Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Physics Letters A 309 (2003) 234–239
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Power series expansion method and its applications to nonlinear wave equation

Shikuo Liu [a], Zuntao Fu [a,b,∗], Shida Liu [a,b], Qiang Zhao [a]

[a] School of Physics, Peking University, Beijing 100871, PR China
[b] LTCS, Department of Mechanics & Engineering Science, Peking University, Beijing 100871, PR China

Received 17 July 2002; received in revised form 20 November 2002; accepted 8 January 2003

Communicated by A.R. Bishop

Abstract

The power series expansion method is proposed and applied, just like the reductive perturbation method, to reduce the complicated nonlinear equation or set of equations to be the one that can be found the exact solutions.

PACS: 01.55.+b; 04.20.Jb; 05.45.Yv

Keywords: Nonlinear wave equation; Power series expansion method; KdV equation; Gardner equation

1. Introduction

During the past three decades, the nonlinear wave researches have made great progress, among which a number of new methods have been proposed to get the exact solutions to nonlinear wave equations. In these methods, the homogeneous balance method [1–3], the hyperbolic tangent function expansion method [4–6], the nonlinear transformation method [7,8], the trial function method [9,10], sine–cosine method [11], the Jacobi elliptic function expansion method [12,13] and so on [14–16] are widely applied to solve nonlinear wave equations exactly and many solutions are obtained. Based on these solutions, the richness of structures is shown to exist in the different nonlinear wave equations. However, there still exist much more complicated nonlinear equations or set of equations can hardly be solved directly, one has to seek their approximate or asymptotic solutions [17,18]. Through the Gardner–Morikawa transformation, the reductive perturbation method [19] is applied to reduce the complicated nonlinear equations or set of equations to solvable ones, such as KdV equation, cylindrical KdV equation, spherical KdV equation and so on. In this Letter, the power series expansion method is proposed and applied to some equations or set of equations, just as the reductive perturbation method, the similar results are given, but contrary to the reductive perturbation method, the power series expansion method is a relatively simple one.

∗ Corresponding author. School of Physics (Building Physics), Peking University, Beijing 100871, PR China.
E-mail address: fuzt@pku.edu.cn (Z. Fu).

0375-9601/03/$ – see front matter
doi:10.1016/S0375-9601(03)00077-X

2. Power series expansion method

Consider a given nonlinear wave equations or set of equations

$$N(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0. \tag{1}$$

First step in this method, the field quantities are divided into the basic and perturbation parts, i.e.,

$$u = U + u' \quad (u' \ll |U|) \tag{2}$$

and write (1) the equations of perturbation quantities. Second step is seeking the travelling wave solutions in the form

$$u' = u'(\xi), \quad \xi = x - ct, \tag{3}$$

where c is the wave speed which is supposed to be a constant. In the third step, the equation of perturbation quantities is reduced to the following form:

$$\frac{d^2u'}{d\xi^2} + F(u')u' = 0, \tag{4}$$

where $F(u')$ is a nonlinear function of u'. Final step is expanding $F(u')$ in power series, the linear equation of (4) is given when the power series of $F(u')$ is taken only the first term; while when the second term is also included, the equation is the ordinary differential equation that KdV corresponds to; when the power series is truncated after the third term, then the resulted equation is the ordinary differential equation that mixed KdV–mKdV (i.e., Gardner) equation corresponds to. So the crucial step is the third one to get the nonlinear function $F(u')$, and then its expansion form.

3. Applications

In this section, we will illustrate the applications of the power series expansion method to some nonlinear wave equations.

3.1. *Boussinesq system of shallow water waves* [20]

$$\begin{cases} u_t + uu_x + gh_x + \frac{H}{3}h_{ttx} = 0, \\ h_t + uh_x + hu_x = 0, \end{cases} \tag{5}$$

here (u, v) is the horizontal velocity, h is the height of free surface, g is the acceleration of gravity.

Setting

$$u = u', \qquad h = H + h' \quad (h' \ll H), \tag{6}$$

then Eq. (5) reduces to

$$\begin{cases} u'_t + u'u'_x + gh'_x + \frac{H}{3}h'_{ttx} = 0, \\ h'_t + u'h'_x + (H + h')u'_x = 0. \end{cases} \tag{7}$$

Supposing that the system (7) has travelling wave solution, i.e.,

$$u' = u'(\xi), \qquad h' = h'(\xi),$$
$$\xi = x - ct \quad (c = \text{constant}) \tag{8}$$

and substituting Eq. (8) into Eq. (7) yields

$$(u' - c)\frac{du'}{d\xi} + g\frac{dh'}{d\xi} + \frac{Hc^2}{3}\frac{d^3h'}{d\xi^3} = 0, \tag{9a}$$

$$(u' - c)\frac{dh'}{d\xi} + (H + h')\frac{du'}{d\xi} = 0. \tag{9b}$$

Integrating Eq. (9b) with respect to ξ once and taking the integration constant as zero, one can get

$$u' = \frac{ch'}{H + h'}. \tag{10}$$

Substituting Eq. (9a) into Eq. (10) yields

$$\frac{d^2}{d\xi^2}\left(\frac{dh'}{d\xi}\right) + F(h')\left(\frac{dh'}{d\xi}\right) = 0, \tag{11}$$

here

$$F(h') = \frac{3g}{Hc^2}\left[1 - \frac{H^2c^2}{g(H + h')^3}\right]. \tag{12}$$

Because of $h' \ll H$, $F(h')$ can be expanded as power series, i.e.,

$$F(h') = \frac{3g}{Hc^2}\left[\left(1 - \frac{c^2}{c_0^2}\right) + \frac{3c^2}{c_0^2}\frac{h'}{H} - \frac{6c^2}{c_0^2}\frac{h'^2}{H^2} + \cdots\right], \tag{13}$$

where $c_0^2 = gH$.

If only the first term of $F(h')$ is chosen, then Eq. (11) is reduced to

$$\frac{d^3h'}{d\xi^3} + \frac{3g}{Hc^2}\left(1 - \frac{c^2}{c_0^2}\right)\frac{dh'}{d\xi} = 0. \tag{14}$$

Obviously, this is a linear equation.

If only the first two terms of $F(h')$ are chosen, then Eq. (11) is reduced to

$$\frac{d^3h'}{d\xi^3}+\frac{3g}{Hc^2}\left[\left(1-\frac{c^2}{c_0^2}\right)+\frac{3c^2}{c_0^2}\frac{h'}{H}\right]\frac{dh'}{d\xi}=0, \tag{15}$$

which is the ordinary differential equation that KdV equation corresponds to. Actually, from Eq. (8) one has

$$\frac{\partial}{\partial t}=-c\frac{d}{d\xi}, \qquad \frac{\partial}{\partial x}=\frac{d}{d\xi}, \tag{16}$$

then (15) can be rewritten as

$$\frac{\partial h'}{\partial t}+c_0\left(1+\frac{3h'}{2H}\right)\frac{\partial h'}{\partial x}+\frac{1}{6}c_0H^2\frac{\partial^3 h'}{\partial x^3}=0. \tag{17}$$

Obviously, this is KdV equation.

If only the first three terms of $F(h')$ are chosen, then Eq. (11) is reduced to

$$\frac{d^3h'}{d\xi^3}+\frac{3g}{Hc^2}\left[\left(1-\frac{c^2}{c_0^2}\right)+\frac{3c^2}{c_0^2}\frac{h'}{H}-\frac{6c^2}{c_0^2}\frac{h'^2}{H^2}\right]\frac{dh'}{d\xi}=0, \tag{18}$$

which is the ordinary differential equation that Gardner (i.e., mixed KdV–mKdV) equation corresponds to, similarly it can be rewritten as

$$\frac{\partial h'}{\partial t}+c_0\left(1+\frac{3h'}{2H}-\frac{3h'^2}{H^2}\right)\frac{\partial h'}{\partial x}+\frac{1}{6}c_0H^2\frac{\partial^3 h'}{\partial x^3}=0, \tag{19}$$

which is Gardner equation.

3.2. KD equations [21]

Konopelchenko–Dubrovsky (KD) equations read

$$\begin{cases} u_t-u_{xxx}-6\beta uu_x+\frac{3}{2}\alpha^2u^2u_x \\ \quad -3v_y+3\alpha vu_x=0, \\ u_y=v_x, \end{cases} \tag{20}$$

where both u and v are the perturbation quantities.

Setting

$$u=u(\theta), \qquad v=v(\theta), \quad \theta=kx+ly-\omega t, \tag{21}$$

then (20) can be rewritten as

$$\begin{cases} -\omega\frac{du}{d\theta}-k^3\frac{d^3u}{d\theta^3}-6\beta ku\frac{du}{d\theta}+\frac{3}{2}\alpha^2ku^2\frac{du}{d\theta} \\ \quad -3l\frac{dv}{d\theta}+3\alpha kv\frac{du}{d\theta}=0, \\ l\frac{du}{d\theta}=k\frac{dv}{d\theta}. \end{cases} \tag{22}$$

Integrating the second equation in (22) with respect to θ once and taking the integration constant as zero result in $lu=kv$, i.e.,

$$v=\frac{l}{k}u. \tag{23}$$

So substituting (23) into the first equation of (22) leads to

$$-k^3\frac{d^3u}{d\theta^3}+3(\alpha l-2\beta k)u\frac{du}{d\theta}+\frac{3}{2}\alpha^2ku^2\frac{du}{d\theta}-\left(\omega+\frac{3l^2}{k}\right)\frac{du}{d\theta}=0. \tag{24}$$

Integrating (24) with respect to θ once and taking the integration constant as zero yields

$$\frac{d^2u}{d\theta^2}+\left[-\frac{\alpha^2}{2k^2}u^2-\frac{3(\alpha l-2\beta k)}{2k^3}u+\frac{\omega+\frac{3l^2}{k}}{k^3}\right]u=0, \tag{25}$$

where

$$F(u)=\frac{\omega+\frac{3l^2}{k}}{k^3}-\frac{3(\alpha l-2\beta k)}{2k^3}u-\frac{\alpha^2}{2k^2}u^2. \tag{26}$$

It is obvious that $F(u)$ is a polynomial of u, if just the first term is chosen, then (25) is reduced to

$$\frac{d^2u}{d\theta^2}+\frac{\omega+\frac{3l^2}{k}}{k^3}u=0, \tag{27}$$

which is a linear equation. If one takes $(\omega+\frac{3l^2}{k})/k^3=1$, then the dispersion relation is

$$\omega=k^3-\frac{3l^2}{k}. \tag{28}$$

If the first two terms of $F(u)$ are chosen, then one get the ordinary differential equation that KdV equation corresponds to, i.e.,

$$\frac{\omega+\frac{3l^2}{k}}{k^3}\frac{du}{d\theta}-\frac{3(\alpha l-2\beta k)}{k^3}u\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}=0. \tag{29}$$

If all three terms of $F(u)$ are chosen, then one get the ordinary differential equation that KdV–mKdV (i.e., Gardner) equation corresponds to, i.e.,

$$\frac{\omega+\frac{3l^2}{k}}{k^3}\frac{du}{d\theta}-\frac{3\alpha^2}{2k^2}u^2\frac{du}{d\theta}-\frac{3(\alpha l-2\beta k)}{k^3}u\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}=0. \tag{30}$$

3.3. *Electron–ion acoustic waves* [22]

$$\begin{cases} \frac{\partial n_e}{\partial t} + \frac{\partial n_e v_i}{\partial x} - \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x}\right)\frac{\partial \ln n_e}{\partial x}\right] = 0, \\ \frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} + \frac{\partial}{\partial x}\ln n_e = 0, \end{cases} \tag{31}$$

where perturbation quantity n_e and v_i are the electronic number density and ionic velocity, respectively.

Assuming that the solutions are given by

$$n_e = n_e(\xi), \qquad v_i = v_i(\xi), \quad \xi = x - ct. \tag{32}$$

Substituting (32) into (31) yields

$$\begin{cases} -c\frac{dn_e}{d\xi} + \frac{dn_e v_i}{d\xi} - \frac{d}{d\xi}\left[(v_i - c)\frac{d^2 \ln n_e}{d\xi^2}\right] = 0, \\ -c\frac{dv_i}{d\xi} + v_i \frac{dv_i}{d\xi} + \frac{d}{d\xi}\ln n_e = 0. \end{cases} \tag{33}$$

Integrating (33) once with respect to ξ and taking the integration constant as zero lead to

$$\begin{cases} -cn_e + n_e v_i - (v_i - c)\frac{d^2 \ln n_e}{d\xi^2} = 0, \\ -cv_i + \frac{1}{2}v_i^2 + \ln n_e = 0. \end{cases} \tag{34}$$

Eliminating v_i from (34) results in

$$\frac{d^2 \ln n_e}{d\xi^2} - n_e = 0. \tag{35}$$

Setting

$$w = \ln n_e, \tag{36}$$

we get

$$\frac{d^2 w}{d\xi^2} + F(w)w = 0, \tag{37}$$

where

$$F(w) = -\frac{1}{w}e^w. \tag{38}$$

Expanding $F(w)$ in power series yields

$$F(w) = -\frac{1}{w}\left(1 + w + \frac{1}{2}w^2 + \frac{1}{6}w^3 + \cdots\right). \tag{39}$$

When the first two terms in $F(w)$ is taken, i.e., $F(w) = -\frac{1}{w}(1 + w)$, then (37) is reduced to

$$\frac{d^2 w}{d\xi^2} - (1 + w) = 0, \tag{40}$$

which is a linear relation.

When the first three terms in $F(w)$ is taken, i.e., $F(w) = -\frac{1}{w}(1 + w + \frac{1}{2}w^2)$, then (37) is reduced to

$$\frac{d^2 w}{d\xi^2} - \left(1 + w + \frac{1}{2}w^2\right) = 0. \tag{41}$$

Differentiating (41) once with respect to ξ leads to

$$\frac{d^3 w}{d\xi^3} - \frac{dw}{d\xi} - w\frac{dw}{d\xi} = 0, \tag{42}$$

which is the corresponding ordinary differential equation of KdV equation.

When the first four terms in $F(w)$ is taken, i.e., $F(w) = -\frac{1}{w}(1 + w + \frac{1}{2}w^2 + \frac{1}{6}w^3)$, then (37) is reduced to

$$\frac{d^2 w}{d\xi^2} - \left(1 + w + \frac{1}{2}w^2 + \frac{1}{6}w^3\right) = 0. \tag{43}$$

Differentiating (43) once with respect to ξ leads to

$$\frac{d^3 w}{d\xi^3} - \frac{dw}{d\xi} - w\frac{dw}{d\xi} - \frac{1}{2}w^2\frac{dw}{d\xi} = 0, \tag{44}$$

which is the corresponding ordinary differential equation of Gardner equation.

3.4. *Inertial wave equations* [23]

$$\begin{cases} u_t + uu_x - f_0 v = 0, \\ v_t + uv_x + f_0 u = 0, \end{cases} \tag{45}$$

where (u, v) are the perturbation velocities, f_0 is a constant.

Assuming that there exists travelling wave solution, i.e.,

$$u = u(\xi), \qquad v = v(\xi),$$
$$\xi = x - ct \quad (c = \text{constant}). \tag{46}$$

Then Eq. (45) can be rewritten as

$$\begin{cases} \frac{du}{d\xi} + F(u)v = 0, \\ \frac{dv}{d\xi} - F(u)u = 0, \end{cases} \tag{47}$$

here

$$F(u) = \frac{f_0}{c - u}. \tag{48}$$

If $u \ll c$, then $F(u)$ can be expanded as

$$F(u) = \frac{f_0}{c - u} = \frac{f_0}{c}\left(1 + \frac{u}{c} + \frac{u^2}{c^2} + \cdots\right). \tag{49}$$

If just the first term in $F(u)$ is taken, Eq. (47) can be reduced to

$$\begin{cases} \frac{du}{d\xi} + \frac{f_0}{c} v = 0, \\ \frac{dv}{d\xi} - \frac{f_0}{c} u = 0, \end{cases} \tag{50}$$

then one gets

$$\frac{d^2u}{d\xi^2} + \frac{f_0^2}{c^2} u = 0. \tag{51}$$

It is obvious that this is a linear equation.

If the first two terms in $F(u)$ is taken, Eq. (47) can be reduced to

$$\begin{cases} \frac{du}{d\xi} + \frac{f_0}{c}\left(1 + \frac{u}{c}\right) v = 0, \\ \frac{dv}{d\xi} - \frac{f_0}{c}\left(1 + \frac{u}{c}\right) u = 0, \end{cases} \tag{52}$$

from which one can get

$$\frac{d^2u}{d\xi^2} + \frac{f_0^2}{c^2} u + \frac{2f_0^2}{c^3} u^2 + \frac{f_0^2}{c^4} u^3 - \frac{f_0^2}{c^3} v^2 - \frac{f_0^2}{c^4} u v^2 = 0. \tag{53}$$

Actually, from Eq. (47), it is easily to obtain that

$$\frac{du}{dv} = -\frac{v}{u}, \tag{54}$$

i.e.,

$$u^2 + v^2 = a^2 \quad (a^2 = \text{constant}), \tag{55}$$

here a^2 is integration constant.

Substituting Eq. (55) into Eq. (53) yields

$$\frac{d^2u}{d\xi^2} + \frac{f_0^2}{c^2}\left(1 - \frac{a^2}{c^2}\right) u + \frac{3f_0^2}{c^3} u^2 + \frac{2f_0^2}{c^4} u^3 - \frac{f_0^2}{c^3} a^2 = 0. \tag{56}$$

Differentiating Eq. (56) with respect to ξ once, the ordinary differential equation that Gardner (i.e., mixed KdV–mKdV) equation can be derived, i.e.,

$$\frac{d^3u}{d\xi^3} + \frac{f_0^2}{c^2}\left(1 - \frac{a^2}{c^2}\right)\frac{du}{d\xi} + \frac{6f_0^2}{c^3} u \frac{du}{d\xi} + \frac{6f_0^2}{c^4} u^2 \frac{du}{d\xi} = 0. \tag{57}$$

Actually, from Eq. (46) one has

$$\frac{\partial}{\partial t} = -c\frac{d}{d\xi}, \qquad \frac{\partial}{\partial x} = \frac{d}{d\xi}, \tag{58}$$

then from Eq. (57), it is easily derived that

$$\frac{\partial u}{\partial t} + \frac{6c^2}{a^2 - c^2}\left(1 + \frac{u}{c}\right) u \frac{\partial u}{\partial x} + \frac{c^5}{f_0^2(a^2 - c^2)} \frac{\partial^3 u}{\partial x^3} = 0. \tag{59}$$

Obviously, this is Gardner (i.e., mixed KdV–mKdV) equation.

4. Conclusion and discussion

In this Letter, the power series expansion method is proposed and applied to reduce some complicated nonlinear equations or set of equations to the exactly solvable nonlinear ones. Usually, the lowest order approximation satisfies the linear relation, and the next order and/or still next order approximations satisfy the celebrated KdV equation and/or Garder equation (i.e., mixed KdV–mKdV equation), respectively. More higher order approximations can be also got if we take more terms, but it is worth noting that here no convergence conditions of power series expansion are considered and this deserves further research.

Acknowledgements

Many thanks are due to anonymous referees for their good suggestions. This Letter is supported by NSFC (No. 40045016) and NSFC (No. 40175016).

References

[1] M.L. Wang, Phys. Lett. A 199 (1995) 169.
[2] M.L. Wang, Y.B. Zhou, Z.B. Li, Phys. Lett. A 216 (1996) 67.
[3] L. Yang, Z. Zhu, Y. Wang, Phys. Lett. A 260 (1999) 55.
[4] E.J. Parkes, B.R. Duffy, Phys. Lett. A 229 (1997) 217.
[5] L. Yang, J. Liu, K. Yang, Phys. Lett. A 278 (2001) 267.
[6] E.G. Fan, Phys. Lett. A 277 (2000) 212.
[7] R. Hirota, J. Math. Phys. 14 (1973) 810.
[8] M. Otwinowski, R. Paul, W.G. Laidlaw, Phys. Lett. A 128 (1988) 483.
[9] N.A. Kudryashov, Phys. Lett. A 147 (1990) 287.
[10] S.K. Liu, Z.T. Fu, S.D. Liu, Q. Zhao, Appl. Math. Mech. 22 (2001) 326.
[11] C.T. Yan, Phys. Lett. A 224 (1996) 77.
[12] S.K. Liu, Z.T. Fu, S.D. Liu, Q. Zhao, Phys. Lett. A 289 (2001) 69.

[13] Z.T. Fu, S.K. Liu, S.D. Liu, Q. Zhao, Phys. Lett. A 290 (2001) 72.
[14] A.V. Porubov, Phys. Lett. A 221 (1996) 391.
[15] A.V. Porubov, M.G. Velarde, J. Math. Phys. 40 (1999) 884.
[16] A.V. Porubov, D.F. Parker, Wave Motion 29 (1999) 97.
[17] A. Jeffrey, T. Kawahara, Asymptotic Method in Nonlinear Wave Theory, Pitman, Boston, 1982.
[18] A.H. Nayfeh, Perturbation Method, Wiley, New York, 1973.
[19] L.G. Redekopp, J. Fluid Mech. 82 (1977) 725.
[20] G.B. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974.
[21] J. Lin, S.Y. Lou, K.L. Wang, Chin. Phys. Lett. 18 (2001) 1173.
[22] R.A. Treumann, W. Baumjohann, Advanced Space Plasma Physics, Imperial College Press, London, 1997.
[23] S.K. Liu, S.D. Liu, Nonlinear Equations in Physics, Peking Univ. Press, Beijing, 2000 (in Chinese).