展开法[11-12], 在 β 依赖于时间的条件下:

$$\beta = \beta(t) = \beta_0 \mathrm{e}^{\pm 2\sqrt{\alpha\gamma}t}, \qquad (4)$$

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求一维 GP 方程(3)的包络周期解和包络孤立波解。

对于一维 GP 方程(3),首先令

$$u = \phi(x,t)\mathrm{e}^{\mathrm{i}\theta(x,t)}, \tag{5}$$

其中 $\phi(x,t)$ 和 $\theta(x,t)$ 是实函数。

式(5)代入式(3)并且分开实部和虚部,得到

$$-\phi\frac{\partial\theta}{\partial t} + \alpha\left[\frac{\partial^2\phi}{\partial x^2} - \phi\left(\frac{\partial\theta}{\partial x}\right)^2\right] + \beta(t)\phi^3 + \gamma x^2\phi = 0, \tag{6a}$$

$$\frac{\partial\phi}{\partial t} + \alpha\phi\frac{\partial^2\theta}{\partial x^2} + 2\alpha\frac{\partial\phi}{\partial x}\frac{\partial\theta}{\partial x} = 0, \tag{6b}$$

这是关于 ϕ 和 θ 的变系数非线性方程组。下面应用扩展的 Jacobi 椭圆函数展开法求解。

1　Jacobi 椭圆正弦函数展开法

依据扩展的 Jacobi 椭圆正弦函数展开法,方程组(6)有如下形式的解:

$$\phi = a_0(t) + a_1(t)\mathrm{sn}\xi,\ \xi = f(t)x + g(t), \tag{7a}$$

$$\theta = b_0(t) + b_1(t)x + b_2(t)x^2, \tag{7b}$$

其中 $a_0(t)$, $a_1(t)$, $b_0(t)$, $b_1(t)$, $b_2(t)$, $f(t)$ 和 $g(t)$ 都是时间 t 的任意函数。snξ 是 Jacobi 椭圆正弦函数。

注意,由式(7)有

$$\frac{\partial\phi}{\partial t} = a_0'(t) + a_1'(t)\mathrm{sn}\xi + a_1(t)(f'(t)x + g'(t))\mathrm{cn}\xi\mathrm{dn}\xi, \tag{8a}$$

$$\frac{\partial\phi}{\partial x} = f(t)a_1(t)\mathrm{cn}\xi\mathrm{dn}\xi, \tag{8b}$$

$$\frac{\partial^2\phi}{\partial x^2} = -\left(1 + m^2\right)f^2(t)a_1(t)\mathrm{sn}\xi + 2m^2f^2(t)a_1(t)\mathrm{sn}^3\xi, \tag{8c}$$

$$\phi^3 = a_0^3(t) + 3a_0^2(t)a_1(t)\mathrm{sn}\xi + 3a_0(t)a_1^2(t)\mathrm{sn}^2\xi + a_1^3(t)\mathrm{sn}^3\xi, \tag{8d}$$

$$\frac{\partial\theta}{\partial t} = b_0'(t) + b_1'(t)x + b_2'(t)x^2, \tag{8e}$$

$$\frac{\partial\theta}{\partial x} = b_1(t) + 2b_2(t)x, \tag{8f}$$

$$\frac{\partial^2\theta}{\partial x^2} = 2b_2(t), \tag{8g}$$

其中 cnξ,dnξ 分别为 Jacobi 椭圆余弦函数和第三类 Jacobi 椭圆函数,m 为模数($0 \le m \le 1$)。

将式(7)和(8)代入式(6)得到

$$\begin{aligned}&-a_0(t)\left[b_0'(t) + \alpha b_1^2(t) - \beta(t)a_0^2(t)\right]\\&-a_0(t)\left[b_1'(t) + 4\alpha b_1(t)b_2(t)\right]x\\&-a_0(t)\left[b_2'(t) + 4\alpha b_2^2(t) - \gamma\right]x^2\\&-a_1(t)\left[b_0'(t) + (1+m^2)\alpha f^2(t) + \alpha b_1^2(t) - 3\beta(t)a_0^2(t)\right]\mathrm{sn}\xi\\&+3\beta(t)a_0(t)a_1^2(t)\mathrm{sn}^2\xi\\&+a_1(t)\left[2\alpha m^2f^2(t) + \beta(t)a_1^2(t)\right]\mathrm{sn}^3\xi\\&-a_1(t)\left[b_1'(t) + 4\alpha b_1(t)b_2(t)\right]x\mathrm{sn}\xi\\&-a_1(t)\left[b_2'(t) + 4\alpha b_2^2(t) - \gamma\right]x^2\mathrm{sn}\xi = 0,\end{aligned} \tag{9a}$$

$$\begin{aligned}&\left[a_0'(t) + 2\alpha a_0(t)b_2(t)\right]\\&+\left[a_1'(t) + 2\alpha a_1(t)b_2(t)\right]\mathrm{sn}\xi + a_1(t)\\&\cdot\left[g'(t) + 2\alpha b_1(t)f(t)\right]\mathrm{cn}\xi\mathrm{dn}\xi\\&+a_1(t)\left[f'(t) + 4\alpha b_2(t)f(t)\right]\\&\cdot x\mathrm{cn}\xi\mathrm{dn}\xi = 0。\end{aligned} \tag{9b}$$

在 $a_1(t) \ne 0$ 的条件下,由式(9)得

$$a_0(t) = 0, \tag{10a}$$

$$b_0'(t) + \left(1+m^2\right)\alpha f^2(t) + \alpha b_1^2(t) = 0, \tag{10b}$$

$$2\alpha m^2f^2(t) + \beta(t)a_1^2(t) = 0, \tag{10c}$$

$$b_1'(t) + 4\alpha b_1(t)b_2(t) = 0, \tag{10d}$$

$$b_2'(t) + 4\alpha b_2^2(t) - \gamma = 0, \tag{10e}$$

$$a_1'(t) + 2\alpha a_1(t)b_2(t) = 0, \tag{10f}$$

$$g'(t) + 2\alpha b_1(t)f(t) = 0, \tag{10g}$$

$$f'(t) + 4\alpha b_2(t)f(t) = 0。 \tag{10h}$$

在上述 8 个方程中,只有式(10e)仅含一个变量 $b_2(t)$,它是 Riccati 方程,至少有如下 4 个简单的解:

$$b_2^{(1)}(t) = -\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}, \tag{11a}$$

$$b_2^{(2)}(t) = \frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}, \tag{11b}$$

$$b_2^{(3)}(t) = \frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}\tanh 2\sqrt{\alpha\gamma}t, \tag{11c}$$

$$b_2^{(4)}(t) = \frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}\coth 2\sqrt{\alpha\gamma}t。 \tag{11d}$$

首先将式(11)代入式(10f),(10d)和(10h),依次得到:

$$a_1^{(1)}(t) = a_{10}\mathrm{e}^{\sqrt{\alpha\gamma}t}, \tag{12a}$$

$$a_1^{(2)}(t) = a_{10}\mathrm{e}^{-\sqrt{\alpha\gamma}t}, \tag{12b}$$

$$a_1^{(3)}(t) = a_{10}(\mathrm{sech}2\sqrt{\alpha\gamma}t)^{1/2}, \tag{12c}$$

$$a_1^{(4)}(t) = a_{10}(\mathrm{csch}2\sqrt{\alpha\gamma}t)^{1/2}, \tag{12d}$$

$$b_1^{(1)}(t) = b_{10}\mathrm{e}^{2\sqrt{\alpha\gamma}t}, \tag{13a}$$

$$b_1^{(2)}(t) = b_{10}\mathrm{e}^{-2\sqrt{\alpha\gamma}t}, \tag{13b}$$

$$b_1^{(3)}(t) = b_{10}\operatorname{sech} 2\sqrt{\alpha\gamma}t, \tag{13c}$$

$$b_1^{(4)}(t) = b_{10}\operatorname{csch} 2\sqrt{\alpha\gamma}t, \tag{13d}$$

$$f^{(1)}(t) = f_0 e^{2\sqrt{\alpha\gamma}t}, \tag{14a}$$

$$f^{(2)}(t) = f_0 e^{-2\sqrt{\alpha\gamma}t}, \tag{14b}$$

$$f^{(3)}(t) = f_0\operatorname{sech} 2\sqrt{\alpha\gamma}t, \tag{14c}$$

$$f^{(4)}(t) = f_0\operatorname{csch} 2\sqrt{\alpha\gamma}t, \tag{14d}$$

其中 a_{10}, b_{10} 和 f_0 是常数。

将式(13)和(14)代入式(10g)得到:

$$g^{(1)}(t) = g_0 e^{4\sqrt{\alpha\gamma}t}\quad\left(g_0 = -\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}b_{10}f_0\right), \tag{15a}$$

$$g^{(2)}(t) = g_0 e^{-4\sqrt{\alpha\gamma}t}\quad\left(g_0 = \frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}b_{10}f_0\right), \tag{15b}$$

$$g^{(3)}(t) = g_0\tanh 2\sqrt{\alpha\gamma}t\quad\left(g_0 = -\sqrt{\frac{\alpha}{\gamma}}b_{10}f_0\right), \tag{15c}$$

$$g^{(4)}(t) = g_0\coth 2\sqrt{\alpha\gamma}t\quad\left(g_0 = \sqrt{\frac{\alpha}{\gamma}}b_{10}f_0\right)。 \tag{15d}$$

将式(12)和(14)代入式(10c)得

$$m^2 = -\frac{\beta_0}{2\alpha}\frac{a_{10}^2}{f_0^2}。 \tag{16}$$

注意上式对

$\beta(t) = \beta_0 e^{2\sqrt{\alpha\gamma}t}$, $\beta(t) = \beta_0 e^{-2\sqrt{\alpha\gamma}t}$, $\beta(t) = \beta_0\operatorname{sech} 2\sqrt{\alpha\gamma}t$, $\beta(t) = \beta_0\operatorname{csch} 2\sqrt{\alpha\gamma}t$ 均成立。

将式(13)和(14)代入(10b)得

$$b_0^{(1)}(t) = -\frac{1}{4}\sqrt{\frac{\alpha}{\gamma}}\left[\left(1+m^2\right)f_0^2 + b_{10}^2\right]e^{4\sqrt{\alpha\gamma}t}, \tag{17a}$$

$$b_0^{(2)}(t) = \frac{1}{4}\sqrt{\frac{\alpha}{\gamma}}\left[\left(1+m^2\right)f_0^2 + b_{10}^2\right]e^{-4\sqrt{\alpha\gamma}t}, \tag{17b}$$

$$b_0^{(3)}(t) = -\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\left[\left(1+m^2\right)f_0^2 + b_{10}^2\right]\cdot\tanh 2\sqrt{\alpha\gamma}t, \tag{17c}$$

$$b_0^{(4)}(t) = \frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\left[\left(1+m^2\right)f_0^2 + b_{10}^2\right]\cdot\coth 2\sqrt{\alpha\gamma}t。 \tag{17d}$$

将式(10a),(11)—(17)代入式(7),可以得到关于 ϕ 和 θ 的下列 4 种解:

$$\left\{\begin{aligned}
&\phi^{(1)} = a_{10}e^{\sqrt{\alpha\gamma}t}\operatorname{sn}\xi,\\
&\theta^{(1)} = -\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}x^2 + b_{10}e^{2\sqrt{\alpha\gamma}t}x - \frac{1}{4}\sqrt{\frac{\alpha}{\gamma}}\left[(1+m^2)f_0^2 + b_{10}^2\right]e^{4\sqrt{\alpha\gamma}t},\\
&\xi = f_0e^{2\sqrt{\alpha\gamma}t}\cdot x - \frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}b_{10}f_0e^{4\sqrt{\alpha\gamma}t},\\
&\beta(t) = \beta_0e^{2\sqrt{\alpha\gamma}t},\ m^2 = -\frac{\beta_0}{2\alpha}\frac{a_{10}^2}{f_0^2},
\end{aligned}\right. \tag{18a}$$

$$\left\{\begin{aligned}
&\phi^{(2)} = a_{10}e^{-\sqrt{\alpha\gamma}t}\operatorname{sn}\xi,\\
&\theta^{(2)} = \frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}x^2 + b_{10}e^{-2\sqrt{\alpha\gamma}t}x + \frac{1}{4}\sqrt{\frac{\alpha}{\gamma}}\left[(1+m^2)f_0^2 + b_{10}^2\right]e^{-4\sqrt{\alpha\gamma}t},\\
&\xi = f_0e^{-2\sqrt{\alpha\gamma}t}\cdot x + \frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}b_{10}f_0e^{-4\sqrt{\alpha\gamma}t},\\
&\beta(t) = \beta_0e^{-2\sqrt{\alpha\gamma}t},\ m^2 = -\frac{\beta_0}{2\alpha}\frac{a_{10}^2}{f_0^2},
\end{aligned}\right. \tag{18b}$$

$$\left\{\begin{aligned}
&\phi^{(3)} = a_{10}\left(\operatorname{sech} 2\sqrt{\alpha\gamma}t\right)^{1/2}\operatorname{sn}\xi,\\
&\theta^{(3)} = \frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}\left(\tanh 2\sqrt{\alpha\gamma}t\right)x^2 + b_{10}\left(\operatorname{sech} 2\sqrt{\alpha\gamma}t\right)x\\
&\qquad - \frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\left[\left(1+m^2\right)f_0^2 + b_{10}^2\right]\tanh 2\sqrt{\alpha\gamma}t,\\
&\xi = f_0\left(\operatorname{sech} 2\sqrt{\alpha\gamma}t\right)\cdot x - \sqrt{\frac{\alpha}{\gamma}}b_{10}f_0\tanh 2\sqrt{\alpha\gamma}t,\\
&\beta(t) = \beta_0\operatorname{sech} 2\sqrt{\alpha\gamma}t,\ m^2 = -\frac{\beta_0}{2\alpha}\frac{a_{10}^2}{f_0^2},
\end{aligned}\right. \tag{18c}$$

$$\left\{\begin{aligned}
&\phi^{(4)} = a_{10}\left(\operatorname{csch} 2\sqrt{\alpha\gamma}t\right)^{1/2}\operatorname{sn}\xi,\\
&\theta^{(4)} = \frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}\left(\coth 2\sqrt{\alpha\gamma}t\right)x^2 + b_{10}\left(\operatorname{csch} 2\sqrt{\alpha\gamma}t\right)x\\
&\qquad + \frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\left[\left(1+m^2\right)f_0^2 + b_{10}^2\right]\coth 2\sqrt{\alpha\gamma}t,\\
&\xi = f_0\left(\operatorname{csch} 2\sqrt{\alpha\gamma}t\right)\cdot x + \sqrt{\frac{\alpha}{\gamma}}b_{10}f_0\coth 2\sqrt{\alpha\gamma}t,\\
&\beta(t) = \beta_0\operatorname{csch} 2\sqrt{\alpha\gamma}t\ m^2 = -\frac{\beta_0}{2\alpha}\frac{a_{10}^2}{f_0^2}。
\end{aligned}\right. \tag{18d}$$

兹记

$$\left\{\begin{aligned}
&f_0 = p,\ b_{10} = k,\ c_g = 2\alpha b_{10} = 2\alpha k,\\
&\omega = \alpha k^2 + (1+m^2)\alpha p^2,
\end{aligned}\right. \tag{19}$$

其中 p, k 分别为波包和载波的波数, ω 为载波的圆频率, c_g 为波包的群速度。将式(18)和(19)代到式(5),就得到一维 GP 方程(3)的下列 4 种包络周期解:

$$
\begin{cases}
u^{(1)} = \left\{ \pm\sqrt{-\dfrac{2\alpha}{\beta_0}}\, m p e^{\sqrt{\alpha\gamma}t} \operatorname{sn} p \cdot \left\{ \left[e^{2\sqrt{\alpha\gamma}t} x - \dfrac{c_g}{4\sqrt{\alpha\gamma}} e^{4\sqrt{\alpha\gamma}t} \right] \right\} \cdot e^{i\left(-\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}x^2 + kxe^{2\sqrt{\alpha\gamma}t} - \frac{\omega}{4\sqrt{\alpha\gamma}}e^{4\sqrt{\alpha\gamma}t}\right)} \right\}, \\
\beta(t) = \beta_0 e^{2\sqrt{\alpha\gamma}t}, \quad m^2 = -\dfrac{\beta_0}{2\alpha}\dfrac{a_{10}^2}{f_0^2},
\end{cases} \tag{20a}
$$

$$
\begin{cases}
u^{(2)} = \left\{ \pm\sqrt{-\dfrac{2\alpha}{\beta_0}}\, m p e^{-\sqrt{\alpha\gamma}t} \operatorname{sn} p \cdot \left\{ \left[e^{-2\sqrt{\alpha\gamma}t} x + \dfrac{c_g}{4\sqrt{\alpha\gamma}} e^{-4\sqrt{\alpha\gamma}t} \right] \right\} \cdot e^{i\left(\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}x^2 + kxe^{-2\sqrt{\alpha\gamma}t} + \frac{\omega}{4\sqrt{\alpha\gamma}}e^{-4\sqrt{\alpha\gamma}t}\right)} \right\}, \\
\beta(t) = \beta_0 e^{-2\sqrt{\alpha\gamma}t}, \quad m^2 = -\dfrac{\beta_0}{2\alpha}\dfrac{a_{10}^2}{f_0^2},
\end{cases} \tag{20b}
$$

$$
\begin{cases}
u^{(3)} = \left\{ \pm\sqrt{-\dfrac{2\alpha}{\beta_0}}\, m p \left(\operatorname{sech} 2\sqrt{\alpha\gamma}t\right)^{1/2} \operatorname{sn} p \left[\left(\operatorname{sech} 2\sqrt{\alpha\gamma}t\right) x - \dfrac{c_g}{2\sqrt{\alpha\gamma}}\left(\tanh 2\sqrt{\alpha\gamma}t\right) \right] \right\} \cdot e^{i\left[\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}\left(\tanh 2\sqrt{\alpha\gamma}t\right)x^2 + k\left(\operatorname{sech} 2\sqrt{\alpha\gamma}t\right)x - \frac{\omega}{2\sqrt{\alpha\gamma}}\tanh 2\sqrt{\alpha\gamma}t\right]}, \\
\beta(t) = \beta_0 \operatorname{sech} 2\sqrt{\alpha\gamma}t, \quad m^2 = -\dfrac{\beta_0}{2\alpha}\dfrac{a_{10}^2}{f_0^2},
\end{cases} \tag{20c}
$$

$$
\begin{cases}
u^{(4)} = \left\{ \pm\sqrt{-\dfrac{2\alpha}{\beta_0}}\, m p \left(\operatorname{csch} 2\sqrt{\alpha\gamma}t\right)^{1/2} \operatorname{sn} p \left[\left(\operatorname{csch} 2\sqrt{\alpha\gamma}t\right) x + \dfrac{c_g}{2\sqrt{\alpha\gamma}}\coth 2\sqrt{\alpha\gamma}t \right] \right\} \cdot e^{i\left[\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}\left(\coth 2\sqrt{\alpha\gamma}t\right)x^2 + k\left(\operatorname{csch} 2\sqrt{\alpha\gamma}t\right)x + \frac{\omega}{2\sqrt{\alpha\gamma}}\coth 2\sqrt{\alpha\gamma}t\right]}, \\
\beta(t) = \beta_0 \operatorname{csch} 2\sqrt{\alpha\gamma}t, \quad m^2 = -\dfrac{\beta_0}{2\alpha}\dfrac{a_{10}^2}{f_0^2}。
\end{cases} \tag{20d}
$$

令 $m \to 1$,则从式(20) 求得一维 GP 方程(3) 的包络孤波解为

$$
\begin{cases}
u^{(1)} = \left\{ \pm\sqrt{-\dfrac{2\alpha}{\beta_0}}\, p e^{\sqrt{\alpha\gamma}t} \tanh p \cdot \left\{ \left[e^{2\sqrt{\alpha\gamma}t} x - \dfrac{c_g}{4\sqrt{\alpha\gamma}} e^{4\sqrt{\alpha\gamma}t} \right] \right\} \cdot e^{i\left(-\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}x^2 + kxe^{2\sqrt{\alpha\gamma}t} - \frac{\omega}{4\sqrt{\alpha\gamma}}e^{4\sqrt{\alpha\gamma}t}\right)} \right\}, \\
\beta(t) = \beta_0 e^{2\sqrt{\alpha\gamma}t},
\end{cases} \tag{21a}
$$

$$
\begin{cases}
u^{(2)} = \left\{ \pm\sqrt{-\dfrac{2\alpha}{\beta_0}}\, p e^{-\sqrt{\alpha\gamma}t} \tanh p \cdot \left\{ \left[e^{-2\sqrt{\alpha\gamma}t} x + \dfrac{c_g}{4\sqrt{\alpha\gamma}} e^{-4\sqrt{\alpha\gamma}t} \right] \right\} \cdot e^{i\left(\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}x^2 + kxe^{-2\sqrt{\alpha\gamma}t} + \frac{\omega}{4\sqrt{\alpha\gamma}}e^{-4\sqrt{\alpha\gamma}t}\right)} \right\}, \\
\beta(t) = \beta_0 e^{-2\sqrt{\alpha\gamma}t},
\end{cases} \tag{21b}
$$

$$
\begin{cases}
u^{(3)} = \left\{ \pm\sqrt{\dfrac{2\alpha}{\beta_0}}\, p \left(\operatorname{sech} 2\sqrt{\alpha\gamma}t\right)^{1/2} \tanh p \left[\left(\operatorname{sech} 2\sqrt{\alpha\gamma}t\right) x - \dfrac{c_g}{2\sqrt{\alpha\gamma}}\left(\tanh 2\sqrt{\alpha\gamma}t\right) \right] \right\} \cdot e^{i\left[\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}\left(\tanh 2\sqrt{\alpha\gamma}t\right)x^2 + k\left(\operatorname{sech} 2\sqrt{\alpha\gamma}t\right)x - \frac{\omega}{2\sqrt{\alpha\gamma}}\tanh 2\sqrt{\alpha\gamma}t\right]}, \\
\beta(t) = \beta_0 \operatorname{sech} 2\sqrt{\alpha\gamma}t,
\end{cases} \tag{21c}
$$

$$
\begin{cases}
u^{(4)} = \left\{ \pm\sqrt{\dfrac{2\alpha}{\beta_0}}\, p \left(\operatorname{csch} 2\sqrt{\alpha\gamma}t\right)^{1/2} \tanh p \left[\left(\operatorname{csch} 2\sqrt{\alpha\gamma}t\right) x + \dfrac{c_g}{2\sqrt{\alpha\gamma}}\coth 2\sqrt{\alpha\gamma}t \right] \right\} \cdot e^{i\left[\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}\left(\coth 2\sqrt{\alpha\gamma}t\right)x^2 + k\left(\operatorname{csch} 2\sqrt{\alpha\gamma}t\right)x + \frac{\omega}{2\sqrt{\alpha\gamma}}\coth 2\sqrt{\alpha\gamma}t\right]}, \\
\beta(t) = \beta_0 \operatorname{csch} 2\sqrt{\alpha\gamma}t。
\end{cases} \tag{21d}
$$

2 Jacobi 椭圆余弦函数和第三类 Jacobi 椭圆函数展开法

对式(6) 中的 ϕ 利用 Jacobi 椭圆余弦函数展开, θ 仍用幂级数展开法,最后可得

$$
\begin{cases}
u^{(1)} = \left\{ \pm\sqrt{\dfrac{2\alpha}{\beta_0}}\, m p e^{\sqrt{\alpha\gamma}t} \operatorname{cn} p \cdot \left\{ \left[e^{2\sqrt{\alpha\gamma}t} x - \dfrac{c_g}{4\sqrt{\alpha\gamma}} e^{4\sqrt{\alpha\gamma}t} \right] \right\} \cdot e^{i\left(-\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}x^2 + kxe^{2\sqrt{\alpha\gamma}t} - \frac{\omega}{4\sqrt{\alpha\gamma}}e^{4\sqrt{\alpha\gamma}t}\right)} \right\}, \\
\beta(t) = \beta_0 e^{2\sqrt{\alpha\gamma}t}, \quad m^2 = \dfrac{\beta_0}{2\alpha}\dfrac{a_{10}^2}{f_0^2},
\end{cases} \tag{22a}
$$

$$
\begin{cases}
u^{(2)} = \left\{ \pm\sqrt{\dfrac{2\alpha}{\beta_0}}\, m p e^{-\sqrt{\alpha\gamma}t} \operatorname{cn} p \cdot \left\{ \left[e^{-2\sqrt{\alpha\gamma}t} x + \dfrac{c_g}{4\sqrt{\alpha\gamma}} e^{-4\sqrt{\alpha\gamma}t} \right] \right\} \cdot e^{i\left(\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}x^2 + kxe^{-2\sqrt{\alpha\gamma}t} + \frac{\omega}{4\sqrt{\alpha\gamma}}e^{-4\sqrt{\alpha\gamma}t}\right)} \right\}, \\
\beta(t) = \beta_0 e^{-2\sqrt{\alpha\gamma}t}, \quad m^2 = \dfrac{\beta_0}{2\alpha}\dfrac{a_{10}^2}{f_0^2},
\end{cases} \tag{22b}
$$

$$
\begin{cases}
u^{(3)} = \left\{ \pm \sqrt{\dfrac{2\alpha}{\beta_0}}\, mp \left(\operatorname{sech} 2\sqrt{\alpha\gamma}t \right)^{1/2} \operatorname{cn} p \left[\left(\operatorname{sech} 2\sqrt{\alpha\gamma}t \right) x - \dfrac{c_g}{2\sqrt{\alpha\gamma}} \left(\tanh 2\sqrt{\alpha\gamma}t \right) \right] \right\} \mathrm{e}^{\mathrm{i}\left[\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}} \left(\tanh 2\sqrt{\alpha\gamma}t \right) x^2 + k \left(\operatorname{sech} 2\sqrt{\alpha\gamma}t \right) x - \frac{\omega}{2\sqrt{\alpha\gamma}} \tanh 2\sqrt{\alpha\gamma}t \right]}, \\
\beta(t) = \beta_0 \operatorname{sech} 2\sqrt{\alpha\gamma}t,\quad m^2 = \dfrac{\beta_0}{2\alpha} \dfrac{a_{10}^2}{f_0^2},
\end{cases}
\tag{22c}
$$

$$
\begin{cases}
u^{(4)} = \left\{ \pm \sqrt{\dfrac{2\alpha}{\beta_0}}\, mp \left(\operatorname{csch} 2\sqrt{\alpha\gamma}t \right)^{1/2} \operatorname{cn} p \left[\left(\operatorname{csch} 2\sqrt{\alpha\gamma}t \right) x - \dfrac{c_g}{2\sqrt{\alpha\gamma}} \coth 2\sqrt{\alpha\gamma}t \right] \right\} \mathrm{e}^{\mathrm{i}\left[\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}} \left(\coth 2\sqrt{\alpha\gamma}t \right) x^2 + k \left(\operatorname{csch} 2\sqrt{\alpha\gamma}t \right) x + \frac{\omega}{2\sqrt{\alpha\gamma}} \coth 2\sqrt{\alpha\gamma}t \right]}, \\
\beta(t) = \beta_0 \operatorname{csch} 2\sqrt{\alpha\gamma}t,\quad m^2 = \dfrac{\beta_0}{2\alpha} \dfrac{a_{10}^2}{f_0^2},
\end{cases}
\tag{22d}
$$

其中 $p = f_0$, $k = b_{10}$, $c_g = 2\alpha b_{10} = 2\alpha k$, $\omega = \alpha k^2 - (2m^2 - 1)\alpha p^2$。

类似地, ϕ 利用第三类 Jacobi 椭圆函数展开, θ 仍用幂级数展开法,最后求得

$$
\begin{cases}
u^{(1)} = \left\{ \pm \sqrt{\dfrac{2\alpha}{\beta_0}}\, p \mathrm{e}^{\sqrt{\alpha\gamma}t} \operatorname{dn} p \left(\mathrm{e}^{2\sqrt{\alpha\gamma}t} x - \dfrac{c_g}{4\sqrt{\alpha\gamma}} \mathrm{e}^{4\sqrt{\alpha\gamma}t} \right) \right\} \mathrm{e}^{\mathrm{i}\left\{ -\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}} x^2 + kx\mathrm{e}^{2\sqrt{\alpha\gamma}t} - \frac{\omega}{4\sqrt{\alpha\gamma}} \mathrm{e}^{4\sqrt{\alpha\gamma}t} \right\}}, \\
\beta(t) = \beta_0 \mathrm{e}^{2\sqrt{\alpha\gamma}t},\quad 1 = \dfrac{\beta_0}{2\alpha} \dfrac{a_{10}^2}{f_0^2},
\end{cases}
\tag{23a}
$$

$$
\begin{cases}
u^{(2)} = \left\{ \pm \sqrt{\dfrac{2\alpha}{\beta_0}}\, p \mathrm{e}^{-\sqrt{\alpha\gamma}t} \operatorname{dn} p \left(\mathrm{e}^{-2\sqrt{\alpha\gamma}t} x + \dfrac{c_g}{4\sqrt{\alpha\gamma}} \mathrm{e}^{-4\sqrt{\alpha\gamma}t} \right) \right\} \mathrm{e}^{\mathrm{i}\left(\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}} x^2 + kx\mathrm{e}^{-2\sqrt{\alpha\gamma}t} + \frac{\omega}{4\sqrt{\alpha\gamma}} \mathrm{e}^{-4\sqrt{\alpha\gamma}t} \right)}, \\
\beta(t) = \beta_0 \mathrm{e}^{-2\sqrt{\alpha\gamma}t},\quad 1 = \dfrac{\beta_0}{2\alpha} \dfrac{a_{10}^2}{f_0^2},
\end{cases}
\tag{23b}
$$

$$
\begin{cases}
u^{(3)} = \left\{ \pm \sqrt{\dfrac{2\alpha}{\beta_0}}\, p \left(\operatorname{sech} 2\sqrt{\alpha\gamma}t \right)^{1/2} \operatorname{dn} p \left[\left(\operatorname{sech} 2\sqrt{\alpha\gamma}t \right) x - \dfrac{c_g}{2\sqrt{\alpha\gamma}} \left(\tanh 2\sqrt{\alpha\gamma}t \right) \right] \right\} \mathrm{e}^{\mathrm{i}\left[\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}} \left(\tanh 2\sqrt{\alpha\gamma}t \right) x^2 + k \left(\operatorname{sech} 2\sqrt{\alpha\gamma}t \right) x - \frac{\omega}{2\sqrt{\alpha\gamma}} \tanh 2\sqrt{\alpha\gamma}t \right]}, \\
\beta(t) = \beta_0 \operatorname{sech} 2\sqrt{\alpha\gamma}t,\quad 1 = \dfrac{\beta_0}{2\alpha} \dfrac{a_{10}^2}{f_0^2},
\end{cases}
\tag{23c}
$$

$$
\begin{cases}
u^{(4)} = \left\{ \pm \sqrt{\dfrac{2\alpha}{\beta_0}}\, p \left(\operatorname{csch} 2\sqrt{\alpha\gamma}t \right)^{1/2} \cdot \operatorname{dn} p \left[\left(\operatorname{csch} 2\sqrt{\alpha\gamma}t \right) x - \dfrac{c_g}{2\sqrt{\alpha\gamma}} \coth 2\sqrt{\alpha\gamma}t \right] \right\} \cdot \mathrm{e}^{\mathrm{i}\left[\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}} \left(\coth 2\sqrt{\alpha\gamma}t \right) x^2 + k \left(\operatorname{csch} 2\sqrt{\alpha\gamma}t \right) x + \frac{\omega}{2\sqrt{\alpha\gamma}} \coth 2\sqrt{\alpha\gamma}t \right]}, \\
\beta(t) = \beta_0 \operatorname{csch} 2\sqrt{\alpha\gamma}t,\quad 1 = \dfrac{\beta_0}{2\alpha} \dfrac{a_{10}^2}{f_0^2}。
\end{cases}
\tag{23d}
$$

其中 $p = f_0$, $k = b_{10}$, $c_g = 2\alpha b_{10} = 2\alpha k$, $\omega = \alpha k^2 - (2 - m^2)\alpha p^2$。

当 $m \to 1$ 时,式(22)和(23)均退化为下列包络孤波解:

$$
\begin{cases}
u^{(1)} = \left\{ \pm \sqrt{\dfrac{2\alpha}{\beta_0}}\, p \mathrm{e}^{\sqrt{\alpha\gamma}t} \operatorname{sech} p \cdot \left(\mathrm{e}^{2\sqrt{\alpha\gamma}t} x - \dfrac{c_g}{4\sqrt{\alpha\gamma}} \mathrm{e}^{4\sqrt{\alpha\gamma}t} \right) \right\} \cdot \mathrm{e}^{\mathrm{i}\left\{ -\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}} x^2 + kx\mathrm{e}^{2\sqrt{\alpha\gamma}t} - \frac{\omega}{4\sqrt{\alpha\gamma}} \mathrm{e}^{4\sqrt{\alpha\gamma}t} \right\}}, \\
\beta(t) = \beta_0 \mathrm{e}^{2\sqrt{\alpha\gamma}t},
\end{cases}
\tag{24a}
$$

$$
\begin{cases}
u^{(2)} = \left\{ \pm \sqrt{\dfrac{2\alpha}{\beta_0}}\, p \mathrm{e}^{-\sqrt{\alpha\gamma}t} \operatorname{sech} p \cdot \left(\mathrm{e}^{-2\sqrt{\alpha\gamma}t} x + \dfrac{c_g}{4\sqrt{\alpha\gamma}} \mathrm{e}^{-4\sqrt{\alpha\gamma}t} \right) \right\} \cdot \mathrm{e}^{\mathrm{i}\left(\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}} x^2 + kx\mathrm{e}^{-2\sqrt{\alpha\gamma}t} + \frac{\omega}{4\sqrt{\alpha\gamma}} \mathrm{e}^{-4\sqrt{\alpha\gamma}t} \right)}, \\
\beta(t) = \beta_0 \mathrm{e}^{-2\sqrt{\alpha\gamma}t},
\end{cases}
\tag{24b}
$$

$$
\begin{cases}
u^{(3)} = \left\{ \pm \sqrt{\dfrac{2\alpha}{\beta_0}}\, p \left(\operatorname{sech} 2\sqrt{\alpha\gamma}t \right)^{1/2} \operatorname{sech} p \cdot \left[\left(\operatorname{sech} 2\sqrt{\alpha\gamma}t \right) x - \dfrac{c_g}{2\sqrt{\alpha\gamma}} \left(\tanh 2\sqrt{\alpha\gamma}t \right) \right] \right\} \cdot \mathrm{e}^{\mathrm{i}\left[\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}} \left(\tanh 2\sqrt{\alpha\gamma}t \right) x^2 + k \left(\operatorname{sech} 2\sqrt{\alpha\gamma}t \right) x - \frac{\omega}{2\sqrt{\alpha\gamma}} \tanh 2\sqrt{\alpha\gamma}t \right]}, \\
\beta(t) = \beta_0 \operatorname{sech} 2\sqrt{\alpha\gamma}t,
\end{cases}
\tag{24c}
$$

$$
\begin{cases}
u^{(4)} = \left\{ \pm \sqrt{\dfrac{2\alpha}{\beta_0}}\, p \left(\operatorname{csch} 2\sqrt{\alpha\gamma}t \right)^{1/2} \operatorname{sech} p \cdot \left[\left(\operatorname{csch} 2\sqrt{\alpha\gamma}t \right) x - \dfrac{c_g}{2\sqrt{\alpha\gamma}} \coth 2\sqrt{\alpha\gamma}t \right] \right\} \cdot \mathrm{e}^{\mathrm{i}\left\{ \frac{1}{2}\sqrt{\frac{\gamma}{\alpha}} \left(\coth 2\sqrt{\alpha\gamma}t \right) x^2 + k \left(\operatorname{csch} 2\sqrt{\alpha\gamma}t \right) x + \frac{\omega}{2\sqrt{\alpha\gamma}} \coth 2\sqrt{\alpha\gamma}t \right\}}, \\
\beta(t) = \beta_0 \operatorname{csch} 2\sqrt{\alpha\gamma}t。
\end{cases}
\tag{24d}
$$

3 结论

在 Landau 系数为 $\beta(t) = \beta_0 e^{2\sqrt{\alpha\gamma}t}, \beta_0 e^{-2\sqrt{\alpha\gamma}t}$, $\beta_0 \mathrm{sech}2\sqrt{\alpha\gamma}t$ 和 $\beta_0 \mathrm{sech}2\sqrt{\alpha\gamma}t$ 的 4 种情况下,应用 Jacobi 椭圆函数展开法,求得一维 GP 方程(3)的包络周期解和包络孤立波解。

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