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Solving Nonlinear Wave Equations by Elliptic Equation*

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Abstract *The elliptic equation is taken as a transformation and applied to solve nonlinear wave equations. It is shown that this method is more powerful to give more kinds of solutions, such as rational solutions, solitary wave solutions, periodic wave solutions and so on, so it can be taken as a generalized method.*

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1 Introduction

Since more and more problems have to involve nonlinearity, it attracts much attention how the nonlinear models can be solved. Many methods have been proposed to construct exact solutions to nonlinear equations. Among them are the sine-cosine method,[1] the homogeneous balance method,[2−4] the hyperbolic function expansion method,[5−7] the Jacobi elliptic function expansion method,[8,9] the nonlinear transformation method,[10,11] the trial function method,[12,13] and others.[14−16]

Among these methods, a transformation is often introduced to simplify solving process. For example, given a nonlinear wave equation

$$N(u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0\,, \tag{1}$$

Yan[1] introduced a transformation

$$u = u(\omega), \quad \mathrm{d}\omega/\mathrm{d}\xi = \sin\omega, \quad \xi = k(x - ct)\,. \tag{2}$$

Fu *et al.*[17] extended this transformation as

$$u = u(\omega), \quad \mathrm{d}\omega/\mathrm{d}\xi = \pm\lambda_0\sqrt{1 - m^2\sin^2\omega}\,. \tag{3}$$

These transformations are applied to solve nonlinear wave equations, and more solutions have been obtained. And in the hyperbolic function expansion methods, a transformation is also needed, thus one can get more kinds of solutions. Fan[5] introduced

$$u = u(w), \qquad \mathrm{d}w/\mathrm{d}\xi = b + w^2\,. \tag{4}$$

Yan *et al.*[18] extended it as

$$u = u(w), \qquad \mathrm{d}w/\mathrm{d}\xi = R(1 + \mu w^2)\,. \tag{5}$$

Actually, Fan and Yan applied the well-known Riccati equation as their transformations.

In this paper, we will consider the elliptic equation[19]

$$y'^2 = \sum_{i=0}^{i=4} a_i y^i, \qquad a_4 \neq 0\,, \tag{6}$$

where $y' = \mathrm{d}y/\mathrm{d}\xi$, and take it as a new transformation to solve nonlinear wave equations. Obviously, equations (4) abd (5) are just special cases of Eq. (6), so application of Eq. (6) to nonlinear wave equations will lead to more kinds of solutions. In the following sections, applications of Eq. (6) to some well-known equations will be given.

2 KdV Equation

KdV equation reads

$$u_t + uu_x + \beta u_{xxx} = 0\,. \tag{7}$$

We seek its travelling wave solutions in the following frame

$$u = u(\xi), \qquad \xi = k(x - ct)\,, \tag{8}$$

here c is wave velocity, k is wave number.

Substituting Eq. (8) into Eq. (7) and integrating once yield

$$-cu + \tfrac{1}{2}u^2 + \beta k^2 u'' = C\,, \tag{9}$$

where C is an integration constant. And then we suppose equation (7) has the following solution

$$u = u(y) = \sum_{j=0}^{n} b_j y^j, \qquad y = y(\xi)\,, \tag{10}$$

where y satisfies the elliptic equation (6), then

$$y'' = \frac{a_1}{2} + a_2 y + \frac{3a_3}{2}y^2 + 2a_4 y^3\,. \tag{11}$$

There n in Eq. (10) can be determined by the partial balance between the highest order derivative terms and the highest degree nonlinear term in Eq. (7). Here we know that the degree of u is

$$O(u) = O(y^n) = n\,, \tag{12}$$

and from Eqs. (6) and (11), one has

$$O(y'^2) = O(y^4) = 4, \quad O(y'') = O(y^3) = 3\,, \tag{13}$$

and actually one can have

$$O(y^{(l)}) = l + 1\,. \tag{14}$$

So one has

$$O(u) = n, \qquad O(u') = n + 1\,,$$
$$O(u'') = n + 2, \quad O(u^{(l)}) = n + l\,. \tag{15}$$

For KdV equation (7), we have $n = 2$, so the ansatz solution of Eq. (10) can be rewritten as

$$u = b_0 + b_1 y + b_2 y^2, \quad b_2 \neq 0\,, \tag{16}$$

then

$$u^2 = b_0^2 + 2b_0b_1y + (2b_0b_2 + b_1^2)y^2 + 2b_1b_2y^3 + b_2^2y^4\,, \tag{17}$$

$$u'' = \left(\tfrac{1}{2}a_1b_1 + 2a_0b_2\right) + (a_2b_1 + 3a_1b_2)y + \left(\tfrac{3}{2}a_3b_1 + 4a_2b_2\right)y^2 + (2a_4b_1 + 5a_3b_2)y^3$$

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$$+ 6a_4b_2y^4 \,. \tag{18}$$

Substituting Eqs. (16), (17), and (18) into Eq. (9) and collecting each order of y yield algebraic equations about coefficients b_j $(j = 0, 1, 2)$ and a_i $(i = 0, 1, 2, 3, 4)$, i.e.,

$$-cb_0 + \tfrac{1}{2}b_0^2 + \beta k^2\big(\tfrac{1}{2}a_1b_1 + 2a_0b_2\big) - C = 0\,, \tag{19a}$$

$$-cb_1 + b_0b_1 + \beta k^2(a_2b_1 + 3a_1b_2) = 0\,, \tag{19b}$$

$$-cb_2 + \tfrac{1}{2}(2b_0b_2 + b_1^2) + \beta k^2\big(\tfrac{3}{2}a_3b_1 + 4a_2b_2\big) = 0\,, \tag{19c}$$

$$b_1b_2 + \beta k^2(2a_4b_1 + 5a_3b_2) = 0\,, \tag{19d}$$

$$\tfrac{1}{2}b_2^2 + 6\beta k^2a_4b_2 = 0\,, \tag{19e}$$

from which we have

$$b_2 = -12\beta k^2a_4, \qquad b_1 = -6\beta k^2a_3\,,$$
$$b_0 = c - 4\beta k^2a_2 + \frac{3\beta k^2a_3^2}{2a_4}\,, \tag{20}$$

at the same time there is

$$a_1 = \frac{\beta k^2a_3}{2a_4}\Big(a_2 - \frac{a_3^2}{2a_4}\Big)\,. \tag{21}$$

So if $a_3 = 0$, then

$$b_1 = a_1 = 0, \quad b_2 = -12\beta k^2a_4, \quad b_0 = c - 4\beta k^2a_2\,, \tag{22}$$

and the transformation (6) takes the following form

$$y'^2 = a_0 + a_2y^2 + a_4y^4\,, \tag{23}$$

which has many kinds of solutions, some of which we will show next.

Case A Consider $a_0 = 0$, then

$$y'^2 = a_2y^2 + a_4y^4\,, \tag{24}$$

which has three kinds of solutions

(a) If $a_2 > 0$ and $a_4 > 0$, the solution is

$$y = \pm\sqrt{\frac{a_2}{a_4}}\mathrm{csch}(\sqrt{a_2}\,\xi)\,, \tag{25}$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2a_2 - 12\beta k^2a_2\mathrm{csch}^2(\sqrt{a_2}\,\xi)\,. \tag{26}$$

(b) If $a_2 > 0$ and $a_4 < 0$, the solution is

$$y = \pm\sqrt{-\frac{a_2}{a_4}}\mathrm{sech}(\sqrt{a_2}\,\xi)\,, \tag{27}$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2a_2 + 12\beta k^2a_2\mathrm{sech}^2(\sqrt{a_2}\,\xi)\,. \tag{28}$$

(c) If $a_2 < 0$ and $a_4 > 0$, the solution is

$$y = \pm\sqrt{-\frac{a_2}{a_4}}\csc(\sqrt{-a_2}\,\xi)\,, \tag{29}$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2a_2 + 12\beta k^2a_2\csc^2(\sqrt{-a_2}\,\xi)\,. \tag{30}$$

Case B Consider $a_0 = a_2 = 0$ and $a_4 > 0$, so $b_0 = c$, then

$$y = \pm\frac{1}{\sqrt{a_4}\,\xi}\,, \tag{31}$$

and

$$u = b_0 + b_2y^2 = c - \frac{12\beta k^2}{\xi^2}\,, \tag{32}$$

this is a rational solution.

Case C Consider transformation (23) directly, from which many more solutions expressed in terms of different elliptic functions[19] can be got.

(i) If $a_0 = 1$, $a_2 = -(1 + m^2)$ and $a_4 = m^2$ (where $0 \le m \le 1$, is called modulus of Jacobi elliptic functions, see Refs. [19] $\sim$ [22]), then the solution is

$$y = \mathrm{sn}(\xi, m)\,, \tag{33}$$

where $\mathrm{sn}(\xi, m)$ is the Jacobi elliptic sine function (see Refs. [19] $\sim$ [22]) and

$$u = b_0 + b_2y^2 = c + 4\beta k^2(1 + m^2)$$
$$- 12\beta k^2m^2\mathrm{sn}^2(\xi, m)\,. \tag{34}$$

(ii) If $a_0 = 1 - m^2$, $a_2 = 2m^2 - 1$ and $a_4 = -m^2$, then the solution is

$$y = \mathrm{cn}(\xi, m)\,, \tag{35}$$

where $\mathrm{cn}(\xi, m)$ is the Jacobi elliptic cosine function (see Refs. [19] $\sim$ [22]) and

$$u = b_0 + b_2y^2 = c - 4\beta k^2(2m^2 - 1)$$
$$+ 12\beta k^2m^2\mathrm{cn}^2(\xi, m)\,. \tag{36}$$

(iii) If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$, and $a_4 = -1$, then the solution is

$$y = \mathrm{dn}(\xi, m)\,, \tag{37}$$

where $\mathrm{dn}(\xi, m)$ is Jacobi elliptic function of the third kind (see Refs. [19] $\sim$ [22]) and

$$u = b_0 + b_2y^2 = c - 4\beta k^2(2 - m^2) + 12\beta k^2\mathrm{dn}^2(\xi, m)\,. \tag{38}$$

(iv) If $a_0 = m^2$, $a_2 = -(1 + m^2)$, and $a_4 = 1$, then the solution is

$$y = \mathrm{ns}(\xi, m) \equiv \frac{1}{\mathrm{sn}(\xi, m)}\,, \tag{39}$$

and

$$u = b_0 + b_2y^2 = c + 4\beta k^2(1 + m^2) - 12\beta k^2\mathrm{ns}^2(\xi, m)\,. \tag{40}$$

(v) If $a_0 = -m^2$, $a_2 = 2m^2 - 1$, and $a_4 = 1 - m^2$, then the solution is

$$y = \mathrm{nc}(\xi, m) \equiv \frac{1}{\mathrm{cn}(\xi, m)}\,, \tag{41}$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2(2m^2 - 1)$$
$$- 12\beta k^2(1 - m^2)\mathrm{nc}^2(\xi, m)\,. \tag{42}$$

(vi) If $a_0 = -1$, $a_2 = 2 - m^2$, and $a_4 = m^2 - 1$, then the solution is

$$y = \mathrm{nd}(\xi, m) \equiv \frac{1}{\mathrm{dn}(\xi, m)}\,, \tag{43}$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2(2 - m^2)$$
$$- 12\beta k^2(m^2 - 1)\mathrm{nd}^2(\xi, m)\,. \tag{44}$$

(vii) If $a_0 = 1$, $a_2 = 2 - m^2$, and $a_4 = 1 - m^2$, then the solution is

$$y = \mathrm{sc}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)}\,, \tag{45}$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2(2 - m^2)$$
$$- 12\beta k^2(1 - m^2)\mathrm{sc}^2(\xi, m)\,. \tag{46}$$

(viii) If $a_0 = 1$, $a_2 = 2m^2 - 1$, and $a_4 = (m^2 - 1)m^2$, then the solution is

$$y = \mathrm{sd}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}\,, \tag{47}$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2(2m^2 - 1) - 12\beta k^2(m^2 - 1)m^2\mathrm{sd}^2(\xi, m)\,. \quad (48)$$

(ix) If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$, and $a_4 = 1$, then the solution is

$$y = \mathrm{cs}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)}\,, \quad (49)$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2(2 - m^2) - 12\beta k^2\mathrm{cs}^2(\xi, m)\,. \quad (50)$$

(x) If $a_0 = 1$, $a_2 = -(1 + m^2)$, and $a_4 = m^2$, then the solution is

$$y = \mathrm{cd}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)}\,, \quad (51)$$

and

$$u = b_0 + b_2y^2 = c + 4\beta k^2(1 + m^2) - 12\beta k^2m^2\mathrm{cd}^2(\xi, m)\,. \quad (52)$$

(xi) If $a_0 = m^2(m^2 - 1)$, $a_2 = 2m^2 - 1$, and $a_4 = 1$, then the solution is

$$y = \mathrm{ds}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}\,, \quad (53)$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2(2m^2 - 1) - 12\beta k^2\mathrm{ds}^2(\xi, m)\,. \quad (54)$$

(xii) If $a_0 = m^2$, $a_2 = -(1 + m^2)$, and $a_4 = 1$, then the solution is

$$y = \mathrm{dc}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)}\,, \quad (55)$$

and

$$u = b_0 + b_2y^2 = c + 4\beta k^2(1 + m^2) - 12\beta k^2\mathrm{dc}^2(\xi, m)\,. \quad (56)$$

Of course, we can get more generalized solutions:

(xiii) If $a_0 = \mu^2A^2$, $a_2 = -\mu^2(1 + m^2)$, and $a_4 = \mu^2m^2/A^2$, then the solution is

$$y = A\mathrm{sn}(\mu\xi, m)\,, \quad (57)$$

and

$$u = b_0 + b_2y^2 = c + 4\beta k^2\mu^2(1 + m^2) - 12\beta k^2\mu^2m^2\mathrm{sn}^2(\mu\xi, m)\,. \quad (58)$$

(xiv) If $a_0 = \mu^2(1 - m^2)A^2$, $a_2 = \mu^2(2m^2 - 1)$, and $a_4 = -\mu^2m^2/A^2$, then the solution is

$$y = A\mathrm{cn}(\mu\xi, m)\,, \quad (59)$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2\mu^2(2m^2 - 1) + 12\beta k^2\mu^2m^2\mathrm{cn}^2(\mu\xi, m)\,. \quad (60)$$

(xv) If $a_0 = \mu^2(1 - m^2)A^2$, $a_2 = \mu^2(2 - m^2)$, and $a_4 = -\mu^2/A^2$, then the solution is

$$y = A\mathrm{dn}(\mu\xi, m)\,, \quad (61)$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2\mu^2(2 - m^2) + 12\beta k^2\mu^2\mathrm{dn}^2(\mu\xi, m)\,. \quad (62)$$

(xvi) If $a_0 = \mu^2m^2A^2$, $a_2 = -\mu^2(1 + m^2)$, and $a_4 = \mu^2/A^2$, then the solution is

$$y = A\mathrm{ns}(\mu\xi, m)\,, \quad (63)$$

and

$$u = b_0 + b_2y^2 = c + 4\beta k^2\mu^2(1 + m^2) - 12\beta k^2\mu^2\mathrm{ns}^2(\mu\xi, m)\,. \quad (64)$$

(xvii) If $a_0 = -\mu^2m^2A^2$, $a_2 = \mu^2(2m^2 - 1)$, and $a_4 = \mu^2(1 - m^2)/A^2$, then the solution is

$$y = A\mathrm{nc}(\mu\xi, m)\,, \quad (65)$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2\mu^2(2m^2 - 1) - 12\beta k^2\mu^2(1 - m^2)\mathrm{nc}^2(\mu\xi, m)\,. \quad (66)$$

(xviii) If $a_0 = -\mu^2A^2$, $a_2 = \mu^2(2 - m^2)$, and $a_4 = \mu^2(m^2 - 1)/A^2$, then the solution is

$$y = A\mathrm{nd}(\mu\xi, m)\,, \quad (67)$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2\mu^2(2 - m^2) - 12\beta k^2\mu^2(m^2 - 1)\mathrm{nd}^2(\mu\xi, m)\,. \quad (68)$$

(xix) If $a_0 = \mu^2A^2$, $a_2 = \mu^2(2 - m^2)$, and $a_4 = \mu^2(1 - m^2)/A^2$, then the solution is

$$y = A\mathrm{sc}(\mu\xi, m)\,, \quad (69)$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2\mu^2(2 - m^2) - 12\beta k^2\mu^2(1 - m^2)\mathrm{sc}^2(\mu\xi, m)\,. \quad (70)$$

(xx) If $a_0 = \mu^2A^2$, $a_2 = \mu^2(2m^2 - 1)$, and $a_4 = \mu^2(m^2 - 1)m^2/A^2$, then the solution is

$$y = A\mathrm{sd}(\mu\xi, m)\,, \quad (71)$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2\mu^2(2m^2 - 1) - 12\beta k^2\mu^2(m^2 - 1)m^2\mathrm{sd}^2(\mu\xi, m)\,. \quad (72)$$

(xxi) If $a_0 = \mu^2(1 - m^2)A^2$, $a_2 = \mu^2(2 - m^2)$, and $a_4 = \mu^2/A^2$, then the solution is

$$y = A\mathrm{cs}(\mu\xi, m)\,, \quad (73)$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2\mu^2(2 - m^2) - 12\beta k^2\mu^2\mathrm{cs}^2(\mu\xi, m)\,. \quad (74)$$

(xxii) If $a_0 = \mu^2A^2$, $a_2 = -\mu^2(1 + m^2)$, and $a_4 = \mu^2m^2/A^2$, then the solution is

$$y = A\mathrm{cd}(\mu\xi, m)\,, \quad (75)$$

and

$$u = b_0 + b_2y^2 = c + 4\beta k^2\mu^2(1 + m^2) - 12\beta k^2\mu^2m^2\mathrm{cd}^2(\mu\xi, m)\,. \quad (76)$$

(xxiii) If $a_0 = \mu^2m^2(m^2 - 1)A^2$, $a_2 = \mu^2(2m^2 - 1)$, and $a_4 = \mu^2/A^2$, then the solution is

$$y = A\mathrm{ds}(\mu\xi, m)\,, \quad (77)$$

and

$$u = b_0 + b_2y^2 = c - 4\beta k^2\mu^2(2m^2 - 1) - 12\beta k^2\mu^2\mathrm{ds}^2(\mu\xi, m)\,. \quad (78)$$

(xxiv) If $a_0 = \mu^2m^2A^2$, $a_2 = -\mu^2(1 + m^2)$, and $a_4 = \mu^2/A^2$, then the solution is

$$y = A\mathrm{dc}(\mu\xi, m)\,, \quad (79)$$

and

$$u = b_0 + b_2y^2 = c + 4\beta k^2\mu^2(1 + m^2) - 12\beta k^2\mu^2\mathrm{dc}^2(\mu\xi, m)\,, \quad (80)$$

where A and μ are constants.

It is known that when $m \to 1$, $\mathrm{sn}(\xi, m) \to \tanh\xi$, $\mathrm{cn}(\xi, m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}(\xi, m) \to \mathrm{sech}\,\xi$ and when $m \to 0$, $\mathrm{sn}(\xi, m) \to \sin\xi$, $\mathrm{cn}(\xi, m) \to \cos\xi$. And among the Jacobi elliptic functions, Jacobi elliptic sine function, Jacobi elliptic cosine functions, and Jacobi elliptic function of the third kind are the three basic ones, and all other Jacobi elliptic functions can be expressed in terms of them. So we can also get more solutions expressed in terms of hyperbolic functions and trigonometric functions.

3 Klein–Gordon Equation

Nonlinear Klein–Gordon equation reads

$$u_{tt} - c_0^2 u_{xx} + \alpha u - \beta u^3 = 0\,. \tag{81}$$

Substituting Eq. (8) into Eq. (81) leads to

$$u'' + \alpha_1 u - \beta_1 u^3 = 0\,, \tag{82}$$

where

$$\alpha_1 = \frac{\alpha}{k^2(c^2 - c_0^2)}, \qquad \beta_1 = \frac{\beta}{k^2(c^2 - c_0^2)}\,. \tag{83}$$

Similarly, assuming that the solutions of Eq. (81) take the form of Eq. (10), we can get $n = 1$ for Eq. (81), i.e.,

$$u = b_0 + b_1 y, \qquad b_1 \neq 0\,, \tag{84}$$

where y satisfies the elliptic equation (6), then substituting Eq. (84) into Eq. (82) leads to

$$b_1 = \pm\sqrt{\frac{2a_4}{\beta_1}}\,, \qquad b_0 = \pm\frac{a_3}{2\beta_1}\sqrt{\frac{\beta_1}{2a_4}}\,, \tag{85}$$

and

$$a_2 = -\alpha_1 + \frac{3a_3^2}{8a_4}, \qquad a_1 = \frac{(a_3^2 - 8\alpha_1 a_4)a_3}{16a_4^2}\,, \tag{86}$$

If $a_3 = 0$, then $b_0 = a_1 = 0$ and

$$b_1 = \pm\sqrt{\frac{2a_4}{\beta_1}}\,, \qquad a_2 = -\alpha_1\,, \tag{87}$$

then the transformation takes the following form

$$y'^2 = a_0 + a_2 u^2 + a_4 u^4\,. \tag{88}$$

This is an elliptic equation, and it also has many kinds of solutions, some of which we will show next.

Case A Consider $a_0 = 0$, then we have two kinds of solutions.

(a) If $a_2 = -\alpha_1 > 0$ and $a_4 > 0$, $\beta_1 > 0$, the solution is

$$y = \pm\sqrt{-\frac{\alpha_1}{a_4}}\mathrm{csch}(\sqrt{-\alpha_1}\,\xi) = \pm\sqrt{-\frac{\alpha}{k^2(c^2 - c_0^2)a_4}}\,\mathrm{csch}\left[\sqrt{-\frac{\alpha}{k^2(c^2 - c_0^2)}}\,\xi\right], \tag{89}$$

and

$$u = b_1 y = \pm\sqrt{-\frac{2\alpha}{\beta}}\,\mathrm{csch}\left[\sqrt{-\frac{\alpha}{k^2(c^2 - c_0^2)}}\,\xi\right]. \tag{90}$$

(b) If $a_2 = -\alpha_1 > 0$ and $a_4 < 0$, $\beta_1 < 0$, the solution is

$$y = \pm\sqrt{\frac{\alpha_1}{a_4}}\mathrm{sech}(\sqrt{-\alpha_1}\,\xi) = \pm\sqrt{\frac{\alpha}{k^2(c^2 - c_0^2)a_4}}\mathrm{sech}\left[\sqrt{-\frac{\alpha}{k^2(c^2 - c_0^2)}}\,\xi\right], \tag{91}$$

and

$$u = b_1 y = \pm\sqrt{\frac{2\alpha}{\beta}}\,\mathrm{sech}\left[\sqrt{-\frac{\alpha}{k^2(c^2 - c_0^2)}}\,\xi\right]. \tag{92}$$

Case B The ansatz just takes the form of Eq. (88), and there exist many kinds of solutions expressed in terms of different Jacobi elliptic functions.[19] We show some generalized solutions just like what we have done in Sec. 2.

(i) If $a_0 = \mu^2 A^2$, $a_2 = -\alpha_1 = -\mu^2(1 + m^2)$ and $a_4 = \mu^2 m^2/A^2$, then the solution is

$$y = A\,\mathrm{sn}\left[\pm\sqrt{\frac{\alpha}{(1 + m^2)k^2(c^2 - c_0^2)}}\,\xi, m\right], \tag{93}$$

and

$$u = b_1 y = \pm\sqrt{\frac{2m^2\alpha}{(1 + m^2)\beta}}\,\mathrm{sn}\left[\pm\sqrt{\frac{\alpha}{(1 + m^2)k^2(c^2 - c_0^2)}}\,\xi, m\right]. \tag{94}$$

(ii) If $a_0 = \mu^2(1 - m^2)A^2$, $a_2 = -\alpha_1 = \mu^2(2m^2 - 1)$ and $a_4 = -\mu^2 m^2/A^2$, then the solution is

$$y = A\,\mathrm{cn}\left[\pm\sqrt{-\frac{\alpha}{(2m^2 - 1)k^2(c^2 - c_0^2)}}\,\xi, m\right], \tag{95}$$

and

$$u = b_1 y = \pm\sqrt{\frac{2m^2\alpha}{(2m^2 - 1)\beta}}\,\mathrm{cn}\left[\pm\sqrt{-\frac{\alpha}{(2m^2 - 1)k^2(c^2 - c_0^2)}}\,\xi, m\right]. \tag{96}$$

(iii) If $a_0 = \mu^2(1 - m^2)A^2$, $a_2 = -\alpha_1 = \mu^2(2 - m^2)$ and $a_4 = -\mu^2/A^2$, then the solution is

$$y = A\,\mathrm{dn}\left[\pm\sqrt{-\frac{\alpha}{(2 - m^2)k^2(c^2 - c_0^2)}}\,\xi, m\right], \tag{97}$$

and

$$u=b_1y=\pm\sqrt{\frac{2\alpha}{(2-m^2)\beta}}\,\mathrm{dn}\left[\pm\sqrt{-\frac{\alpha}{(2-m^2)k^2(c^2-c_0^2)}}\,\xi,m\right]. \tag{98}$$

(iv) If $a_0=\mu^2m^2A^2$, $a_2=-\alpha_1=-\mu^2(1+m^2)$ and $a_4=\mu^2/A^2$, then the solution is

$$y=A\,\mathrm{ns}\left[\pm\sqrt{\frac{\alpha}{(1+m^2)k^2(c^2-c_0^2)}}\,\xi,m\right], \tag{99}$$

and

$$u=b_1y=\pm\sqrt{\frac{2\alpha}{(1+m^2)\beta}}\,\mathrm{ns}\left[\pm\sqrt{\frac{\alpha}{(1+m^2)k^2(c^2-c_0^2)}}\,\xi,m\right]. \tag{100}$$

(v) If $a_0=-\mu^2m^2A^2$, $a_2=-\alpha_1=\mu^2(2m^2-1)$ and $a_4=\mu^2(1-m^2)/A^2$, then the solution is

$$y=A\,\mathrm{nc}\left[\pm\sqrt{-\frac{\alpha}{(2m^2-1)k^2(c^2-c_0^2)}}\,\xi,m\right], \tag{101}$$

and

$$u=b_1y=\pm\sqrt{-\frac{2(1-m^2)\alpha}{(2m^2-1)\beta}}\,\mathrm{nc}\left[\pm\sqrt{-\frac{\alpha}{(2m^2-1)k^2(c^2-c_0^2)}}\,\xi,m\right]. \tag{102}$$

(vi) If $a_0=-\mu^2A^2$, $a_2=-\alpha_1=\mu^2(2-m^2)$ and $a_4=\mu^2(m^2-1)/A^2$, then the solution is

$$y=A\,\mathrm{nd}\left[\pm\sqrt{-\frac{\alpha}{(2-m^2)k^2(c^2-c_0^2)}}\,\xi,m\right], \tag{103}$$

and

$$u=b_1y=\pm\sqrt{\frac{2(1-m^2)\alpha}{(2-m^2)\beta}}\,\mathrm{nd}\left[\pm\sqrt{-\frac{\alpha}{(2-m^2)k^2(c^2-c_0^2)}}\,\xi,m\right]. \tag{104}$$

(vii) If $a_0=\mu^2A^2$, $a_2=-\alpha_1=\mu^2(2-m^2)$ and $a_4=\mu^2(1-m^2)/A^2$, then the solution is

$$y=A\,\mathrm{sc}\left[\pm\sqrt{-\frac{\alpha}{(2-m^2)k^2(c^2-c_0^2)}}\,\xi,m\right], \tag{105}$$

and

$$u=b_1y=\pm\sqrt{-\frac{2(1-m^2)\alpha}{(2-m^2)\beta}}\,\mathrm{sc}\left[\pm\sqrt{-\frac{\alpha}{(2-m^2)k^2(c^2-c_0^2)}}\,\xi,m\right]. \tag{106}$$

(viii) If $a_0=\mu^2A^2$, $a_2=-\alpha_1=\mu^2(2m^2-1)$ and $a_4=\mu^2(m^2-1)m^2/A^2$, then the solution is

$$y=A\,\mathrm{sd}\left[\pm\sqrt{-\frac{\alpha}{(2m^2-1)k^2(c^2-c_0^2)}}\,\xi,m\right], \tag{107}$$

and

$$u=b_1y=\pm\sqrt{\frac{2m^2(1-m^2)\alpha}{(2m^2-1)\beta}}\,\mathrm{sd}\left[\pm\sqrt{-\frac{\alpha}{(2m^2-1)k^2(c^2-c_0^2)}}\,\xi,m\right]. \tag{108}$$

(ix) If $a_0=\mu^2(1-m^2)A^2$, $a_2=-\alpha_1=\mu^2(2-m^2)$ and $a_4=\mu^2/A^2$, then the solution is

$$y=A\,\mathrm{cs}\left[\pm\sqrt{-\frac{\alpha}{(2-m^2)k^2(c^2-c_0^2)}}\,\xi,m\right], \tag{109}$$

and

$$u=b_1y=\pm\sqrt{-\frac{2\alpha}{(2-m^2)\beta}}\,\mathrm{cs}\left[\pm\sqrt{-\frac{\alpha}{(2-m^2)k^2(c^2-c_0^2)}}\,\xi,m\right]. \tag{110}$$

(x) If $a_0=\mu^2A^2$, $a_2=-\alpha_1=-\mu^2(1+m^2)$ and $a_4=\mu^2m^2/A^2$, then the solution is

$$y=A\,\mathrm{cd}\left[\pm\sqrt{\frac{\alpha}{(1+m^2)k^2(c^2-c_0^2)}}\,\xi,m\right], \tag{111}$$

and

$$u=b_1y=\pm\sqrt{\frac{2m^2\alpha}{(1+m^2)\beta}}\,\mathrm{cd}\left[\pm\sqrt{\frac{\alpha}{(1+m^2)k^2(c^2-c_0^2)}}\,\xi,m\right]. \tag{112}$$

(xi) If $a_0 = \mu^2 m^2 (m^2 - 1) A^2$, $a_2 = -\alpha_1 = \mu^2 (2m^2 - 1)$ and $a_4 = \mu^2 / A^2$, then the solution is

$$y = A\,\mathrm{ds}\left[\pm\sqrt{-\frac{\alpha}{(2m^2-1)k^2(c^2-c_0^2)}}\,\xi, m\right], \tag{113}$$

and

$$u = b_1 y = \pm\sqrt{-\frac{2\alpha}{(2m^2-1)\beta}}\,\mathrm{ds}\left[\pm\sqrt{-\frac{\alpha}{(2m^2-1)k^2(c^2-c_0^2)}}\,\xi, m\right]. \tag{114}$$

(xii) If $a_0 = \mu^2 m^2 A^2$, $a_2 = -\alpha_1 = -\mu^2 (1 + m^2)$ and $a_4 = \mu^2 / A^2$, then the solution is

$$y = A\,\mathrm{dc}\left[\pm\sqrt{\frac{\alpha}{(1+m^2)k^2(c^2-c_0^2)}}\,\xi, m\right], \tag{115}$$

and

$$u = b_1 y = \pm\sqrt{\frac{2\alpha}{(1+m^2)\beta}}\,\mathrm{dc}\left[\pm\sqrt{\frac{\alpha}{(1+m^2)k^2(c^2-c_0^2)}}\,\xi, m\right]. \tag{116}$$

Of course, we can have more solutions if we do not take $a_3 = 0$, we do not discuss this here for sententiousness.

4 Conclusion

In this paper, we considered the elliptic equation as a new transformation to solve nonlinear wave equations. More kinds of solutions can be got from there, including rational solutions, solitary wave solutions constructed in terms of hyperbolic functions, periodic solutions expressed in terms of trigonometric functions and periodic solutions dealing with elliptic functions. If $a_4 = 1$ and $a_0 = a_2^2/4$ in Eq. (23) or Eq. (88), then

$$y' = \frac{a_2}{2} + y^2, \tag{117}$$

this just recovers transformation (4) given by Fan.[5] And if we take $a_0 = R^2$, $a_2 = 2\mu R^2$ and $a_4 = \mu^2 R^2$, then transformation (23) or (88) also recovers the transformation (5) given by Yan.[18] So it is obvious that transformations (4) and (5) are just special cases of Eq. (6). But, by applying transformations (4) and (5) to solve nonlinear wave equations, the periodic solutions expressed in terms of elliptic functions cannot be obtained.

By applying the transformation (6) to some nonlinear wave equations, the obtained solutions consist of those from the hyperbolic tangent expansion method,[5−7] the Jacobi elliptic function expansion method,[8,9] the nonlinear transformation method[10,11] and the trial function method,[12,13] so it can be taken as a unified method, and more applications to solving other nonlinear wave equations are also applicable.

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