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New special structures to the (2 + 1)-dimensional breaking soliton equations

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Abstract
Applying the Jacobi elliptic function expansion method to the (2 + 1)-dimensional breaking soliton equations (BSE for short), we derive some types of exact solutions to the BSE. Based on the derived solutions, we obtain some special structures, such as spatially localized or periodic excitations. For specific choices, we show some features of the (2 + 1)-dimensional BSE, such as breathers or breather lattice solutions.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

In his paper [1], Kudryashov pointed out that one of the most exciting recent advances in nonlinear science and theoretical physics has been the development of methods to look for exact solutions to nonlinear differential equations. Among these methods, the homogeneous balance method [2], the hyperbolic tangent function expansion method [3, 4], the nonlinear transformation method [5, 6], the trial function method [7, 8], the sine–cosine method [9], the Jacobi elliptic function expansion method [10–12], the auxiliary equation and mapping method [13, 14] and the Exp-function method [15] have been widely applied to solve (1 + 1)-dimensional nonlinear wave equations extensively, and (1 + 1)-dimensional nonlinear wave equations have been studied quite well, both in theoretical and in experimental aspects. For (2 + 1)- and higher-dimensional equations, the situation is less clear. Since there are more rich structures in (2 + 1)- or higher-dimensional equations than in (1 + 1)-dimensional equations, more work needs to be done to discover these new structures.

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In this paper, based on Jacobi elliptic functions and the Lamé function [16–19], the Jacobi elliptic function expansion method is applied to the (2 + 1)-dimensional breaking soliton equations (BSE) to derive new solutions with special structures.

2. Breaking soliton equations and special structures

The (2 + 1)-dimensional BSE

$$u_t + \beta u_{xxy} + 4\beta u v_x + 4\beta u_x v = 0, \tag{1a}$$

$$u_y = v_x \tag{1b}$$

describe the (2 + 1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. BSE have been studied in detail by many researchers: Mei solved BSE by the projective Riccati equation expansion method [20], Peng obtained two general solutions to BSE by the singular manifold method [21], Zhang constructed nontraveling wave solutions to BSE by a generalized auxiliary equation method [22], and Dai derived BSE chaotic behaviors by the mapping method [23]. The structures of (2 + 1)-dimensional BSE are rich and there are

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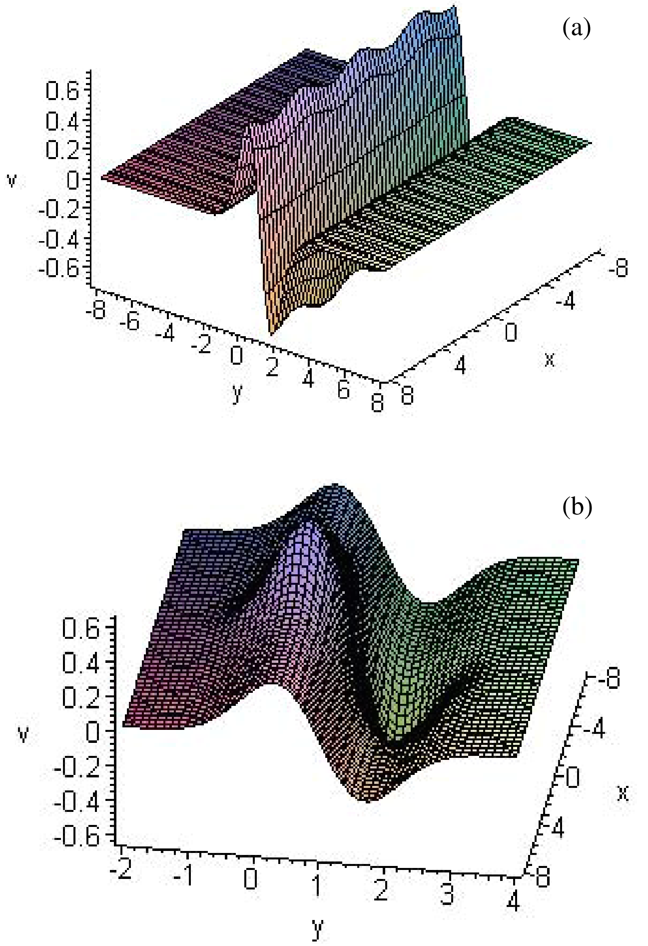

Figure 1. A typical spatial structure of equation (9) (a) for $m = 0.9$ and (b) for $m = 1.0$.

still more structures to be discovered. Next we will show that the Jacobi elliptic function expansion method can be applied to solve BSE to derive more solutions that have not been reported.

In order to solve equation (1), the following ansatz solution,

$$u = \sum_{i=0}^{n_1} a_i F^i, \quad a_{n_1} \neq 0, \tag{2a}$$

$$v = \sum_{i=0}^{n_2} b_i F^i, \quad b_{n_2} \neq 0, \tag{2b}$$

to BSE will be assumed, where $\xi = kx + \eta$, $\eta = \eta(y, t)$, $a_i = a_i(y, t)$, $b_i = b_i(y, t)$ and F is a solution to the elliptic equation

$$F'(\xi)^2 = \left[\frac{\mathrm{d}F}{\mathrm{d}\xi}\right]^2 = A_0 + A_2 F^2 + A_4 F^4. \tag{3}$$

Substituting equation (2) into (1) and balancing the nonlinear term and dispersive term will result in

$$u = a_0 + a_1 F + a_2 F^2, \quad a_2 \neq 0, \tag{4a}$$

$$v = b_0 + b_1 F + b_2 F^2, \quad b_2 \neq 0. \tag{4b}$$

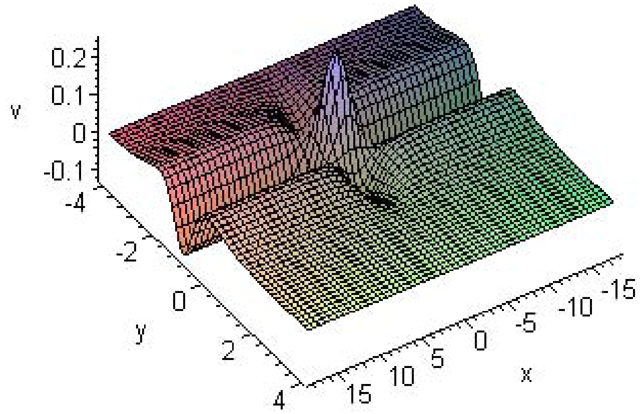

Figure 2. A typical spatial structure of equation (12) for $m = 1.0$.

From equations (1), (3) and (4), the expansion coefficients can be determined as

$$\begin{aligned} &a_0 = \text{constant}, \quad a_1 = 0, \quad a_2 = -\frac{3}{2}k^2 A_4, \\ &b_0 = -\frac{1}{4\beta k}(\eta_t + 4\beta a_0 \eta_y + 4\beta k^2 A_2 \eta_y), \\ &b_1 = 0, \quad b_2 = -\frac{3}{2}k A_4 \eta_y, \end{aligned} \tag{5}$$

where $\eta = \eta(y, t)$ is an arbitrary function of t and y.

Owing to the arbitrariness of function η, we may obtain a diversity of exact solutions to equation (1) by choosing this function. From equation (5), one can see that there are rich coherent structures for the field v. We will show that by choosing a different F solution and arbitrary function η, more special structures can be found.

For brevity, we will set the constant a_0 as zero, so the solution to v can be rewritten as

$$v = -\frac{1}{4\beta k}(\eta_t + 4\beta k^2 A_2 \eta_y) - \frac{3}{2}k A_4 \eta_y F^2. \tag{6}$$

Case 1. If $A_0 = \frac{1}{4}$, $A_2 = -\frac{2-m^2}{2}$, $A_4 = \frac{m^4}{4}$, then F takes the solution

$$F = \frac{\mathrm{sn}(\xi, m)}{1 + \mathrm{dn}(\xi, m)} \tag{7}$$

with the modulus $0 \leqslant m \leqslant 1$ [24, 25].

If we select η as

$$\eta = \mathrm{e}^{-(-y+t)^2}, \tag{8}$$

then it follows from equation (6) that

$$v = \left[\frac{1}{2\beta k} + k(2 - m^2) - \frac{3\,km^4\,\mathrm{sn}^2(\xi, m)}{4(1 + \mathrm{dn}(\xi, m))^2}\right](t - y)\mathrm{e}^{-(y-t)^2} \tag{9}$$

with $\xi = kx + \eta(y, t)$.

This is a new structure for BSE; figures 1(a) and (b) show the structures for $v(x, y, t)$ of equation (9), when the parameter is chosen as

$$\beta = k = 1 \tag{10}$$

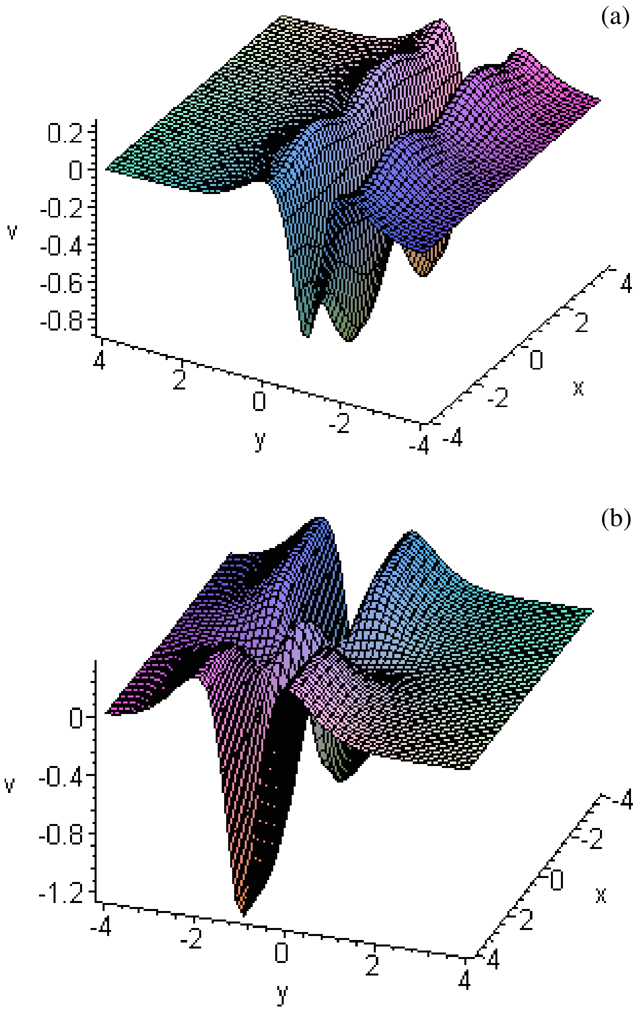

Figure 3. A typical spatial structure of equation (14) (a) for $m = 0.5$ and (b) for $m = 1.0$.

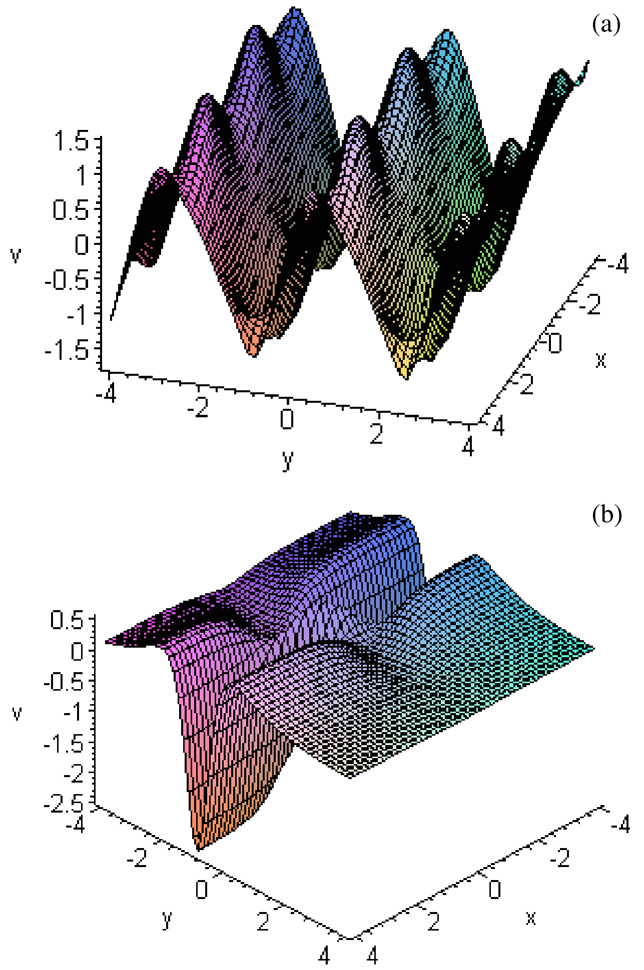

Figure 4. A typical spatial structure of equation (16) (a) for $m = 0.5$ and (b) for $m = 1.0$.

with $m = 0.9$ and $m = 1.0$ at $t = 1$, respectively. Obviously, in the x-axis the solution maintains periodic behavior in figure 1(a) and when $m = 1$ it is localized in figure 1(b). Another feature is the obvious spatial breather behavior found in figure 1(b).

Case 2. If F admits the same solution as (7), but η admits a breather solution [26]

$$\eta = \frac{\sin(y+t)}{\cosh(2y+2t)}, \tag{11}$$

then it follows from equation (6) that

$$v = -\frac{\cos(y+t)\cosh(2y+2t) - 2\sin(y+t)\sinh(2y+2t)}{\cosh^2(2y+2t)} \times \left[\frac{1}{4\beta k} - \frac{k(2-m^2)}{2} + \frac{3\,km^4\,\mathrm{sn}^2(\xi,m)}{8(1+\mathrm{dn}(\xi,m))^2}\right] \tag{12}$$

with $\xi = kx + \eta(y, t)$.

This is another new structure for BSE; figure 2 shows the structure for $v(x, y, t)$ of equation (12), when the parameter is chosen as equation (10) with $m = 1.0$ at $t = 1$. Obviously, in the x-axis the solution maintains solitonic behavior but obvious spatial breather behavior is found in the y-axis.

Case 3. If $A_0 = m^2 - 1$, $A_2 = 2 - m^2$, $A_4 = -1$, then F takes the solution

$$F = \mathrm{dn}(\xi, m) \tag{13}$$

with the modulus $0 \leqslant m \leqslant 1$. If η admits a breather solution as equation (11), then it follows from equation (6) that

$$v = -\frac{\cos(y+t)\cosh(2y+2t) - 2\sin(y+t)\sinh(2y+2t)}{\cosh^2(2y+2t)} \times \left[\frac{1}{4\beta k} + k(2-m^2) - \frac{3k\mathrm{dn}^2(\xi,m)}{2}\right] \tag{14}$$

with $\xi = kx + \eta(y, t)$.

This is another new structure for BSE; figures 3(a) and (b) show the structures for $v(x, y, t)$ of equation (14), when the parameter is chosen as equation (10) with $m = 0.5$ and $m = 1.0$ at $t = 1$, respectively. Obviously, the feature shown in figure 3(b) is different from that given in figure 2; this is a new structure.

Case 4. If F admits the same solution as (13), but η admits a breather lattice solution [27]

$$\eta = \mathrm{dn}(y+t, m)\,\mathrm{sn}(2y+2t, m), \tag{15}$$

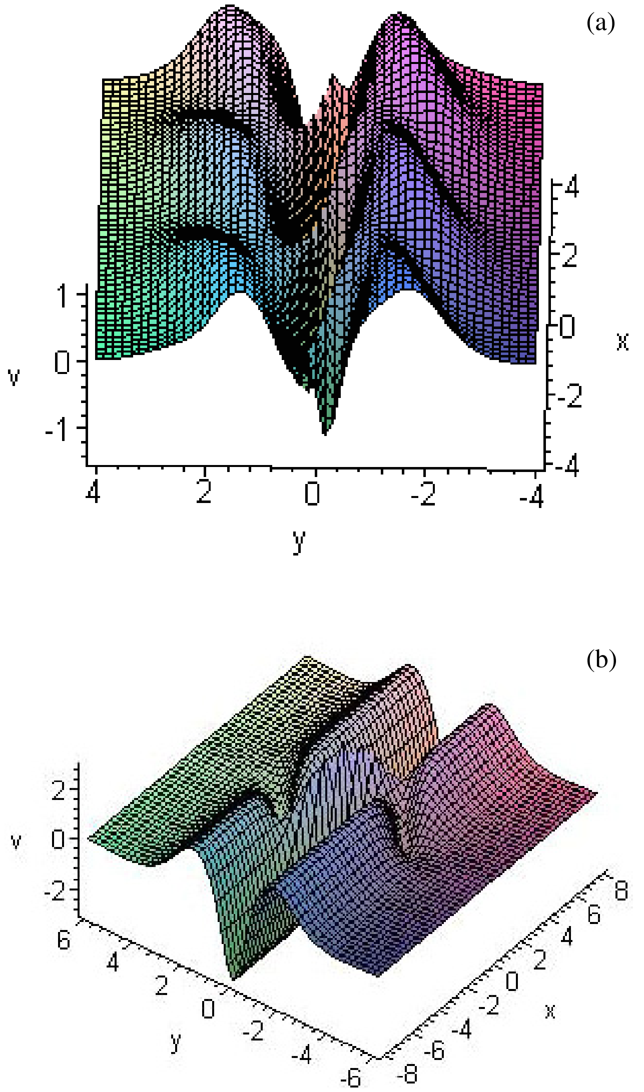

Figure 5. A typical spatial structure of equation (18) (a) for $m = 0.5$ and (b) for $m = 1.0$.

then it follows from equation (6) that

$$v = [2\mathrm{dn}(y+t, m)\mathrm{cn}(2y+2t, m)\mathrm{dn}(2y+2t, m) - m^2\mathrm{sn}(y+t, m)\mathrm{cn}(y+t, m)\mathrm{sn}(2y+2t, m)] \times \left[-\frac{1}{4\beta k} + k(2-m^2) + \frac{3k\mathrm{dn}^2(\xi, m)}{2}\right] \tag{16}$$

with $\xi = kx + \eta(y, t)$.

This is another new structure for BSE; figures 4(a) and (b) show the structures for $v(x, y, t)$ of equation (16), when the parameter is chosen as equation (10) with $m = 0.5$ and $m = 1.0$ at $t = 1$, respectively. Obviously, the feature shown in figure 4(a) is periodic in both the x- and y-axis, and maintains breather lattice structures in the y-axis. If the breather lattice solution (15) is changed to $\eta = \mathrm{dn}(y+t, m)\,\mathrm{sn}(y+t, m)$, which is a kind of Lamé function [16–19], or other kinds of Lamé function [16–19], a similar structure can still be found.

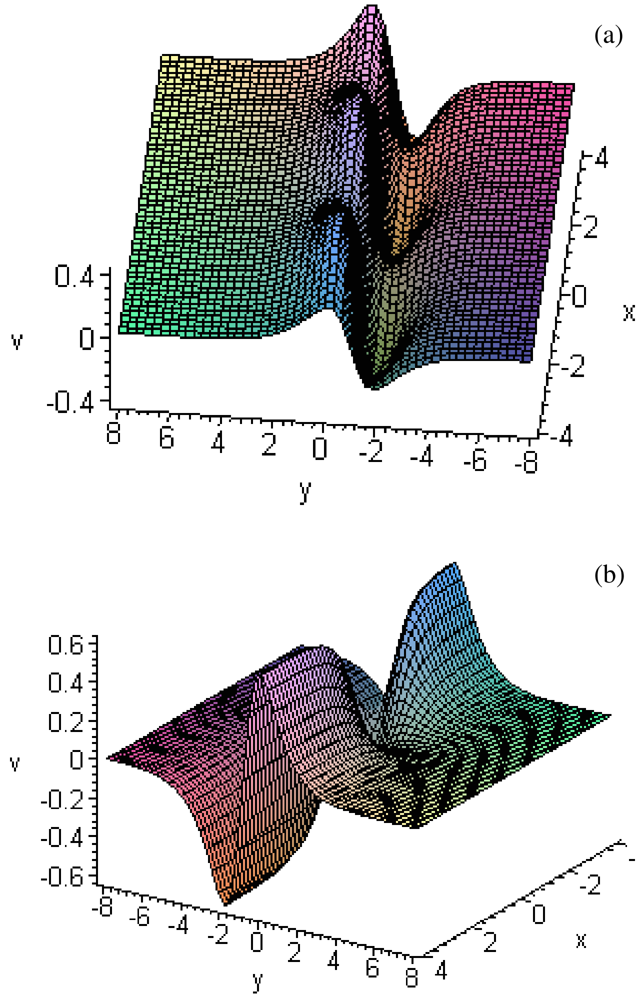

Figure 6. A typical spatial structure of equation (20) (a) for $m = 0.5$ and (b) for $m = 1.0$.

Case 5. If F admits the same solution as (13), but η admits another kind of breather solution [28]

$$\eta = 4\tan^{-1}\left[\frac{\sin(y+t)}{\cosh(2y+2t)}\right], \tag{17}$$

then it follows from equation (6) that

$$v = \frac{1}{\beta k}[\cos(y-t)\cosh(y+t) + \sin(y-t)\sinh(y+t)] - [4k(2-m^2) - 6k\mathrm{dn}^2(\xi, m)][\cos(y-t)) \times \cosh(y+t) - \sin(y-t)\sinh(y+t)] \tag{18}$$

with $\xi = kx + \eta(y, t)$.

This is another new structure for BSE; figures 5(a) and (b) show the structures for $v(x, y, t)$ of equation (18), when the parameter is chosen as equation (10) with $m = 0.5$ and $m = 1.0$ at $t = 1$, respectively. Obviously, the features shown in figures 5(a) and (b) are different from those given in figures 3(a) and (b).

Case 6. If F admits the same solution as (13), but η admits a soliton solution

$$\eta = \mathrm{sech}(y+t), \tag{19}$$

then it follows from equation (6) that

$$v = \text{sech}(y+t)\tanh(y+t) \times \left[-\frac{1}{4\beta k} + k(2-m^2) + \frac{3k\,\text{dn}^2(\xi, m)}{2}\right] \quad (20)$$

with $\xi = kx + \eta(y, t)$.

This is another new structure for BSE; figures 6(a) and (b) show the structures for $v(x, y, t)$ of equation (20), where the parameter is chosen as equation (10) with $m = 0.5$ and $m = 1.0$ at $t = 1$, respectively. Obviously, the feature shown in figure 6(a) is periodic in the x-axis and maintains band soliton structure in the y-axis.

3. Conclusion and discussion

In this paper, the Jacobi elliptic function expansion method has been applied to the $(2+1)$-dimensional BSE, and certain special structures have been obtained because of the existence of arbitrary function $\eta(y, t)$. This indicates that the Jacobi elliptic function expansion method is a powerful method not only in $(1+1)$-dimensional nonlinear equations but also in $(2+1)$-dimensional or higher-dimensional nonlinear equations. More applications of this method to other $(2+1)$-dimensional or higher-dimensional nonlinear equations in order to derive more new structures need to be studied further.

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