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# Envelope Periodic Solutions to Coupled Nonlinear Equations*

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***Abstract*** *The envelope periodic solutions to some nonlinear coupled equations are obtained by means of the Jacobi elliptic function expansion method. And these envelope periodic solutions obtained by this method can degenerate to the envelope shock wave solutions and/or the envelope solitary wave solutions.*

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**Key words:** Jacobi elliptic function, nonlinear coupled equations, envelope periodic solution

## 1 Introduction

Numerous nonlinear models (nonlinear equations) are proposed to understand the physical mechanism in different physical problems. Generally speaking, the exact analytical solutions to these nonlinear equations are hardly obtained, and many numerical methods have to be applied to solve these nonlinear equations. Fortunately, a few exact analytical solutions for some nonlinear equations can be found under certain conditions, for example, the solitary wave solutions, shock wave solutions and periodic solutions, and so on. Applying these analytical solutions, we can check the reliability of numerical solutions in the same control parameters. Much effort has been spent on the construction of exact solutions of nonlinear equations. A number of methods have been proposed, such as the homogeneous balance method,[1−3] the hyperbolic tangent function expansion method,[4−6] the nonlinear transformation method,[7,8] the trial function method,[9,10] and sine-cosine method.[11] These methods, however, can only obtain the shock and solitary wave solutions or the periodic solutions with the elementary functions,[1−12] but cannot get the generalized periodic solutions of nonlinear equations. Although Porubov *et al.*[13−15] have obtained some periodic solutions to some nonlinear equations, they used the Weierstrass elliptic function and involved complicated deduction. The Jacobi elliptic function expansion method[16,17] has been proposed and applied to obtain the generalized periodic solutions to some nonlinear equations and their corresponding shock wave and solitary wave solutions. In this paper, this method is applied to get the envelope periodic solutions and corresponding envelope shock or solitary wave solutions to some coupled nonlinear equations.

## 2 Envelope Periodic Solutions for Coupled NLS Equations

The coupled nonlinear Schrödinger (NLS) equations read

$$\mathrm{i}\frac{\partial u}{\partial t}+\alpha\frac{\partial^2 u}{\partial x^2}+[\beta_1|u|^2+\beta_2|v|^2]u=0\,,$$
$$\mathrm{i}\frac{\partial v}{\partial t}+\alpha\frac{\partial^2 v}{\partial x^2}+[\beta_1|v|^2+\beta_2|u|^2]v=0\,. \tag{1}$$

We seek the following wave packet solutions

$$u=\phi(\xi)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,,\quad v=\psi(\xi)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,,\quad \xi=p(x-c_gt)\,. \tag{2}$$

Substituting Eq. (2) into Eqs. (1) yields

$$\alpha p^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}+\mathrm{i}p(2\alpha k-c_g)\frac{\mathrm{d}\phi}{\mathrm{d}\xi}+(\omega-\alpha k^2)\phi+\beta_1\phi^3+\beta_2\phi\psi^2=0\,,$$
$$\alpha p^2\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2}+\mathrm{i}p(2\alpha k-c_g)\frac{\mathrm{d}\psi}{\mathrm{d}\xi}+(\omega-\alpha k^2)\psi+\beta_1\psi^3+\beta_2\phi^2\psi=0\,. \tag{3}$$

Taking $2\alpha k=c_g$, $\omega-\alpha k^2=-\gamma$ $(\gamma>0)$, we have

$$\alpha p^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}-\gamma\phi+\beta_1\phi^3+\beta_2\phi\psi^2=0\,,\qquad \alpha p^2\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2}-\gamma\psi+\beta_1\psi^3+\beta_2\phi^2\psi=0\,. \tag{4}$$

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Next, we solve Eq. (4) using Jacobi elliptic function expansions.

### 2.1 *Jacobi Elliptic Sine Function Expansion*

By the Jacobi elliptic function expansion method, $\phi(\xi)$ and $\psi(\xi)$ can be expressed as a finite series of Jacobi elliptic function, $\mathrm{sn}\,\xi$, i.e.,

$$\phi(\xi)=\sum_{j=0}^{n_1}a_j\mathrm{sn}^j\xi\,,\qquad \psi(\xi)=\sum_{j=0}^{n_2}b_j\mathrm{sn}^j\xi\,,\tag{5}$$

we can select $n_1$ and $n_2$ to balance the derivative term of the highest order and nonlinear term in Eq. (4) and get the ansatz solution of Eq. (4) in terms of $\mathrm{sn}\,\xi$

$$\phi(\xi)=a_0+a_1\mathrm{sn}\,\xi\,,\qquad \psi(\xi)=b_0+b_1\mathrm{sn}\,\xi\,.\tag{6}$$

Substituting Eq. (6) into Eqs. (4), one can get

$$\begin{aligned}&(-\gamma+\beta_1a_0^2+\beta_2b_0^2)a_0+\{[-\gamma+3\beta_1a_0^2-(1+m^2)\alpha p^2]a_1+\beta_2(2a_0b_1+a_1b_0)b_0\}\mathrm{sn}\,\xi\\&\quad+[3\beta_1a_0a_1^2+\beta_2(a_0b_1+2a_1b_0)b_1]\mathrm{sn}^2\xi+(\beta_1a_1^2+2m^2\alpha p^2+\beta_2b_1^2)a_1\mathrm{sn}^3\xi=0\,,\\&(-\gamma+\beta_1b_0^2+\beta_2a_0^2)a_0+\{[-\gamma+3\beta_1b_0^2-(1+m^2)\alpha p^2]b_1+\beta_2(2a_1b_0+a_0b_1)a_0\}\mathrm{sn}\,\xi\\&\quad+[3\beta_1b_0b_1^2+\beta_2(a_1b_0+2a_0b_1)a_1]\mathrm{sn}^2\xi+(\beta_1b_1^2+2m^2\alpha p^2+\beta_2a_1^2)b_1\mathrm{sn}^3\xi=0\,,\end{aligned}\tag{7}$$

and then set the coefficients of the different powers for $\mathrm{sn}\,\xi$ to be zeros to obtain the algebraic equations about $a_j$ and $b_j$,

$$\begin{aligned}&(-\gamma+\beta_1a_0^2+\beta_2b_0^2)a_0=0\,, && [-\gamma+3\beta_1a_0^2-(1+m^2)\alpha p^2]a_1+\beta_2(2a_0b_1+a_1b_0)b_0=0\,,\\&3\beta_1a_0a_1^2+\beta_2(a_0b_1+2a_1b_0)b_1=0\,, && (\beta_1a_1^2+2m^2\alpha p^2+\beta_2b_1^2)a_1=0\,,\\&(-\gamma+\beta_1b_0^2+\beta_2a_0^2)a_0=0\,, && [-\gamma+3\beta_1b_0^2-(1+m^2)\alpha p^2]b_1+\beta_2(2a_1b_0+a_0b_1)a_0=0\,,\\&3\beta_1b_0b_1^2+\beta_2(a_1b_0+2a_0b_1)a_1=0\,, && (\beta_1b_1^2+2m^2\alpha p^2+\beta_2a_1^2)b_1=0\,,\end{aligned}\tag{8}$$

from which the coefficients $a_j$ and $b_j$ and corresponding constraints can be determined as

$$a_0=b_0=0\,,\qquad p^2=-\frac{\gamma}{(1+m^2)\alpha}\,,\qquad a_1=b_1=\pm\sqrt{-\frac{2m^2\alpha p^2}{\beta_1+\beta_2}}\,.\tag{9}$$

Then the envelope periodic solutions for $u$ and $v$ are

$$u=v=\pm\sqrt{\frac{2m^2\gamma}{(1+m^2)(\beta_1+\beta_2)}}\,\mathrm{sn}\sqrt{-\frac{\gamma}{(1+m^2)\alpha}}\,(x-c_gt)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\tag{10}$$

and when $m\to1$, corresponding envelope solitary wave solutions are

$$u=v=\pm\sqrt{\frac{\gamma}{\beta_1+\beta_2}}\,\tanh\sqrt{-\frac{\gamma}{2\alpha}}\,(x-c_gt)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,.\tag{11}$$

### 2.2 *Jacobi Elliptic Cosine Function Expansion*

$\phi(\xi)$ and $\psi(\xi)$ can be expressed as a finite series of Jacobi cosine elliptic function, $\mathrm{cn}\,\xi$, i.e.,

$$\phi(\xi)=\sum_{j=0}^{n_1}a_j\mathrm{cn}^j\xi\,,\qquad \psi(\xi)=\sum_{j=0}^{n_2}b_j\mathrm{cn}^j\xi\,,\tag{12}$$

and the ansatz solution of Eq. (4) in terms of $\mathrm{cn}\,\xi$ is

$$\phi(\xi)=a_0+a_1\mathrm{cn}\,\xi\,,\qquad \psi(\xi)=b_0+b_1\mathrm{cn}\,\xi\,.\tag{13}$$

Similarly, another kind of envelope periodic solutions can be obtained, for $\mathrm{cn}\,\xi$, which are

$$u=v=\pm\sqrt{\frac{2m^2\gamma}{(\beta_1+\beta_2)(2m^2-1)}}\,\mathrm{cn}\sqrt{\frac{\gamma}{\alpha(2m^2-1)}}\,(x-c_gt)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,,\tag{14}$$

and the limited solutions are

$$u = v = \pm\sqrt{\frac{2\gamma}{\beta_1+\beta_2}}\,\mathrm{sech}\sqrt{\frac{\gamma}{\alpha}}\,(x-c_g t)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{15}$$

### 2.3 *Jacobi Elliptic Function of the Third Kind Expansion*

$\phi(\xi)$ and $\psi(\xi)$ can be expressed as a finite series of Jacobi elliptic function of the third kind, $\mathrm{dn}\,\xi$, i.e.,

$$\phi(\xi) = \sum_{j=0}^{n_1} a_j \mathrm{dn}^j\xi\,, \qquad \psi(\xi) = \sum_{j=0}^{n_2} b_j \mathrm{dn}^j\xi\,, \tag{16}$$

and the ansatz solution of Eq. (4) in terms of $\mathrm{dn}\,\xi$ is

$$\phi(\xi) = a_0 + a_1\mathrm{dn}\,\xi\,, \qquad \psi(\xi) = b_0 + b_1\mathrm{dn}\,\xi\,. \tag{17}$$

Similarly, another kind of envelope periodic solutions can be obtained, for $\mathrm{dn}\,\xi$, which are

$$u = v = \pm\sqrt{\frac{2m^2\gamma}{(\beta_1+\beta_2)(2-m^2)}}\,\mathrm{dn}\,\sqrt{\frac{\gamma}{\alpha(2-m^2)}}\,(x-c_g t)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \tag{18}$$

and the limited solutions are the same one as Eq. (15).

### 2.4 *Jacobi Elliptic Function* $\mathbf{cs}\,\boldsymbol{\xi}$ *Expansion*

$\phi(\xi)$ and $\psi(\xi)$ can be expressed as a finite series of Jacobi elliptic function, $\mathrm{cs}\,\xi$, i.e.,

$$\phi(\xi) = \sum_{j=0}^{n_1} a_j \mathrm{cs}^j\xi\,, \qquad \psi(\xi) = \sum_{j=0}^{n_2} b_j \mathrm{cs}^j\xi\,, \qquad \mathrm{cs}\,\xi = \frac{\mathrm{cn}\,\xi}{\mathrm{sn}\,\xi} \tag{19}$$

and the ansatz solution of Eq. (4) in terms of $\mathrm{cs}\,\xi$ is

$$\phi(\xi) = a_0 + a_1\mathrm{cs}\,\xi\,, \qquad \psi(\xi) = b_0 + b_1\mathrm{cs}\,\xi\,. \tag{20}$$

Similarly, another kind of envelope periodic solutions can be obtained, for $\mathrm{cs}\,\xi$, which are

$$u = v = \pm\sqrt{\frac{2m^2\gamma}{(\beta_1+\beta_2)(2-m^2)}}\,\mathrm{cs}\,\sqrt{\frac{\gamma}{\alpha(2-m^2)}}\,(x-c_g t)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \tag{21}$$

and the limited solutions are

$$u = v = \pm\sqrt{\frac{2\gamma}{\beta_1+\beta_2}}\,\mathrm{csch}\,\sqrt{\frac{\gamma}{\alpha}}\,(x-c_g t)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{22}$$

The above analysis shows that the expansion in terms of different Jacobi elliptic functions can lead to different envelope periodic solutions and envelope solitary wave solutions, and more new results can be got in these expansions.

## 3 Envelope Periodic Solutions for Coupled Nonlinear Schrödinger–KdV Equations

The coupled nonlinear Schrödinger–KdV equations read

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} - \frac{\partial |v|^2}{\partial x} = 0\,, \qquad \mathrm{i}\frac{\partial v}{\partial t} + \alpha\frac{\partial^2 v}{\partial x^2} - \delta uv = 0\,. \tag{23}$$

We seek its following wave packet solutions

$$u = u(\xi)\,, \qquad v = \phi(\xi)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \qquad \xi = p(x-c_g t)\,, \tag{24}$$

where both $\phi(\xi)$ and $u(\xi)$ are real functions.

Substituting Eq. (24) into Eqs. (23) yields

$$-c_g\frac{\mathrm{d}u}{\mathrm{d}\xi} + u\frac{\mathrm{d}u}{\mathrm{d}\xi} + \beta p^2\frac{\mathrm{d}^3 u}{\mathrm{d}\xi^3} - \frac{\mathrm{d}\phi^2}{\mathrm{d}\xi} = 0\,, \tag{25a}$$

$$\alpha p^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + \mathrm{i}p(2\alpha k - c_g)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + (\omega - \alpha k^2)\phi - \delta u\phi = 0\,. \tag{25b}$$

Integrating Eq. (25a) once and taking integration constant as zero, setting $c_g = 2\alpha k$ and $\omega - \alpha k^2 = -\gamma$ in Eq. (25b), then equations (25a) and (25b) take the following forms

$$-c_g u + \frac{1}{2}u^2 + \beta p^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} - \phi^2 = 0\,, \qquad \alpha p^2 \frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} - \gamma\phi - \delta u\phi = 0\,. \tag{26}$$

By the Jacobi elliptic function expansion method, $\phi(\xi)$ and $u(\xi)$ can be expressed as a finite series of Jacobi elliptic function, $\operatorname{sn}\xi$, i.e.,

$$u(\xi) = \sum_{j=0}^{n_1} a_j \operatorname{sn}^j\xi\,, \qquad \phi(\xi) = \sum_{j=0}^{n_2} b_j \operatorname{sn}^j\xi\,. \tag{27}$$

We can select $n_1$ and $n_2$ to balance the derivative term of the highest order and nonlinear terms in Eq. (26) and get the ansatz solution of Eq. (26) in terms of $\operatorname{sn}\xi$,

$$u(\xi) = a_0 + a_1\operatorname{sn}\xi + a_2\operatorname{sn}^2\xi\,, \qquad \phi(\xi) = b_0 + b_1\operatorname{sn}\xi + b_2\operatorname{sn}^2\xi\,. \tag{28}$$

It is obvious that there are the following formula

$$\begin{aligned}
&\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} = 2a_2 - (1+m^2)a_1\operatorname{sn}\xi - 4(1+m^2)a_2\operatorname{sn}^2\xi + 2m^2a_1\operatorname{sn}^3\xi + 6m^2a_2\operatorname{sn}^4\xi\,,\\
&\frac{\mathrm{d}^2 \phi}{\mathrm{d}\xi^2} = 2b_2 - (1+m^2)b_1\operatorname{sn}\xi - 4(1+m^2)b_2\operatorname{sn}^2\xi + 2m^2b_1\operatorname{sn}^3\xi + 6m^2b_2\operatorname{sn}^4\xi\,,\\
&u^2 = a_0^2 + 2a_0a_1\operatorname{sn}\xi + (2a_0a_2 + a_1^2)\operatorname{sn}^2\xi + 2a_1a_2\operatorname{sn}^3\xi + a_2^2\operatorname{sn}^4\xi\,,\\
&\phi^2 = b_0^2 + 2b_0b_1\operatorname{sn}\xi + (2b_0b_2 + b_1^2)\operatorname{sn}^2\xi + 2b_1b_2\operatorname{sn}^3\xi + b_2^2\operatorname{sn}^4\xi\,,\\
&u\phi = a_0b_0 + (a_0b_1 + a_1b_0)\operatorname{sn}\xi + (a_0b_2 + a_1b_1 + a_2b_0)\operatorname{sn}^2\xi + (a_1b_2 + a_2b_1)\operatorname{sn}^3\xi + a_2b_2\operatorname{sn}^4\xi\,.
\end{aligned} \tag{29}$$

Then substituting Eqs. (28) and (29) into Eq. (26) leads to

$$\begin{aligned}
&[-c_g a_0 + a_0^2/2 + 2\beta p^2 a_2 - b_0^2] + [-c_g a_1 + a_0a_1 - \beta p^2(1+m^2)a_1 - 2b_0b_1]\operatorname{sn}\xi\\
&\quad + [-c_g a_2 + (2a_0a_2 + a_1^2)/2 - 4\beta p^2(1+m^2)a_2 - (2b_0b_2 + b_1^2)]\operatorname{sn}^2\xi\\
&\quad + [a_1a_2 + 2m^2\beta p^2 a_1 - 2b_1b_2]\operatorname{sn}^3\xi + [a_2^2/2 + 6m^2\beta p^2 a_2 - b_2^2]\operatorname{sn}^4\xi = 0\,,\\
&[2\alpha p^2 b_2 - \gamma b_0 - \delta a_0 b_0] + [-\alpha p^2(1+m^2)b_1 - \gamma b_1 - \delta(a_0b_1 + a_1b_0)]\operatorname{sn}\xi\\
&\quad + [-4\alpha p^2(1+m^2)b_2 - \gamma b_2 - \delta(a_0b_2 + a_1b_1 + a_2b_0)]\operatorname{sn}^2\xi\\
&\quad + [2\alpha p^2 m^2 b_1 - \delta(a_1b_2 + a_2b_1)]\operatorname{sn}^3\xi + [6\alpha p^2m^2b_2 - \delta a_2b_2]\operatorname{sn}^4\xi = 0\,.
\end{aligned} \tag{30}$$

Setting the coefficients of each power of $\operatorname{sn}\xi$ to be zero, one can get the algebraic equations about $a_i$ and $b_i$ $(i = 0, 1, 2)$, i.e.,

$$\begin{aligned}
&-c_g a_0 + a_0^2/2 + 2\beta p^2 a_2 - b_0^2 = 0\,, \qquad -c_g a_1 + a_0a_1 - \beta p^2(1+m^2)a_1 - 2b_0b_1 = 0\,,\\
&-c_g a_2 + (2a_0a_2 + a_1^2)/2 - 4\beta p^2(1+m^2)a_2 - (2b_0b_2 + b_1^2) = 0\,, \qquad a_1a_2 + 2m^2\beta p^2 a_1 - 2b_1b_2 = 0\,,\\
&a_2^2/2 + 6m^2\beta p^2 a_2 - b_2^2 = 0\,, \qquad 2\alpha p^2 b_2 - \gamma b_0 - \delta a_0 b_0 = 0\,,\\
&-\alpha p^2(1+m^2)b_1 - \gamma b_1 - \delta(a_0b_1 + a_1b_0) = 0\,, \qquad -4\alpha p^2(1+m^2)b_2 - \gamma b_2 - \delta(a_0b_2 + a_1b_1 + a_2b_0) = 0\,,\\
&2\alpha p^2m^2b_1 - \delta(a_1b_2 + a_2b_1) = 0\,, \qquad 6\alpha p^2m^2b_2 - \delta a_2b_2 = 0\,,
\end{aligned} \tag{31}$$

from which one can determine the coefficients

$$\begin{aligned}
&a_2 = \frac{6m^2\alpha p^2}{\delta}\,, \qquad b_2 = \pm 6m^2p^2\sqrt{\frac{\alpha^2 + 2\alpha\beta\delta}{2\delta^2}}\,, \qquad a_1 = b_1 = 0\,,\\
&a_0 = \frac{\alpha}{2(\alpha+\beta\delta)}\Big\{c_g + 4\beta p^2(1+m^2) - \frac{(\alpha + 2\beta\delta)}{\delta}\Big[4p^2(1+m^2) + \frac{\gamma}{\alpha}\Big]\Big\},\\
&b_0 = \pm\frac{1}{2\delta}\sqrt{\frac{2\delta^2}{\alpha^2 + 2\alpha\beta\delta}}\ [a_0 - c_g - 4\beta p^2(1+m^2)]\,.
\end{aligned} \tag{32}$$

So the periodic solutions for the coupled nonlinear Schrödinger–KdV equations are

$$u = a_0 + \frac{6m^2\alpha p^2}{\delta}\mathrm{sn}^2[\,p(x - c_g t)]\,, \tag{33a}$$

$$v = b_0 \pm 6m^2p^2\sqrt{\frac{\alpha^2 + 2\alpha\beta\delta}{2\delta^2}}\mathrm{sn}^2[\,p(x - c_g t)]\,\mathrm{e}^{\,\mathrm{i}(kx-\omega t)}\,, \tag{33b}$$

where $a_0$ and $b_0$ are given in Eqs. (32), and equation (33b) is envelope periodic solutions. When $m \to 1$, equations (33a) and (33b) degenerate to soliton and envelope soliton solution, respectively. Moreover, there are relations

$$\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1\,, \qquad m^2\mathrm{sn}^2\xi + \mathrm{dn}^2\xi = 1\,, \tag{34}$$

so one can get the same results by using $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$ or $\mathrm{dn}\,\xi$ expansion in solving the coupled nonlinear Schrödinger-KdV equations.

## 4 Envelope Periodic Solutions for Coupled Nonlinear Klein–Gordon–Schrödinger Equations

The coupled nonlinear Klein–Gordon–Schrödinger equations read

$$\frac{\partial^2 u}{\partial t^2} - c_0^2\frac{\partial^2 u}{\partial x^2} + f_0^2 u - \gamma|v|^2 = 0\,, \qquad \mathrm{i}\frac{\partial v}{\partial t} + \alpha\frac{\partial^2 v}{\partial x^2} + \beta uv = 0\,. \tag{35}$$

We seek its following wave packet solutions

$$u = u(\xi)\,, \qquad v = \phi(\xi)\,\mathrm{e}^{\,\mathrm{i}(kx-\omega t)}\,, \qquad \xi = p(x - c_g t)\,, \tag{36}$$

where both $\phi(\xi)$ and $u(\xi)$ are real functions.

Substituting Eq. (36) into Eqs. (35) yields

$$p^2(c_g^2 - c_0^2)\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + f_0^2 u - \gamma\phi^2 = 0\,, \tag{37a}$$

$$\alpha p^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + \mathrm{i}p(2\alpha k - c_g)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + (\omega - \alpha k^2)\phi + \beta u\phi = 0\,. \tag{37b}$$

Setting $c_g = 2\alpha k$ and $\omega - \alpha k^2 = -\delta$ in Eq. (37b), then equations (37a) and (37b) take the following forms

$$p^2(c_g^2 - c_0^2)\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + f_0^2 u - \gamma\phi^2 = 0\,, \qquad \alpha p^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} - \delta\phi + \beta u\phi = 0\,. \tag{38}$$

By the Jacobi elliptic function expansion method, $\phi(\xi)$ and $u(\xi)$ can be expressed as a finite series of Jacobi elliptic function $\mathrm{sn}\,\xi$, i.e.,

$$u(\xi) = \sum_{j=0}^{n_1} a_j\mathrm{sn}^j\xi\,, \qquad \phi(\xi) = \sum_{j=0}^{n_2} b_j\mathrm{sn}^j\xi\,. \tag{39}$$

We can select $n_1$ and $n_2$ to balance the derivative term of the highest order and nonlinear term in Eq. (38) and get the ansatz solution of Eq. (38) in terms of $\mathrm{sn}\,\xi$,

$$u(\xi) = a_0 + a_1\mathrm{sn}\,\xi + a_2\mathrm{sn}^2\xi\,, \qquad \phi(\xi) = b_0 + b_1\mathrm{sn}\,\xi + b_2\mathrm{sn}^2\xi\,. \tag{40}$$

Similarly, substituting Eq. (40) into Eq. (38) will lead to the following results

$$a_2 = -\frac{6m^2\alpha p^2}{\beta}\,, \qquad b_2 = \pm 6m^2p^2\sqrt{-\frac{\alpha(c_g^2 - c_0^2)}{\beta\gamma}}\,, \qquad a_1 = b_1 = 0\,,$$

$$a_0 = \frac{\alpha}{f_0^2\beta}\Big\{12m^2p^4(c_g^2 - c_0^2) - \frac{[f_0^2 - 4(1+m^2)p^2(c_g^2 - c_0^2)]}{4(c_g^2 - c_0^2)}\Big\},$$

$$b_0 = \pm\frac{[f_0^2 - 4(1+m^2)p^2(c_g^2 - c_0^2)]}{2(c_g^2 - c_0^2)}\sqrt{-\frac{\alpha(c_g^2 - c_0^2)}{\beta\gamma}}\,. \tag{41}$$

So the periodic solutions for the coupled nonlinear Klein–Gordon–Schrödinger equations are

$$u = \frac{\alpha}{f_0^2\beta}\Big\{12m^2p^4(c_g^2 - c_0^2) - \frac{[f_0^2 - 4(1+m^2)p^2(c_g^2 - c_0^2)]}{4(c_g^2 - c_0^2)}\Big\} - \frac{6m^2\alpha p^2}{\beta}\mathrm{sn}^2[\,p(x - c_g t)]\,, \tag{42a}$$

$$v = \pm \frac{[f_0^2 - 4(1+m^2)p^2(c_g^2 - c_0^2)]}{2(c_g^2 - c_0^2)} \sqrt{-\frac{\alpha(c_g^2 - c_0^2)}{\beta\gamma}} \pm 6m^2p^2 \sqrt{-\frac{\alpha(c_g^2 - c_0^2)}{\beta\gamma}}\, \mathrm{sn}^2[\,p(x - c_g t)]\, \mathrm{e}^{\mathrm{i}(kx - \omega t)}\,, \tag{42b}$$

where equation (42b) is envelope periodic solutions. When $m \to 1$, equations (42a) and (42b) degenerate to soliton and envelope soliton solution, respectively.

## 5 Conclusion

In this paper, the exact envelope periodic solutions to some coupled nonlinear equations are obtained by use of Jacobi elliptic function expansion method. The envelope periodic solutions got by this method can degenerate as the envelope shock wave and envelope solitary wave solutions. Similarly, this solving process can be applied to other nonlinear equations, such as Landau–Lifshitz equations, Kadomtsev–Petviashvili equation and some others.

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