Available online at www.sciencedirect.com
SCIENCE DIRECT®
Communications in Nonlinear Science
and Numerical Simulation 8 (2003) 67–75
Communications in
Nonlinear Science and
Numerical Simulation
www.elsevier.com/locate/cnsns

The JEFE method and periodic solutions of two kinds of nonlinear wave equations

Zuntao Fu [a,b,*], Shikuo Liu [a], Shida Liu [a,b], Qiang Zhao [a]

[a] *School of Physics, Peking University, Beijing 100871, China*
[b] *LTCS, Department of Mechanics and Engineering Science, Peking University, Beijing 100871, China*

Received 22 May 2002; accepted 17 July 2002

Abstract

The Jacobi elliptic function expansion (JEFE) method is applied to construct the exact periodic solutions to two kinds of nonlinear wave equations, such as BBM equation, fifth-order dispersive equation, Kawahara equation, modified Kawahara equation, second-order BO equation and symmetrical-regular long wave equation. The corresponding shock wave solutions or solitary wave solutions are obtained as special cases of the periodic solutions. It is shown that this method is very powerful for some nonlinear wave equations, and its applying domain is given.

PACS: 01.55.+b; 04.20.Jb; 05.45.Yv
Keywords: Jacobi elliptic function; Nonlinear wave equation; Periodic solution

1. Introduction

In Ref. [1], the Jacobi elliptic function expansion (JEFE) method was proposed to construct periodic solutions to some nonlinear wave equations. It is interesting that solutions in their limit obtained by this method, the shock or solitary wave solutions, are just the same as the results given by a number of methods, such as the homogeneous balance method [2–4], the hyperbolic tangent expansion method [5–7], the nonlinear transformation method [8,9], the trial function method [10,11] and sine–cosine method [12]. Among these methods, none deals with special

* Corresponding author. Address: School of Physics, Peking University, Beijing 100871, China. Tel.: +86-010-62757122; fax: +86-010-62757944.
E-mail address: fuzt@pku.edu.cn (Z. Fu).

1007-5704/03/$ - see front matter
PII: S1007-5704(02)00082-5

functions. Actually, the hyperbolic tangent expansion method is just a special case of JEFE method under certain conditions. Although Porubov et al. [13–15] have obtained some exact periodic solutions to some nonlinear wave equations, they used the Weierstrass elliptic function and involved complicated deducing. In this paper, the details and the applying domains of JEFE method are discussed, and periodic solutions and corresponding limited solutions are obtained for some nonlinear wave equations.

2. JEFE method and its applying domain

Consider a given nonlinear wave equation

$$N(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{1}$$

its traveling wave solutions take the following form:

$$u = u(\xi), \quad \xi = k(x - ct), \tag{2}$$

which can be expressed as a finite series of Jacobi elliptic function, $\mathrm{sn}\,\xi$, by JEFE method

$$u(\xi) = \sum_{j=0}^{n} a_j \mathrm{sn}^j \xi, \tag{3}$$

where k and c are the wavenumber and wave speed, respectively. It is known that there are the following relations between the elliptic functions:

$$\mathrm{cn}^2\xi = 1 - \mathrm{sn}^2\xi, \quad \mathrm{dn}^2\xi = 1 - m^2\mathrm{sn}^2\xi, \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{sn}\,\xi = \mathrm{cn}\,\xi\mathrm{dn}\,\xi, \quad \frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{cn}\,\xi = -\mathrm{sn}\,\xi\mathrm{dn}\,\xi, \quad \frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{dn}\,\xi = -m^2\mathrm{sn}\,\xi\mathrm{cn}\,\xi,$$

where $\mathrm{cn}\,\xi$ and $\mathrm{dn}\,\xi$ are the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind, respectively, and m is the modulus $(0 < m < 1)$.

We have

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = \sum_{j=0}^{n} (j a_j \mathrm{sn}^{j-1}\xi)\mathrm{cn}\,\xi\mathrm{dn}\,\xi, \tag{5}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = \sum_{j=0}^{n} [(j-1)ja_j\mathrm{sn}^{j-2}\xi - (1+m^2)j^2a_j\mathrm{sn}^j\xi + m^2(j+1)ja_j\mathrm{sn}^{j+2}\xi], \tag{6}$$

$$\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} = \sum_{j=0}^{n} [(j-2)(j-1)ja_j\mathrm{sn}^{j-3}\xi - (1+m^2)j^3a_j\mathrm{sn}^{j-1}\xi + m^2(j+2)(j+1)ja_j\mathrm{sn}^{j+1}\xi]\mathrm{cn}\,\xi\mathrm{dn}\,\xi, \tag{7}$$

$$\begin{aligned}\frac{\mathrm{d}^4u}{\mathrm{d}\xi^4} = \sum_{j=0}^{n} \{&(j-3)(j-2)(j-1)ja_j\mathrm{sn}^{j-4}\xi - 2(1+m^2)(j^2-2j+2)(j-1)ja_j\mathrm{sn}^{j-2}\xi \\ &+ [2m^2(j^2+5) + (1+m^2)^2j^2]j^2a_j\mathrm{sn}^j\xi - m^2(1+m^2)j(j+1)[j^2+(j+2)^2]a_j\mathrm{sn}^{j+2}\xi \\ &+ m^4(j+3)(j+2)(j+1)ja_j\mathrm{sn}^{j+4}\xi\},\end{aligned} \tag{8}$$

$$\frac{d^5u}{d\xi^5} = \sum_{j=0}^{n}\{(j-4)(j-3)(j-2)(j-1)ja_j\mathrm{sn}^{j-5}\xi - 2(1+m^2)(j^2-2j+2)(j-2)(j-1)ja_j\mathrm{sn}^{j-3}\xi$$
$$+ [2m^2(j^2+5)+(1+m^2)^2j^2]j^3a_j\mathrm{sn}^{j-1}\xi - m^2(1+m^2)j(j+1)(j+2)$$
$$\times [j^2+(j+2)^2]a_j\mathrm{sn}^{j+1}\xi + m^4(j+4)(j+3)(j+2)(j+1)ja_j\mathrm{sn}^{j+3}\xi\}\mathrm{cn}\,\xi\mathrm{dn}\,\xi. \quad (9)$$

It is obvious that the factor $\mathrm{cn}\,\xi\mathrm{dn}\,\xi$ exits only in the odd derivative terms, so there is good alternate formalism between odd and even derivative terms, i.e.

$$\frac{d^{(2m)}u}{d\xi^{(2m)}} = f_m(\mathrm{sn}\,\xi), \quad \frac{d^{(2m+1)}u}{d\xi^{(2m+1)}} = g_m(\mathrm{sn}\,\xi)\mathrm{cn}\,\xi\mathrm{dn}\,\xi, \quad m = 0,1,2,\ldots$$

Here, we define $d^0u/d\xi^0 = u(\xi)$, $f_m(\mathrm{sn}\,\xi)$ and $g_m(\mathrm{sn}\,\xi)$ are m-order polynomials in term of $\mathrm{sn}\,\xi$, respectively.

The highest power order of $u(\xi)$ is equal to n, i.e.

$$O(u(\xi)) = n, \quad (10)$$

and the highest power order of $du/d\xi$ can be taken as

$$O\left(\frac{du}{d\xi}\right) = n+1.$$

We have

$$O\left(u\frac{du}{d\xi}\right) = 2n+1, \quad O\left(\frac{d^2u}{d\xi^2}\right) = n+2, \quad O\left(\frac{d^3u}{d\xi^3}\right) = n+3. \quad (11)$$

If every sum of derivative order of every term in the Eq. (1) is odd or even synchronously, we can select n in (3) to balance the highest order of derivative term and nonlinear term in Eq. (1). Thus we can obtain the finite series periodic solutions to this kind of equations. Actually, the number of this kind of equations is very large. Such as KdV equation, mKdV equation, Benjamin–Bona–Mahony (BBM) equation, modified BBM equation, fifth-order dispersive equation, Kawahara equation, modified Kawahara equation, $(2i+1)$-order KdV equation, every sum of derivative order of every term is odd in all these equations. And for Boussinesq equation, nonlinear Klein–Gordon equation, symmetrical-regular long wave equation, second-order Benjamin–Ono (BO) equation, Bretherton equation, every sum of derivative order of every term is even in all these equations.

Therefore, the equations mentioned above can be classified as follows: the equations, whose sum of derivative order of every term is odd, are set as the first kind; and the equations, whose sum of derivative order of every term is even, are set as the second kind. In Sections 3 and 4, we illustrate the applications of JEFE method to these two kinds of nonlinear wave equations.

It is known that $\mathrm{sn}\,\xi \to \tanh\xi$ when $m \to 1$, thus (3) degenerates as the following form:

$$u(\xi) = \sum_{j=0}^{n} a_j \tanh^j \xi.$$

So we can also obtain solitary or shock wave solutions by using JEFE method.

3. Applications to the first kind of equations

3.1. Benjamin–Bona–Mahony (BBM) equation

$$\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial x}+p\frac{\partial^3 u}{\partial t\partial x^2}=0. \tag{12}$$

Substituting (2) into (12), we have

$$(c_0-c)\frac{\mathrm{d}u}{\mathrm{d}\xi}+u\frac{\mathrm{d}u}{\mathrm{d}\xi}-pck^2\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}=0. \tag{13}$$

Thus we can deduce from (10) and (11) that

$$\mathrm{O}\left(u\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)=2n+1,\quad \mathrm{O}\left(\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}\right)=n+3.$$

Balance of these orders gives

$$n=2.$$

So the BBM Eq. (12) may have the following form travelling wave solution

$$u(\xi)=a_0+a_1\mathrm{sn}\,\xi+a_2\mathrm{sn}^2\xi \tag{14}$$

and from (14), (5) and (7), we have

$$\frac{\mathrm{d}u}{\mathrm{d}\xi}=(a_1+2a_2\mathrm{sn}\,\xi)\mathrm{cn}\,\xi\mathrm{dn}\,\xi, \tag{15}$$

$$u\frac{\mathrm{d}u}{\mathrm{d}\xi}=[a_0a_1+(a_1^2+2a_0a_2)\mathrm{sn}\,\xi+3a_1a_2\mathrm{sn}^2\xi+2a_2^2\mathrm{sn}^3\xi]\mathrm{cn}\,\xi\mathrm{dn}\,\xi, \tag{16}$$

$$\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}=[-(1+m^2)a_1-8(1+m^2)a_2\mathrm{sn}\,\xi+6m^2a_1\mathrm{sn}^2\xi+24m^2a_2\mathrm{sn}^3\xi]\mathrm{cn}\,\xi\mathrm{dn}\,\xi. \tag{17}$$

Substituting (15)–(17) into (13) yields

$$(c_0-c)a_1+a_0a_1+(1+m^2)pck^2a_1=0,$$

$$2(c_0-c)a_2+(a_1^2+2a_0a_2)+8(1+m^2)pck^2a_2=0,$$

$$a_1a_2-2m^2pck^2a_1=0,$$

$$a_2^2-12m^2pck^2a_2=0.$$

Thus we can determine the coefficients

$$a_1=0,\quad a_2=12m^2pck^2,\quad a_0=c-c_0-4(1+m^2)pck^2, \tag{18}$$

where k and c are arbitrary constants.

Substituting (18) into (14), a final solution is

$$u = c - c_0 - 4(1+m^2)pck^2 + 12m^2pck^2\mathrm{sn}^2 k(x-ct), \tag{19}$$

which is the exact periodic solution of BBM Eq. (12).

Taking $m = 1$, then (19) is reduced to

$$u = c - c_0 - 8pck^2 + 12pck^2 \tanh^2 \xi,$$

which is the solitary wave solution of BBM equation.

Similarly, this method can be applied to other equations of the first kind.

3.2. Fifth-order dispersive equation

$$\frac{\partial u}{\partial t} + p\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + q\frac{\partial^5 u}{\partial x^5} = 0.$$

It is easily determined that its ansatz solution is (14), too. Applying (14), (5), (6) and (9), we can determine that

$$a_1 = 0, \quad a_2 = -\frac{60qm^2k^2}{p}, \quad c = 16qk^4(1-m^2+m^4),$$

where k and a_0 are arbitrary constants.

The periodic solution of fifth-order dispersive equation can be written as

$$u = a_0 - \frac{60qm^2k^2}{p}\mathrm{sn}^2 k[x - 16qk^4(1-m^2+m^4)t]$$

and its corresponding solitary wave solution is

$$u = a_0 - \frac{60qk^2}{p}\tanh^2 k[x - 16qk^4 t].$$

It is obvious that sum of derivative order of every term in $(2i+1)$-order KdV or mKdV equation

$$\frac{\partial u}{\partial t} + c_1 u^l \frac{\partial u}{\partial x} + \sum_{i=1}^{M} c_{2i+1}\frac{\partial^{(2i+1)} u}{\partial x^{(2i+1)}} = 0, \quad M = 1, 2, 3, \ldots \tag{20}$$

is odd, so JEFE method can be easily applied to it. In Section 3.3, we will show detailed applications to $(2i+1)$-order KdV or mKdV equation, including Kawahara equation, modified Kawahara equation.

3.3. Kawahara equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + p\frac{\partial^3 u}{\partial x^3} + q\frac{\partial^5 u}{\partial x^5} = 0. \tag{21}$$

Obviously, here $l = 1$ and $M = 2$ in (20). Its ansatz solution is

$$u(\xi) = a_0 + a_1\mathrm{sn}\,\xi + a_2\mathrm{sn}^2\xi + a_3\mathrm{sn}^3\xi + a_4\mathrm{sn}^4\xi. \tag{22}$$

Substituting (22) into (21) leads to the following solution

$$u = c - pk^2\left[\frac{936qm^2k^2}{p-52(1+m^2)qk^2} - 4(1+m^2)\right] - qk^4\left[72m^2 + 16(1+m^2)^2 - \frac{18720qm^2(1+m^2)k^2}{p-52(1+m^2)qk^2}\right] - 1680qm^4k^4\mathrm{sn}^4k(x-ct) - \frac{280m^2k^2[p-52(1+m^2)qk^2]}{13}\mathrm{sn}^2k(x-ct), \tag{23}$$

which is the exact periodic solution of Kawahara equation (21).

Taking $m = 1$, then (23) is reduced to

$$u = c - \frac{69p^2}{169q} + \frac{210p^2}{169q}\tanh^2\left[\pm\sqrt{\frac{-p}{52q}}(x-ct)\right] - \frac{105p^2}{169q}\tanh^4\left[\pm\sqrt{\frac{-p}{52q}}(x-ct)\right],$$

which is the solitary wave solution of Kawahara equation, and $k^2 = -p/52q$.

3.4. Modified Kawahara equation

$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} + p\frac{\partial^3 u}{\partial x^3} + q\frac{\partial^5 u}{\partial x^5} = 0, \tag{24}$$

here $l = 2$ and $M = 2$, and it is easily determined that $n = 2$ and its ansatz solution is (14).

Substituting (14) into (24) yields the following result

$$u = \pm\frac{20q(1+m^2)k^2 - p}{\sqrt{-10q}} \pm 6\sqrt{-10q}m^2k^2\mathrm{sn}^2k(x-ct)$$

and its corresponding solitary wave solution is

$$u = \pm\frac{40qk^2 - p}{\sqrt{-10q}} \pm 6\sqrt{-10q}k^2\tanh^2 k\left[x - \left(1 - \frac{p^2}{10q} - 24qk^4\right)t\right].$$

4. Applications to the second kind of equations

4.1. Second-order Benjamin–Ono equation

$$\frac{\partial^2 u}{\partial t^2} + q\frac{\partial^2 u^2}{\partial x^2} + r\frac{\partial^4 u}{\partial x^4} = 0. \tag{25}$$

Its ansatz solution is (14). Applying (5), (6) and (8), we have

$$\frac{d^2u}{d\xi^2} = [2a_0a_1 + (2a_1^2 + 4a_0a_2)\mathrm{sn}\,\xi + 6a_1a_2\mathrm{sn}^2\xi + 4a_2^2\mathrm{sn}^3\xi]\mathrm{cn}\,\xi\mathrm{dn}\,\xi, \tag{26}$$

$$\frac{d^4u}{d\xi^4} = -8(1+m^2)a_2 + [(1+m^2)^2 + 12m^2]a_1\mathrm{sn}\,\xi + 8[2(1+m^2)^2 + 9m^2]a_2\mathrm{sn}^2\xi - 20m^2(1+m^2)a_1\mathrm{sn}^3\xi - 120m^2(1+m^2)a_2\mathrm{sn}^4\xi + 24m^4a_1\mathrm{sn}^5\xi + 120m^4a_2\mathrm{sn}^6\xi, \tag{27}$$

$$\frac{d^2u^2}{d\xi^2} = 2(a_1^2 + 2a_0a_2) - 2[(1+m^2)a_0 - 6a_2]a_1 \mathrm{sn}\,\xi - 4[(1+m^2)a_1^2 + 2(1+m^2)a_0a_2 - 3a_2^2]\mathrm{sn}^2\xi$$
$$+ 2[2m^2a_0 - 9(1+m^2)a_2]a_1\mathrm{sn}^3\xi + 2[3m^2a_1^2 + 6m^2a_0a_2 - 8(1+m^2)a_2^2]\mathrm{sn}^4\xi$$
$$+ 24m^2a_1a_2\mathrm{sn}^5\xi + 20m^2a_2^2\mathrm{sn}^6\xi. \tag{28}$$

Substituting (26)–(28) into (25) yields

$$2c^2a_2 + 2q(a_1^2 + 2a_0a_2) - 4rk^2(1+m^2)a_2 = 0,$$
$$(1+m^2)c^2a_1 + 2q[(1+m^2)a_0 - 6a_2]a_1 - rk^2[(1+m^2)^2 + 12m^2]a_1 = 0,$$
$$c^2(1+m^2)a_2 + q[(1+m^2)^2a_1^2 + 2(1+m^2)a_0a_2 - 3a_2^2] - 2rk^2[2(1+m^2)^2 + 9m^2]a_2 = 0,$$
$$c^2m^2a_1 + q[2m^2a_0 - 9(1+m^2)a_2]a_1 - 10rk^2m^2(1+m^2)a_1 = 0,$$
$$3c^2m^2a_2 + q[3m^2a_1^2 + 6m^2a_0a_2 - 8(1+m^2)a_2^2] - 60rk^2m^2(1+m^2)a_2 = 0,$$
$$qm^2a_1a_2 + rk^2m^4a_1 = 0,$$
$$qm^2a_2^2 + 6rk^2m^4a_2 = 0,$$

from which it is determined that

$$a_1 = 0, \quad a_2 = -\frac{6rm^2k^2}{q}, \quad a_0 = \frac{4(1+m^2)rk^2 - c^2}{2q},$$

where c and k are arbitrary constants. Thus the periodic solution of (25) is

$$u = \frac{4(1+m^2)rk^2 - c^2}{2q} - \frac{6rm^2k^2}{q}\mathrm{sn}^2k(x-ct),$$

its corresponding solitary wave solution is

$$u = \frac{8rk^2 - c^2}{2q} - \frac{6rk^2}{q}\tanh^2 k(x-ct).$$

4.2. Symmetrical-regular long wave equation

$$\frac{\partial^2 u}{\partial t^2} + p\frac{\partial^2 u}{\partial x^2} + q\frac{\partial^2 u^2}{\partial t\partial x} + r\frac{\partial^4 u}{\partial t^2\partial x^2} = 0. \tag{29}$$

Its ansatz solution is (14).

Substituting (26)–(28) into (29) yields

$$a_1 = 0, \quad a_2 = \frac{6rm^2k^2c}{q}, \quad a_0 = \frac{c^2 + p - 4r(1+m^2)k^2c^2}{2qc},$$

where c and k are arbitrary constants. Thus the periodic solution of (29) is

$$u = \frac{c^2 + p - 4r(1+m^2)k^2c^2}{2qc} + \frac{6rm^2k^2c}{q}\mathrm{sn}^2 k(x - ct)$$

and its corresponding solitary wave solution is

$$u = \frac{c^2 + p - 8rk^2c^2}{2qc} + \frac{6rk^2c}{q}\tanh^2 k(x - ct).$$

5. Conclusion

The JEFE method is applied to some nonlinear wave equations and its applying domain is given. The periodic solutions obtained by the JEFE method can be deduced as the shock wave or solitary wave solutions in the limit condition. In this paper, the JEFE method is applied to $(2i+1)$-order KdV equations only when $i = 2$; when $i = 1$, i.e. KdV equation, its results can be found in Ref. [1]. Actually, more mathematical softwares, such as Mathematica and Maple, dealing with symbol computation have been applied to the boring algebraic operations, so $(2i+1)$-order KdV equations with large value i can be easily solved similarly. Actually, this method can be applied to obtain solutions to more nonlinear wave equations, as long as the sum of derivative order of every term is odd or even simultaneously in the nonlinear wave equations.

In this paper, only even n in (3) is considered, so the expansion about other Jacobi elliptic functions such as $\mathrm{cn}\,\xi$ and $\mathrm{dn}\,\xi$ gives the same results as that about $\mathrm{sn}\,\xi$ for the relations (4). However, we can get different solutions when we use different JEFE in some cases, especially when n is odd in the (3), the details about $n = 1$ can be found in [16] (for mKdV equation and nonlinear Klein–Gordon equation). Of course, the above discussions can be applied to coupled equations, see [1,17,18] (for a variant of the Boussinesq equations) for example.

Acknowledgements

Many thanks are due to Prof. Zhi Li for his suggestions. This paper is supported by the state key projects of the National Science Foundation of China (no. 40045016 and no. 40175016).

References

[1] Liu SK, Fu ZT, Liu SD, et al. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys Lett A 2001;289:69–74.
[2] Wang ML. Solitary wave solutions for variant Boussinesq equations. Phys Lett A 1995;199:169–72.
[3] Wang ML, Zhou YB, Li ZB. Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. Phys Lett A 1996;216:67–75.
[4] Yang L, Zhu Z, Wang Y. Exact solutions of nonlinear equations. Phys Lett A 1999;260:55–9.
[5] Yang L, Liu J, Yang K. Exact solutions of nonlinear PDE, nonlinear transformations and reduction of nonlinear PDE to a quadrature. Phys Lett A 2001;278:267–70.

[6] Parkes EJ, Duffy BR. Travelling solitary wave solutions to a compound KdV-Burgers equation. Phys Lett A 1997;229:217–20.
[7] Fan EG. Extended tanh-function method and its applications to nonlinear equations. Phys Lett A 2000;277:212–8.
[8] Hirota R. Exact N-solutions of the wave equation of long waves in shallow water and in nonlinear lattices. J Math Phys 1973;14:810–4.
[9] Kudryashov NA. Exact solutions of the generalized Kuramoto–Sivashinsky equation. Phys Lett A 1990;147:287–91.
[10] Otwinowski M, Paul R, Laidlaw WG. Exact travelling wave solutions of a class of nonlinear diffusion equations by reduction to a quadrature. Phys Lett A 1988;128:483–7.
[11] Liu SK, Fu ZT, et al. A simple fast mehod in finding particular solutions of some nonlinear PDE. Appl Math Mech 2001;22:326–31.
[12] Yan CT. A simple transformation for nonlinear waves. Phys Lett A 1996;224:77–84.
[13] Porubov AV. Periodical solution to the nonlinear dissipative equation for surface waves in a convecting liquid layer. Phys Lett A 1996;221:391–4.
[14] Porubov AV, Velarde MG. Exact periodic solutions of the complex Ginzburg–Landau equation. J Math Phys 1999;40:884–96.
[15] Porubov AV, Parker DF. Some general periodic solutions to coupled nonlinear Schrödinger equations. Wave Motion 1999;29:97–108.
[16] Fu ST, Liu SK, Liu SD, et al. New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations. Phys Lett A 2001;290:72–6.
[17] Liu SD, Fu ST, et al. The envelope periodic solutions to nonlinear wave equations with Jacobi elliptic function. Acta Phys Sin 2002;51:718–22 (in Chinese).
[18] Liu SK, Fu ST, et al. New periodic solutions to one kind of nonlinear wave equations. Acta Phys Sin 2002;51:10–4 (in Chinese).