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Solutions to Generalized mKdV Equation*

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Abstract *A transformation is introduced for generalized mKdV (GmKdV for short) equation and Jacobi elliptic function expansion method is applied to solve it. It is shown that GmKdV equation with a real number parameter can be solved directly by using Jacobi elliptic function expansion method when this transformation is introduced, and periodic solution and solitary wave solution are obtained. Then the generalized solution to GmKdV equation deduces to some special solutions to some well-known nonlinear equations, such as KdV equation, mKdV equation, when the real parameter is set specific values.*

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1 Introduction

A number of problems in branches of physics, mathematics, and other interdisciplinary sciences are described in terms of suitable nonlinear models, such as nonlinear Schrödinger equations in plasma physics,[1] KdV equation in shallow water model,[2] and so on. Recently, special attention has been devoted in literature to solving nonlinear evolution equations. Many methods have been proposed to construct exact solutions to nonlinear equations. Among them are the function transformation method,[3,4] the homogeneous balance method,[5,6] the hyperbolic function expansion method,[7,8] the Jacobi elliptic function expansion method,[9,10] the nonlinear transformation method,[11,12] the trial function method,[13,14] and others.[15−18]

Actually, these methods are all just suitable for solving some special kinds of nonlinear evolution equations. No method can work for all kinds of nonlinear evolution equations directly. For example, the solutions obtained from the trial function method[13,14] are just some special ones, which are fewer than those obtained from some expansion methods. But for expansion methods, such as the function transformation method,[3,4] the homogeneous balance method,[5,6] the hyperbolic function expansion method,[7,8] and the Jacobi elliptic function expansion method,[9,10] the expansion order must be a positive integer. However, for more nonlinear evolution equations, the expansion order (obtained from the partial balance between the highest degree nonlinear terms and the highest order derivative terms) is not a positive integer. It may be a negative integer, or it may be just a real number. When the expansion order is not a positive integer, the expansion method cannot be applied to solve the corresponding nonlinear equation directly. Then some kinds of transformations are needed.

In this paper, we will consider this case. A transformation for GmKdV equation is introduced and then Jacobi elliptic expansion method is applied to transformed equation to derive solutions to GmKdV equation indirectly.

The GmKdV equation considered here is introduced by Fedele,[19] which reads

$$\frac{\partial u}{\partial t} + \alpha u^\gamma \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0\,, \tag{1}$$

where u is a real function, and α, β, and γ are real numbers. We seek its travelling wave solution, i.e.

$$u = u(\xi)\,, \quad \xi = k(x - ct)\,, \tag{2}$$

where k and c are wave number and wave speed, respectively. Substituting Eq. (2) into Eq. (3) yields

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi} + \alpha u^\gamma \frac{\mathrm{d}u}{\mathrm{d}\xi} + \beta k^2 \frac{\mathrm{d}^3 u}{\mathrm{d}\xi^3} = 0\,. \tag{3}$$

Applying expansion method, if we take the expansion order of u as $O(u) = n$ and $O(\mathrm{d}u/\mathrm{d}\xi) = n + 1$, then partial balance between the highest degree nonlinear term and the highest order derivative term leads to $n = 2/\gamma$. Obviously, when γ is a real number, $2/\gamma$ must not be an integer, so expansion method cannot be applied to solve Eq. (3) directly. In order to solve Eq. (3), we introduce a transformation,

$$u = v^{2/\gamma}\,. \tag{4}$$

Considering the transformation (4), equation (3) can be rewritten as

$$-cv^2\frac{\mathrm{d}v}{\mathrm{d}\xi} + \alpha v^4 \frac{\mathrm{d}v}{\mathrm{d}\xi} + \beta k^2\Big[\Big(\frac{2}{\gamma} - 1\Big)\Big(\frac{2}{\gamma} - 2\Big)\Big(\frac{\mathrm{d}v}{\mathrm{d}\xi}\Big)^3 + 3\Big(\frac{2}{\gamma} - 1\Big) v \frac{\mathrm{d}v}{\mathrm{d}\xi}\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} + v^2 \frac{\mathrm{d}^3 v}{\mathrm{d}\xi^3}\Big] = 0\,. \tag{5}$$

In the next sections, we will apply Jacobi elliptic function expansion method to solve Eq. (5), and then obtain solutions to GmKdV equation (1).

2 Solutions to GmKdV Equation

2.1 *Jacobi Elliptic Cosine Function Expansion Solutions*

Firstly, we consider Jacobi elliptic cosine function expansion solutions to Eq. (5), i.e.

$$v = \sum_{i=0}^{n} a_i \mathrm{cn}^i \xi\,, \tag{6}$$

where $\mathrm{cn}\,\xi$ is Jacobi elliptic cosine function.[20−23]

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Substituting Eq. (6) into (5) and partial balance between the highest degree nonlinear term and the highest order derivative term lead to $n=1$, so the ansatz solution can be written as

$$v = a_0 + a_1 \mathrm{cn}\,\xi\,, \tag{7}$$

from which we can get

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = -a_1 \mathrm{sn}\,\xi \mathrm{dn}\,\xi\,, \tag{8}$$

and

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} = a_1 \mathrm{cn}\,\xi[(2m^2-1) - 2m^2\mathrm{cn}^2\xi]\,, \tag{9}$$

and

$$\frac{\mathrm{d}^3 v}{\mathrm{d}\xi^3} = a_1[(1-2m^2) + 6m^2\mathrm{cn}^2\xi]\,\mathrm{sn}\,\xi \mathrm{dn}\,\xi\,, \tag{10}$$

where $\mathrm{sn}\,\xi$ and $\mathrm{dn}\,\xi$ are Jacobi elliptic sine function and Jacobi elliptic function of the third kind.[20−23]

Substituting Eqs. (7) $\sim$ (10) into Eq. (5) leads to

$$\begin{aligned}
&\Big\{ca_0^2a_1 - \alpha a_0^4 a_1 + \beta k^2\Big[-\Big(\frac{2}{\gamma}-1\Big)\Big(\frac{2}{\gamma}-2\Big)(1-m^2)a_1^3 + (1-2m^2)a_0^2a_1\Big]\Big\}\\
&\quad + \Big\{2ca_0a_1^2 - 4\alpha a_0^3a_1^2 + \beta k^2\Big[-3\Big(\frac{2}{\gamma}-1\Big)(2m^2-1)a_0a_1^2 + 2(1-2m^2)a_0a_1^2\Big]\Big\}\mathrm{cn}\,\xi\\
&\quad + \Big\{ca_1^3 - 6\alpha a_0^2a_1^3 + \beta k^2\Big[-\Big(\frac{2}{\gamma}-1\Big)\Big(\frac{2}{\gamma}-2\Big)(2m^2-1)a_1^3\\
&\quad - 3\Big(\frac{2}{\gamma}-1\Big)(2m^2-1)a_1^3 + (1-2m^2)a_1^3 + 6m^2a_0^2a_1\Big]\Big\}\mathrm{cn}^2\xi\\
&\quad + \Big\{-4\alpha a_0a_1^4 + \beta k^2\Big[6\Big(\frac{2}{\gamma}-1\Big)m^2a_0a_1^2 + 12m^2a_0a_1^2\Big]\Big\}\mathrm{cn}^3\xi\\
&\quad + \Big\{-\alpha a_1^5 + \beta k^2\Big[\Big(\frac{2}{\gamma}-1\Big)\Big(\frac{2}{\gamma}-2\Big)m^2a_1^3 + 6\Big(\frac{2}{\gamma}-1\Big)m^2a_1^3 + 6m^2a_1^3\Big]\Big\}\mathrm{cn}^4\xi = 0\,.
\end{aligned} \tag{11}$$

Because of the arbitrariness of ξ, in order to let Eq. (11) have solution, there must be the following algebraic equations

$$ca_0^2a_1 - \alpha a_0^4a_1 + \beta k^2\Big[-\Big(\frac{2}{\gamma}-1\Big)\Big(\frac{2}{\gamma}-2\Big)(1-m^2)a_1^3 + (1-2m^2)a_0^2a_1\Big] = 0\,, \tag{12a}$$

$$2ca_0a_1^2 - 4\alpha a_0^3a_1^2 + \beta k^2\Big[-3\Big(\frac{2}{\gamma}-1\Big)(2m^2-1)a_0a_1^2 + 2(1-2m^2)a_0a_1^2\Big] = 0\,, \tag{12b}$$

$$\begin{aligned}
&ca_1^3 - 6\alpha a_0^2a_1^3 + \beta k^2\Big[-\Big(\frac{2}{\gamma}-1\Big)\Big(\frac{2}{\gamma}-2\Big)(2m^2-1)a_1^3\\
&\qquad - 3\Big(\frac{2}{\gamma}-1\Big)(2m^2-1)a_1^3 + (1-2m^2)a_1^3 + 6m^2a_0^2a_1\Big] = 0\,,
\end{aligned} \tag{12c}$$

$$-4\alpha a_0a_1^4 + \beta k^2\Big[6\Big(\frac{2}{\gamma}-1\Big)m^2a_0a_1^2 + 12m^2a_0a_1^2\Big] = 0\,, \tag{12d}$$

$$-\alpha a_1^5 + \beta k^2\Big[\Big(\frac{2}{\gamma}-1\Big)\Big(\frac{2}{\gamma}-2\Big)m^2a_1^3 + 6\Big(\frac{2}{\gamma}-1\Big)m^2a_1^3 + 6m^2a_1^3\Big] = 0\,, \tag{12e}$$

from which one has

$$a_0 = 0\,,\qquad \gamma = 1\,,\qquad c = 4(2m^2-1)\beta k^2\,,\qquad a_1 = \pm\sqrt{\frac{12m^2\beta k^2}{\alpha}}\,, \tag{13}$$

or

$$a_0 = 0\,,\qquad \gamma = 2\,,\qquad c = (2m^2-1)\beta k^2\,,\qquad a_1 = \pm\sqrt{\frac{6m^2\beta k^2}{\alpha}}\,, \tag{14}$$

or

$$a_0 = 0\,,\qquad m^2 = 1\,,\qquad c = \frac{4\beta k^2}{\gamma^2}\,,\qquad a_1 = \pm\sqrt{\frac{\beta k^2}{\alpha}\Big(\frac{2}{\gamma}+1\Big)\Big(\frac{2}{\gamma}+2\Big)}\,. \tag{15}$$

And then the corresponding solutions are

$$v_1 = \pm\sqrt{12m^2\beta k^2/\alpha}\;\mathrm{cn}\,\xi\,, \tag{16}$$

$$v_2 = \pm\sqrt{6m^2\beta k^2/\alpha}\;\mathrm{cn}\,\xi\,, \tag{17}$$

and

$$v_3 = \pm\sqrt{(\beta k^2/\alpha)(2/\gamma+1)(2/\gamma+2)}\;\mathrm{sech}\,\xi\,. \tag{18}$$

Considering the transformation (4), the final solutions are

$$u_1 = \frac{12m^2\beta k^2}{\alpha}\;\mathrm{cn}^2\xi\,, \tag{19}$$

$$u_2 = \pm\sqrt{\frac{6m^2\beta k^2}{\alpha}}\;\mathrm{cn}\,\xi\,, \tag{20}$$

and

$$u_3 = \Big[\pm\sqrt{\frac{\beta k^2}{\alpha}\Big(\frac{2}{\gamma}+1\Big)\Big(\frac{2}{\gamma}+2\Big)}\;\mathrm{sech}\,\xi\Big]^{2/\gamma}\,. \tag{21}$$

Here it should be noted that for some cases depending on the value of γ, the signs $\pm$ in Eq. (21) should be replaced by $+$.

Equations (19) and (20) are periodic solutions to Eq. (1) with $\gamma = 1$ and $\gamma = 2$, respectively. When $m = 1$, they degenerate to solitary wave solutions,

$$u_4 = \frac{12\beta k^2}{\alpha}\mathrm{sech}^2\xi\,, \tag{22}$$

and

$$u_5 = \pm\sqrt{\frac{6\beta k^2}{\alpha}}\,\mathrm{sech}\,\xi\,. \tag{23}$$

For Eq. (21), we know that γ is any real number, so it can be taken as a generalized solitary wave solution to Eq. (1). For example, when $\gamma = 3, 4, 6, 8, 16$, the corresponding solutions are

$$u_6 = \Big(\frac{40\beta k^2}{9\alpha}\Big)^{1/3}\mathrm{sech}^{2/3}\xi\,, \tag{24}$$

$$u_7 = \Big(\frac{15\beta k^2}{4\alpha}\Big)^{1/4}\mathrm{sech}^{1/2}\xi\,, \tag{25}$$

$$u_8 = \pm\Big(\frac{28\beta k^2}{9\alpha}\Big)^{1/6}\mathrm{sech}^{1/3}\xi\,, \tag{26}$$

$$u_9 = \Big(\frac{45\beta k^2}{16\alpha}\Big)^{1/8}\mathrm{sech}^{1/4}\xi\,, \tag{27}$$

and

$$u_{10} = \Big[\frac{153\beta k^2}{64\alpha}\Big]^{1/16}\mathrm{sech}^{1/8}\xi\,. \tag{28}$$

When $\gamma = 1/3, 1/4, 1/6, 1/8$, the corresponding solutions are

$$u_{11} = \Big(\frac{56\beta k^2}{\alpha}\Big)^{3}\mathrm{sech}^{6}\xi\,, \tag{29}$$

$$u_{12} = \Big(\frac{90\beta k^2}{\alpha}\Big)^{4}\mathrm{sech}^{8}\xi\,, \tag{30}$$

$$u_{13} = \Big(\frac{182\beta k^2}{\alpha}\Big)^{6}\mathrm{sech}^{12}\xi\,, \tag{31}$$

$$u_{14} = \Big(\frac{306\beta k^2}{\alpha}\Big)^{8}\mathrm{sech}^{16}\xi\,. \tag{32}$$

2.2 *Jacobi Elliptic Sine Function Expansion Solutions*

Here we consider Jacobi elliptic sine function expansion solutions to Eq. (5), i.e.

$$v = \sum_{i=0}^{n} a_i \mathrm{sn}^i\xi\,. \tag{33}$$

Similarly the ansatz solution can be written as

$$v = a_0 + a_1\mathrm{sn}\,\xi\,, \tag{34}$$

from which we can get

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = a_1\mathrm{cn}\,\xi\mathrm{dn}\,\xi\,, \tag{35}$$

$$\frac{\mathrm{d}^2v}{\mathrm{d}\xi^2} = -a_1\mathrm{sn}\,\xi[(1+m^2) - 2m^2\mathrm{sn}^2\xi]\,, \tag{36}$$

and

$$\frac{\mathrm{d}^3v}{\mathrm{d}\xi^3} = -a_1[(1+m^2) - 6m^2\mathrm{sn}^2\xi]\mathrm{cn}\,\xi\mathrm{dn}\,\xi\,. \tag{37}$$

Substituting Eqs. (34) $\sim$ (37) into Eq. (5) leads to the following algebraic equations

$$-ca_0^2a_1 + \alpha a_0^4a_1 + \beta k^2\Big[\Big(\frac{2}{\gamma}-1\Big)\Big(\frac{2}{\gamma}-2\Big)a_1^3 - (1+m^2)a_0^2a_1\Big] = 0\,, \tag{38a}$$

$$-2ca_0a_1^2 + 4\alpha a_0^3a_1^2 + \beta k^2\Big[-3\Big(\frac{2}{\gamma}-1\Big)(1+m^2)a_0a_1^2 - 2(1+m^2)a_0a_1^2\Big] = 0\,, \tag{38b}$$

$$-ca_1^3 + 6\alpha a_0^2a_1^3 + \beta k^2\Big[-\Big(\frac{2}{\gamma}-1\Big)\Big(\frac{2}{\gamma}-2\Big)(1+m^2)a_1^3 - 3\Big(\frac{2}{\gamma}-1\Big)(1+m^2)a_1^3 - (1+m^2)a_1^3 + 6m^2a_0^2a_1\Big] = 0\,, \tag{38c}$$

$$4\alpha a_0a_1^4 + \beta k^2\Big[6\Big(\frac{2}{\gamma}-1\Big)m^2a_0a_1^2 + 12m^2a_0a_1^2\Big] = 0\,, \tag{38d}$$

$$\alpha a_1^5 + \beta k^2\Big[\Big(\frac{2}{\gamma}-1\Big)\Big(\frac{2}{\gamma}-2\Big)m^2a_1^3 + 6\Big(\frac{2}{\gamma}-1\Big)m^2a_1^3 + 6m^2a_1^3\Big] = 0\,, \tag{38e}$$

from which one has

$$a_0 = 0\,,\qquad \gamma = 1\,,\qquad c = -4(1+m^2)\beta k^2\,,\qquad a_1 = \pm\sqrt{-\frac{12m^2\beta k^2}{\alpha}}\,, \tag{39}$$

or

$$a_0 = 0\,,\qquad \gamma = 2\,,\qquad c = -(1+m^2)\beta k^2\,,\qquad a_1 = \pm\sqrt{-\frac{6m^2\beta k^2}{\alpha}}\,. \tag{40}$$

So their corresponding solutions are

$$v_4 = \pm\sqrt{-\frac{12m^2\beta k^2}{\alpha}}\,\mathrm{sn}\,\xi \tag{41}$$

and

$$v_5 = \pm\sqrt{-\frac{6m^2\beta k^2}{\alpha}}\,\mathrm{sn}\,\xi\,, \tag{42}$$

and then

$$u_{15} = -\frac{12m^2\beta k^2}{\alpha}\,\mathrm{sn}^2\xi \tag{43}$$

and

$$u_{16} = \pm\sqrt{-\frac{6m^2\beta k^2}{\alpha}}\,\mathrm{sn}\,\xi\,. \tag{44}$$

Equations (43) and (44) are periodic solutions to Eq. (1) with $\gamma = 1$ and $\gamma = 2$, respectively. When $m = 1$, they degenerate to solitary wave solutions

$$u_{17} = -\frac{12\beta k^2}{\alpha}\tanh^2\xi \tag{45}$$

and

$$u_{18} = \pm\sqrt{-\frac{6\beta k^2}{\alpha}}\,\tanh\xi\,. \tag{46}$$

2.3 *Jacobi Elliptic Function of the Third Kind Expansion Solutions*

In this section, we consider Jacobi elliptic function of the third kind expansion solutions to Eq. (5), i.e.

$$v = \sum_{i=0}^{n} a_i \mathrm{dn}^i\xi\,. \tag{47}$$

Similarly the ansatz solution can be written as

$$v = a_0 + a_1 \mathrm{dn}\,\xi\,, \tag{48}$$

from which we can get

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = -m^2 a_1 \mathrm{sn}\,\xi \mathrm{cn}\,\xi\,, \tag{49}$$

$$\frac{\mathrm{d}^2v}{\mathrm{d}\xi^2} = -a_1 \mathrm{dn}\,\xi[(m^2-2) + 2\,\mathrm{dn}^2\xi]\,, \tag{50}$$

and

$$\frac{\mathrm{d}^3v}{\mathrm{d}\xi^3} = a_1 m^2[(m^2-2) + 6\,\mathrm{dn}^2\xi]\,\mathrm{sn}\,\xi \mathrm{cn}\,\xi\,. \tag{51}$$

Substituting Eq. (48) $\sim$ (51) into Eq. (5) yields

$$a_0 = 0\,, \quad \gamma = 1\,, \quad c = 4(2-m^2)\beta k^2\,, \quad a_1 = \pm\sqrt{12\beta k^2/\alpha}\,, \tag{52}$$

or

$$a_0 = 0\,, \quad \gamma = 2\,, \quad c = (2-m^2)\beta k^2\,, \quad a_1 = \pm\sqrt{6\beta k^2/\alpha}\,, \tag{53}$$

or

$$a_0 = 0\,, \quad m^2 = 1\,, \quad c = 4\beta k^2/\gamma^2\,, \quad a_1 = \pm\sqrt{\frac{\beta k^2}{\alpha}\Big(\frac{2}{\gamma}+1\Big)\Big(\frac{2}{\gamma}+2\Big)}\,. \tag{54}$$

So their corresponding solutions are

$$v_6 = \pm\sqrt{12\beta k^2/\alpha}\,\mathrm{dn}\,\xi \tag{55}$$

and

$$v_7 = \pm\sqrt{6\beta k^2/\alpha}\,\mathrm{dn}\,\xi\,, \tag{56}$$

and then

$$u_{19} = (12\beta k^2/\alpha)\,\mathrm{dn}^2\xi \tag{57}$$

and

$$u_{20} = \pm\sqrt{6\beta k^2/\alpha}\,\mathrm{dn}\,\xi\,. \tag{58}$$

Equations (57) and (58) are periodic solutions to Eq. (1) with $\gamma = 1$ and $\gamma = 2$, respectively. When $m = 1$, they degenerate to solitary wave solutions (22) and (23), respectively.

For the case of Eq. (54), the solution is the same one as Eq. (18) and the final solution is just the same one as Eq. (21), so the same discussion can be obtained as from Eqs. (24) $\sim$ (32).

3 Conclusion

In this paper, we introduce a new transformation and apply it to transform GmKdV equation into the one solvable in terms of Jacobi elliptic function expansion method, directly. Many solutions are obtained for this generalized mKdV equation, such as solitary wave solutions constructed in terms of hyperbolic functions and periodic solutions expressed in terms of periodic solutions dealing with elliptic functions. Some of them are not given in literature to our knowledge. Of course, the similar transformations for other nonlinear wave equations can also be constructed, which makes the Jacobi elliptic function expansion method applicable directly to more nonlinear wave equations.

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