Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

Chaos, Solitons and Fractals 20 (2004) 301–309

CHAOS
SOLITONS & FRACTALS

www.elsevier.com/locate/chaos

New kinds of solutions to Gardner equation

Zuntao Fu *, Shida Liu, Shikuo Liu

School of Physics (Building Physics), Peking University, Beijing 100871, China

Accepted 18 July 2003

Abstract

On the basis of analysis to the projective Riccati equations, an intermediate transformation in expansion method is constructed. And this transformation is applied to solve Gardner equation, there many new kinds of travelling wave solutions including solitary wave solution are obtained, in which some are found for the first time.

1. Introduction

The Gardner equation reads

$$u_t + \gamma u u_x + \alpha u^2 u_x + \beta u_{xxx} = 0 \tag{1}$$

where γ, α and β are real constants, and u is a real function. Eq. (1) is also called combined KdV–mKdV equation, and it is widely applied in various branches of physics, such as solid-state physics, plasma physics, fluid physics, quantum field theory and so on [1–3]. Many methods have been applied to solved Eq. (1), such as, Wadati's inverse scattering transform and Hirota methods [1,2], Coffey's series expansion method [4], Mohamad's direct method [5], Lou's mapping method [6] and Zhang's leading-order analysis method and direct method [7]. The application of these methods results in many kinds of exact solutions, including travelling wave solutions and various types solitary wave solutions. In this paper, we will apply a new method to solve Eq. (1), there more kinds of solutions are derived, among them some are found for the first time.

2. Formal solutions to Gardner equation

The travelling wave solutions of Eq. (1) take the following form

$$u(x,t) = u(\xi), \quad \xi = x - ct \tag{2}$$

where c is wave speed. After substituting Eq. (2) into Eq. (1), we have

$$-cu_\xi + \gamma u u_\xi + \alpha u^2 u_\xi + \beta u_{\xi\xi\xi} = 0 \tag{3}$$

i.e.

$$-c + \frac{\gamma}{2}u^2 + \frac{\alpha}{3}u^3 + \beta u_{\xi\xi} = c_0 \tag{4}$$

where c_0 is an integration constant.

* Corresponding author. Tel.: +86-106-276-7184; fax: +86-106-275-1615.
E-mail address: fuzt@pku.edu.cn (Z. Fu).

0960-0779/$ - see front matter
doi:10.1016/S0960-0779(03)00383-7

In order to solve Eq. (4), a crucial ansatz

$$u(\xi) = \sum_{i=1}^{n} f^{i-1}(\xi)[A_i f(\xi) + B_i g(\xi)] + A_0, \quad A_n^2 + B_n^2 \neq 0 \tag{5}$$

is introduced, and $f(\xi)$ and $g(\xi)$ are nonzero solutions to the projective Riccati equations [8–10]

$$f'(\xi) = pf(\xi)g(\xi) \tag{6a}$$

$$g'(\xi) = q + pg^2(\xi) - rf(\xi) \tag{6b}$$

where $p \neq 0$ is a real constant, q and r are two real constants. When $p = -1$ and $q = 1$, Eqs. (6) reduce to the coupled equations given in the references [8,9], and when $p = \pm 1$ and $q \geqslant 0$, Eqs. (6) reduce to the coupled equations given in the reference [10].

For Eqs. (6), there is a relation between f and g

$$g^2 = -\frac{1}{p}\left[q - 2rf + \frac{r^2+\delta}{q} f^2\right] \tag{7}$$

where $\delta = \pm 1$.

Applying above expansion method, if taking the expansion order of u as $O(u) = n$ and considering the relations (6) and (7), then we have $O(\frac{du}{d\xi}) = n + 1$, so partial balance between the highest degree nonlinear term and the highest order derivative term leads to $n = 1$. Obviously, the formal solution can be written as

$$u = A_0 + A_1 f(\xi) + B_1 g(\xi), \quad A_1^2 + B_1^2 \neq 0 \tag{8}$$

Considering the relation (7), from Eq. (8) one has

$$u^2 = \left[A_0^2 - \frac{q}{p}B_1^2\right] + 2A_0B_1g + \left[2A_0A_1 + \frac{2r}{p}B_1^2\right]f + 2A_1B_1fg + \left[A_1^2 - \frac{(r^2+\delta)}{pq}B_1^2\right]f^2 \tag{9}$$

$$\begin{aligned} u^3 = {} & \left[A_0^3 - \frac{3q}{p}A_0B_1^2\right] + \left[3A_0^2B_1 - \frac{q}{p}B_1^3\right]g + \left[3A_0^2A_1 - \frac{3q}{p}A_1B_1^2 + \frac{6r}{p}A_0B_1^2\right]f + \left[6A_0A_1B_1 + \frac{2r}{p}B_1^3\right]fg \\ & + \left[3A_0A_1^2 + \frac{6r}{p}A_1B_1^2 - \frac{3(r^2+\delta)}{pq}A_0B_1^2\right]f^2 + \left[3A_1^2B_1 - \frac{(r^2+\delta)}{pq}B_1^3\right]f^2g + \left[A_1^3 - \frac{3(r^2+\delta)}{pq}A_1B_1^2\right]f^3 \end{aligned} \tag{10}$$

and

$$\frac{d^2u}{d\xi^2} = -pqA_1f + prB_1fg + 3prA_1f^2 - \frac{2p(r^2+\delta)}{q}B_1f^2g - \frac{2p(r^2+\delta)}{q}A_1f^3 \tag{11}$$

Substituting Eqs. (8)–(11) into Eq. (4) yields

$$\begin{aligned} & \left[-cA_0 + \frac{\gamma}{2}\left(A_0^2 - \frac{q}{p}B_1^2\right) + \frac{\alpha}{3}\left(A_0^3 - \frac{3q}{p}A_0B_1^2\right) - c_0\right] + \left[-cA_1 + \gamma\left(2A_0A_1 + \frac{2r}{p}B_1^2\right)\right. \\ & \left. + \frac{\alpha}{3}\left(-\frac{3q}{p}A_1B_1^2 + \frac{6r}{p}A_0B_1^2 + 3A_0^2A_1\right) - \beta pqA_1\right]f + \left[-cB_1 + \gamma A_0B_1 + \frac{\alpha}{3}\left(-\frac{q}{p}B_1^3 + 3A_0^2B_1\right)\right]g \\ & + \left[\gamma A_1B_1 + \frac{\alpha}{3}\left(\frac{2r}{p}B_1^3 + 6A_0A_1B_1\right) + \beta prB_1\right]fg + \left[\frac{\gamma}{2}\left(A_1^2 - \frac{(r^2+\delta)}{pq}B_1^2\right)\right. \\ & \left. + \frac{\alpha}{3}\left(\frac{6r}{p}A_1B_1^2 - \frac{3(r^2+\delta)}{pq}A_0B_1^2 + 3A_0A_1^2\right) + 3\beta prA_1\right]f^2 + \left[\frac{\alpha}{3}\left(3A_1^2B_1 - \frac{(r^2+\delta)}{pq}B_1^3\right) - \frac{2\beta p(r^2+\delta)}{q}B_1\right]f^2g \\ & + \left[\frac{\alpha}{3}\left(A_1^3 - \frac{3(r^2+\delta)}{pq}A_1B_1^2\right) - \frac{2\beta p(r^2+\delta)}{q}A_1\right]f^3 = 0 \end{aligned} \tag{12}$$

The arbitrariness of the argument ξ results in the following algebraic equations:

$$-cA_0 + \frac{\gamma}{2}\left(A_0^2 - \frac{q}{p}B_1^2\right) + \frac{\alpha}{3}\left(A_0^3 - \frac{3q}{p}A_0B_1^2\right) - c_0 = 0 \tag{13a}$$

$$-cA_1+\gamma(A_0A_1+\frac{r}{p}B_1^2)+\frac{\alpha}{3}\left(-\frac{3q}{p}A_1B_1^2+\frac{6r}{p}A_0B_1^2+3A_0^2A_1\right)-\beta pqA_1=0 \tag{13b}$$

$$-cB_1+\gamma A_0B_1+\frac{\alpha}{3}\left(-\frac{q}{p}B_1^3+3A_0^2B_1\right)=0 \tag{13c}$$

$$\gamma A_1B_1+\frac{\alpha}{3}\left(\frac{2r}{p}B_1^3+6A_0A_1B_1\right)+\beta prB_1=0 \tag{13d}$$

$$\frac{\gamma}{2}\left(A_1^2-\frac{(r^2+\delta)}{pq}B_1^2\right)+\frac{\alpha}{3}\left(\frac{6r}{p}A_1B_1^2-\frac{3(r^2+\delta)}{pq}A_0B_1^2+3A_0A_1^2\right)+3\beta prA_1=0 \tag{13e}$$

$$\frac{\alpha}{3}\left(3A_1^2B_1-\frac{(r^2+\delta)}{pq}B_1^3\right)-\frac{2\beta p(r^2+\delta)}{q}B_1=0 \tag{13f}$$

$$\frac{\alpha}{3}\left(A_1^3-\frac{3(r^2+\delta)}{pq}A_1B_1^2\right)-\frac{2\beta p(r^2+\delta)}{q}A_1=0 \tag{13g}$$

from which the parameters can be determined, for example, for $\delta=-1$, there are the following solutions:

Case 1: If $A_1=0$, $A_0=-\frac{\gamma}{2\alpha}$, $r=0$, then

$$B_1=\pm\sqrt{-\frac{6\beta p^2}{\alpha}},\quad pq=\frac{c}{2\beta}+\frac{\gamma^2}{8\alpha\beta} \tag{14}$$

obviously, there is a constraint $\alpha\beta<0$.

Case 2: If $A_1=0$, $A_0=-\frac{\gamma}{2\alpha}$, $r\neq 0$, then

$$B_1=\pm\sqrt{-\frac{3\beta p^2}{2\alpha}},\quad pq=\frac{2c}{\beta}+\frac{\gamma^2}{2\alpha\beta},\quad r=\pm 1 \tag{15}$$

obviously, there is the constraint $\alpha\beta<0$, too.

Case 3: If $B_1=0$, $A_0=-\frac{\gamma}{2\alpha}$, then

$$A_1=\pm\sqrt{\frac{24\beta^2p^2}{4\alpha c+\gamma^2}},\quad pq=-\frac{c}{\beta}-\frac{\gamma^2}{4\alpha\beta},\quad r=0 \tag{16}$$

Case 4: If $B_1=0$, $A_0\neq-\frac{\gamma}{2\alpha}$, then

$$A_1=\pm\sqrt{\frac{6\beta^2p^2(r^2+2)}{\alpha(2c+\gamma)}},\quad A_0=-\frac{\gamma}{2\alpha}\pm\sqrt{\frac{3(2c+\gamma)r^2}{2\alpha(r^2+2)}},\quad pq=\frac{(2c+\gamma)(r^2-1)}{\beta(r^2+2)} \tag{17}$$

with constraints $r\neq 0$ and $r^2\neq 1$.

Case 5: If $A_0=-\frac{\gamma}{2\alpha}$, $A_1\neq 0$, $B_1\neq 0$, then

$$A_1=\pm\sqrt{\frac{3\beta^2p^2(r^2-1)}{4\alpha c+\gamma^2}},\quad B_1=\pm\sqrt{-\frac{3\beta p^2}{2\alpha}},\quad pq=\frac{2c}{\beta}+\frac{\gamma^2}{2\alpha\beta} \tag{18}$$

with constraint $r^2\neq 1$.

For $\delta=1$, there are the following solutions:

Case 1: If $A_1=0$, $A_0=-\frac{\gamma}{2\alpha}$, $r=0$, then

$$B_1=\pm\sqrt{-\frac{6\beta p^2}{\alpha}},\quad pq=\frac{c}{2\beta}+\frac{\gamma^2}{8\alpha\beta} \tag{19}$$

obviously, there is the constraint $\alpha\beta<0$, too.

Case 2: If $B_1=0$, $A_0=-\frac{\gamma}{2\alpha}$, then

$$A_1=\pm\sqrt{-\frac{24\beta^2p^2}{4\alpha c+\gamma^2}},\quad pq=-\frac{c}{\beta}-\frac{\gamma^2}{4\alpha\beta},\quad r=0 \tag{20}$$

Case 3: If $B_1 = 0$, $A_0 \neq -\frac{\gamma}{2\alpha}$, then

$$A_1 = \pm\sqrt{\frac{6\beta^2 p^2(r^2-2)}{\alpha(2c+\gamma)}}, \quad A_0 = -\frac{\gamma}{2\alpha} \pm \sqrt{\frac{3r^2(2c+\gamma)}{2\alpha(r^2-2)}}, \quad pq = \frac{(r^2+1)(2c+\gamma)}{\beta(r^2-2)} \tag{21}$$

with constraints $r \neq 0$ and $r^2 \neq 2$.

Case 4: If $A_0 = -\frac{\gamma}{2\alpha}$, $A_1 \neq 0$, $B_1 \neq 0$, then

$$A_1 = \pm\sqrt{\frac{3\beta^2 p^2(r^2+1)}{4\alpha c+\gamma^2}}, \quad B_1 = \pm\sqrt{-\frac{3\beta p^2}{2\alpha}}, \quad pq = \frac{2c}{\beta} + \frac{\gamma^2}{2\alpha\beta} \tag{22}$$

Eqs. (14)–(22) combining with Eq. (8) reach the formal solutions to Eq. (1).

3. Analysis to the projective Riccati equations

In order to obtain exact solutions to Eq. (1), we must derive the expression of $f(\xi)$ and $g(\xi)$.

Next, we will analyze the solutions to Eqs. (6). From Eq. (6a), one has

$$g = \frac{1}{p}\frac{f'}{f} \tag{23}$$

Substituting Eq. (23) into Eq. (6b) leads to

$$f''f - 2f'^2 - pqf^2 + prf^3 = 0 \tag{24}$$

In order to solve Eq. (24), we introduce the following transformation

$$f = \frac{1}{w} \tag{25}$$

then

$$\frac{f'}{f} = -\frac{w'}{w}, \quad g = -\frac{1}{p}\frac{w'}{w} \tag{26}$$

and

$$w'' + pqw - pr = 0 \tag{27}$$

For the solutions to Eq. (27), two basic cases must be considered. The first basic case is

Case A: $q \neq 0$

There are still two cases must be considered, the first one is

Case A1: $pq < 0$

Then we can assume $k^2 = -pq$, Eq. (27) can be rewritten as

$$w'' - k^2 w - pr = 0 \tag{28}$$

and the general solution to Eq. (28) is

$$w = a_0 + a_1 \sinh k\xi + a_2 \cosh k\xi \tag{29}$$

where $a_0 = r/q$, i.e.

$$w = \frac{r}{q} + a_1 \sinh(\sqrt{-pq}\xi) + a_2 \cosh(\sqrt{-pq}\xi) \tag{30}$$

Considering the relation in Eq. (26), here we select two special solutions from Eq. (30), the first one is

$$w = \frac{r}{q} + \frac{1}{q}\sinh(\sqrt{-pq}\xi) \tag{31}$$

then

$$f_1 = \frac{1}{w} = \frac{q}{r + \sinh(\sqrt{-pq}\xi)} \tag{32}$$

and

$$g_1 = -\frac{1}{p}\frac{w'}{w} = -\frac{1}{p}\frac{\sqrt{-pq}\cosh(\sqrt{-pq}\xi)}{r+\sinh(\sqrt{-pq}\xi)} \tag{33}$$

From Eqs. (32) and (33) one can derive the relation between $f(\xi)$ and $g(\xi)$ is (7) with $\delta = 1$.

The second one is

$$w = \frac{r}{q} + \frac{1}{q}\cosh(\sqrt{-pq}\xi) \tag{34}$$

then

$$f_2 = \frac{1}{w} = \frac{q}{r+\cosh(\sqrt{-pq}\xi)} \tag{35}$$

and

$$g_2 = -\frac{1}{p}\frac{w'}{w} = -\frac{1}{p}\frac{\sqrt{-pq}\sinh(\sqrt{-pq}\xi)}{r+\cosh(\sqrt{-pq}\xi)} \tag{36}$$

From Eqs. (35) and (36) one can derive the relation between $f(\xi)$ and $g(\xi)$ is (7) with $\delta = -1$.

Case A2: $pq > 0$

Then we can assume $k^2 = pq$, Eq. (27) can be rewritten as

$$w'' + k^2 w - pr = 0 \tag{37}$$

and the general solution to Eq. (37) is

$$w = a_0 + a_1 \sin k\xi + a_2 \cos k\xi \tag{38}$$

where $a_0 = r/q$, i.e.

$$w = \frac{r}{q} + a_1 \sin(\sqrt{pq}\xi) + a_2 \cos(\sqrt{pq}\xi) \tag{39}$$

Considering the relation in Eq. (26), here we also select two special solutions from Eq. (39), the first one is

$$w = \frac{r}{q} + \frac{1}{q}\sin(\sqrt{pq}\xi) \tag{40}$$

then

$$f_3 = \frac{1}{w} = \frac{q}{r+\sin(\sqrt{pq}\xi)} \tag{41}$$

and

$$g_3 = -\frac{1}{p}\frac{w'}{w} = -\frac{1}{p}\frac{\sqrt{pq}\cos(\sqrt{pq}\xi)}{r+\sin(\sqrt{pq}\xi)} \tag{42}$$

From Eqs. (41) and (42) one can derive the relation between $f(\xi)$ and $g(\xi)$ is (7) with $\delta = -1$.

The second one is

$$w = \frac{r}{q} + \frac{1}{q}\cos(\sqrt{pq}\xi) \tag{43}$$

then

$$f_4 = \frac{1}{w} = \frac{q}{r+\cos(\sqrt{pq}\xi)} \tag{44}$$

and

$$g_4 = -\frac{1}{p}\frac{w'}{w} = \frac{1}{p}\frac{\sqrt{pq}\sin(\sqrt{pq}\xi)}{r+\cos(\sqrt{pq}\xi)} \tag{45}$$

From Eqs. (44) and (45) one can derive the relation between $f(\xi)$ and $g(\xi)$ is (7) with $\delta = -1$.

The second basic case is

Case B: $q = 0$

Then Eq. (27) can be rewritten as

$$w'' - pr = 0 \tag{46}$$

its general solution is

$$w = \frac{pr}{2}\xi^2 + a_1\xi + a_0 \tag{47}$$

where a_1 and a_0 are two arbitrary real constants.

From Eq. (47), one has

$$f_5 = \frac{1}{w} = \frac{1}{\frac{pr}{2}\xi^2 + a_1\xi + a_0} \tag{48}$$

and

$$g_5 = -\frac{1}{p}\frac{w'}{w} = -\frac{1}{p}\frac{pr\xi + a_1}{\frac{pr}{2}\xi^2 + a_1\xi + a_0} \tag{49}$$

4. Exact solutions to Gardner equation

Combining the results (32), (33), (35), (36), (41), (42), (44) and (45) with (8) and results from (14)–(22), we can obtain many kinds of exact travelling wave solutions to Gardner equation (1), for example,

Type 1: For $\delta = -1$, if $\alpha\beta < 0$ and $\frac{c}{2\beta} + \frac{\gamma^2}{8\alpha\beta} < 0$, then the solution to Gardner equation (1) is

$$u_1 = A_0 + B_1 g = -\frac{\gamma}{2\alpha} \mp \sqrt{\frac{3(4\alpha c+\gamma^2)}{4\alpha^2}}\tanh\left(\sqrt{-\frac{c}{2\beta} - \frac{\gamma^2}{8\alpha\beta}}\xi\right) \tag{50}$$

Type 2: For $\delta = -1$, if $\alpha\beta < 0$ and $\frac{c}{2\beta} + \frac{\gamma^2}{8\alpha\beta} > 0$, then the solution to Gardner equation (1) is

$$u_2 = A_0 + B_1 g = -\frac{\gamma}{2\alpha} \mp \sqrt{-\frac{3(4\alpha c+\gamma^2)}{4\alpha^2}}\cot\left(\sqrt{\frac{c}{2\beta} + \frac{\gamma^2}{8\alpha\beta}}\xi\right) \tag{51}$$

and

$$u_3 = A_0 + B_1 g = -\frac{\gamma}{2\alpha} \pm \sqrt{-\frac{3(4\alpha c+\gamma^2)}{4\alpha^2}}\tan\left(\sqrt{\frac{c}{2\beta} + \frac{\gamma^2}{8\alpha\beta}}\xi\right) \tag{52}$$

Type 3: For $\delta = -1$, if $\alpha\beta < 0$ and $\frac{2c}{\beta} + \frac{\gamma^2}{2\alpha\beta} < 0$, then the solution to Gardner equation (1) is

$$u_4 = A_0 + B_1 g = -\frac{\gamma}{2\alpha} \mp \sqrt{\frac{3(4\alpha c+\gamma^2)}{4\alpha^2}}\frac{\sinh\left(\sqrt{-\frac{2c}{\beta} - \frac{\gamma^2}{2\alpha\beta}}\xi\right)}{\cosh\left(\sqrt{-\frac{2c}{\beta} - \frac{\gamma^2}{2\alpha\beta}}\xi\right) \pm 1} \tag{53}$$

Type 4: For $\delta = -1$, if $\alpha\beta < 0$ and $\frac{2c}{\beta} + \frac{\gamma^2}{2\alpha\beta} > 0$, then the solution to Gardner equation (1) is

$$u_5 = A_0 + B_1 g = -\frac{\gamma}{2\alpha} \mp \sqrt{-\frac{3(4\alpha c+\gamma^2)}{4\alpha^2}}\frac{\cos\left(\sqrt{\frac{2c}{\beta} + \frac{\gamma^2}{2\alpha\beta}}\xi\right)}{\sin\left(\sqrt{\frac{2c}{\beta} + \frac{\gamma^2}{2\alpha\beta}}\xi\right) \pm 1} \tag{54}$$

and

$$u_6 = A_0 + B_1 g = -\frac{\gamma}{2\alpha} \pm \sqrt{-\frac{3(4\alpha c + \gamma^2)}{4\alpha^2}} \frac{\sin\left(\sqrt{\frac{2c}{\beta} + \frac{\gamma^2}{2\alpha\beta}}\xi\right)}{\cos\left(\sqrt{\frac{2c}{\beta} + \frac{\gamma^2}{2\alpha\beta}}\xi\right) \pm 1} \tag{55}$$

Type 5: For $\delta = -1$, if $4\alpha c + \gamma^2 > 0$ and $\frac{c}{\beta} + \frac{\gamma^2}{4\alpha\beta} > 0$, then the solution to Gardner equation (1) is

$$u_7 = A_0 + A_1 f = -\frac{\gamma}{2\alpha} \mp \sqrt{\frac{3(4\alpha c + \gamma^2)}{2\alpha^2}} \operatorname{sech}\left(\sqrt{\frac{c}{\beta} + \frac{\gamma^2}{4\alpha\beta}}\xi\right) \tag{56}$$

Type 6: For $\delta = -1$, if $4\alpha c + \gamma^2 > 0$ and $\frac{c}{\beta} + \frac{\gamma^2}{4\alpha\beta} < 0$, then the solution to Gardner equation (1) is

$$u_8 = A_0 + A_1 f = -\frac{\gamma}{2\alpha} \pm \sqrt{\frac{3(4\alpha c + \gamma^2)}{2\alpha^2}} \csc\left(\sqrt{-\frac{c}{\beta} - \frac{\gamma^2}{4\alpha\beta}}\xi\right) \tag{57}$$

and

$$u_9 = A_0 + A_1 f = -\frac{\gamma}{2\alpha} \pm \sqrt{\frac{3(4\alpha c + \gamma^2)}{2\alpha^2}} \sec\left(\sqrt{-\frac{c}{\beta} - \frac{\gamma^2}{4\alpha\beta}}\xi\right) \tag{58}$$

Type 7: For $\delta = -1$, if $\alpha(2c + \gamma) > 0$ and $(r^2 - 1)(2c + \gamma)\beta < 0$, then the solution to Gardner equation (1) is

$$u_{10} = A_0 + A_1 f = -\frac{\gamma}{2\alpha} \pm \sqrt{\frac{3(2c+\gamma)r^2}{2\alpha(r^2+2)}} \pm \frac{(r^2-1)(2c+\gamma)}{p\beta(r^2+2)} \sqrt{\frac{6\beta^2 p^2 (r^2+2)}{\alpha(2c+\gamma)}} \frac{1}{\cosh\left(\sqrt{\frac{(1-r^2)(2c+\gamma)}{\beta(r^2+2)}}\xi\right) + r} \tag{59}$$

with constraints $r \neq 0$ and $r^2 \neq 1$.

Type 8: For $\delta = -1$, if $\alpha(2c + \gamma) > 0$ and $(r^2 - 1)(2c + \gamma)\beta > 0$, then the solution to Gardner equation (1) is

$$u_{11} = A_0 + A_1 f = -\frac{\gamma}{2\alpha} \pm \sqrt{\frac{3(2c+\gamma)r^2}{2\alpha(r^2+2)}} \pm \frac{(r^2-1)(2c+\gamma)}{p\beta(r^2+2)} \sqrt{\frac{6\beta^2 p^2 (r^2+2)}{\alpha(2c+\gamma)}} \frac{1}{\sin\left(\sqrt{\frac{(r^2-1)(2c+\gamma)}{\beta(r^2+2)}}\xi\right) + r} \tag{60}$$

and

$$u_{12} = A_0 + A_1 f = -\frac{\gamma}{2\alpha} \pm \sqrt{\frac{3(2c+\gamma)r^2}{2\alpha(r^2+2)}} \pm \frac{(r^2-1)(2c+\gamma)}{p\beta(r^2+2)} \sqrt{\frac{6\beta^2 p^2 (r^2+2)}{\alpha(2c+\gamma)}} \frac{1}{\cos\left(\sqrt{\frac{(r^2-1)(2c+\gamma)}{\beta(r^2+2)}}\xi\right) + r} \tag{61}$$

with constraints $r \neq 0$ and $r^2 \neq 1$.

Type 9: For $\delta = -1$, if $\alpha\beta < 0$, $4\alpha c + \gamma^2 > 0$ and $r^2 > 1$, then the solution to Gardner equation (1) is

$$\begin{aligned} u_{13} &= A_0 + A_1 f + B_1 g \\ &= -\frac{\gamma}{2\alpha} \pm \frac{4\alpha c + \gamma^2}{2p\alpha\beta} \sqrt{\frac{3\beta^2 p^2 (r^2-1)}{4\alpha c + \gamma^2}} \frac{1}{\cosh\left(\sqrt{-\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right) + r} \mp \sqrt{\frac{3(4\alpha c + \gamma^2)}{4\alpha^2}} \frac{\sinh\left(\sqrt{-\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right)}{\cosh\left(\sqrt{-\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right) + r} \end{aligned} \tag{62}$$

with a constraint $r^2 \neq 1$.

Type 10: For $\delta = -1$, if $\alpha\beta < 0$, $4\alpha c + \gamma^2 < 0$ and $r^2 < 1$, then the solution to Gardner equation (1) is

$$\begin{aligned} u_{14} &= A_0 + A_1 f + B_1 g \\ &= -\frac{\gamma}{2\alpha} \pm \frac{4\alpha c + \gamma^2}{2p\alpha\beta} \sqrt{\frac{3\beta^2 p^2 (r^2-1)}{4\alpha c + \gamma^2}} \frac{1}{\sin\left(\sqrt{\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right) + r} \mp \sqrt{-\frac{3(4\alpha c + \gamma^2)}{4\alpha^2}} \frac{\cos\left(\sqrt{\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right)}{\sin\left(\sqrt{\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right) + r} \end{aligned} \tag{63}$$

and

$$u_{15} = A_0 + A_1 f + B_1 g$$
$$= -\frac{\gamma}{2\alpha} \pm \frac{4\alpha c+\gamma^2}{2p\alpha\beta}\sqrt{\frac{3\beta^2p^2(r^2-1)}{4\alpha c+\gamma^2}}\frac{1}{\cos\left(\sqrt{\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right)+r} \pm \sqrt{-\frac{3(4\alpha c+\gamma^2)}{4\alpha^2}}\frac{\sin\left(\sqrt{\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right)}{\cos\left(\sqrt{\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right)+r} \quad (64)$$

with a constraint $r^2 \neq 1$.

Type 11: For $\delta = 1$, if $\alpha\beta < 0$ and $4\alpha c + \gamma^2 > 0$, then the solution to Gardner equation (1) is

$$u_{16} = A_0 + B_1 g = -\frac{\gamma}{2\alpha} \mp \sqrt{\frac{3(4\alpha c+\gamma^2)}{4\alpha^2}}\coth\left(\sqrt{-\frac{4\alpha c+\gamma^2}{8\alpha\beta}}\xi\right) \quad (65)$$

Type 12: For $\delta = 1$, if $4\alpha c + \gamma^2 < 0$ and $\alpha\beta < 0$, then the solution to Gardner equation (1) is

$$u_{17} = A_0 + A_1 f = -\frac{\gamma}{2\alpha} \mp \sqrt{-\frac{3(4\alpha c+\gamma^2)}{2\alpha^2}}\mathrm{csch}\left(\sqrt{\frac{4\alpha c+\gamma^2}{4\alpha\beta}}\xi\right) \quad (66)$$

Type 13: For $\delta = 1$, if $\alpha(2c+\gamma)(r^2-2) > 0$ and $(r^2-2)(2c+\gamma)\beta < 0$, then the solution to Gardner equation (1) is

$$u_{18} = A_0 + A_1 f = -\frac{\gamma}{2\alpha} \pm \sqrt{\frac{3(2c+\gamma)r^2}{2\alpha(r^2-2)}} \pm \frac{(r^2+1)(2c+\gamma)}{p\beta(r^2-2)}\sqrt{\frac{6\beta^2p^2(r^2-2)}{\alpha(2c+\gamma)}}\frac{1}{\sinh\left(\sqrt{\frac{(1+r^2)(2c+\gamma)}{\beta(2-r^2)}}\xi\right)+r} \quad (67)$$

with constraints $r \neq 0$ and $r^2 \neq 2$.

Type 14: For $\delta = 1$, if $\alpha\beta < 0$ and $4\alpha c + \gamma^2 > 0$, then the solution to Gardner equation (1) is

$$u_{19} = A_0 + A_1 f + B_1 g$$
$$= -\frac{\gamma}{2\alpha} \pm \frac{4\alpha c+\gamma^2}{2\alpha\beta p}\sqrt{\frac{3\beta^2p^2(r^2+1)}{4\alpha c+\gamma^2}}\frac{1}{\sinh\left(\sqrt{-\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right)+r} \mp \sqrt{\frac{3(4\alpha c+\gamma^2)}{4\alpha^2}}\frac{\cosh\left(\sqrt{-\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right)}{\sinh\left(\sqrt{-\frac{4\alpha c+\gamma^2}{2\alpha\beta}}\xi\right)+r} \quad (68)$$

Obviously, the solutions u_1, u_2, u_3, u_7, u_8, u_9, u_{16} and u_{17} are usual solitary wave solutions and periodic solutions expressed by sine–cosine functions, which can be found in the usual expansion methods, such as the function transformation method [11,12], the hyperbolic function expansion method [13,14], the Jacobi elliptic function expansion method [15,16] and the sine–cosine method [17], and some solutions are also given in references [1,2,4–7]. But the solutions u_4, u_5, u_6, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{18} and u_{19} can not be obtained in these expansion methods, these solutions are new type solitary wave solutions or new type periodic solutions expressed by sine–cosine functions, some of them have not been found before.

5. Conclusion

In this paper, we introduce a new transformation from the solutions to the projective Riccati equations and apply it to solve Gardner equation. Many solutions are obtained for this Gardner equation, such as solitary wave solutions constructed in terms of hyperbolic functions, periodic solutions expressed in terms of sine and cosine functions, some solutions are not given in literatures to our knowledge. Of course, this transformation can be also applied to other nonlinear wave equations.

Acknowledgements

Many thanks are due to supports from National Natural Science Foundation of China (no. 40045016) and National Natural Science Foundation of China (no. 40175016).

References

[1] Wadati M. J Phys Soc Jpn 1975;38:673.
[2] Wadati M. J Phys Soc Jpn 1975;38:681.
[3] Konno K, Ichikawa YH. J Phys Soc Jpn 1974;37:1631.
[4] Coffey MW. SIAM J Appl Math 1990;50:1580.
[5] Mohamad MN. Math Meth Appl Sci 1992;15:73.
[6] Lou SY, Chen LL. Math Meth Appl Sci 1994;17:339.
[7] Zhang JF. Int J Theor Phys 1998;37(5):1541.
[8] Conte R, Musette M. J Phys A: Math Gen 1992;25:2609.
[9] Zhang GX, Li ZB, Duan YS. Sci China: A 2000;30(12):1103.
[10] Yan ZY. Chaos, Solitons & Fractals 2003;16:759.
[11] Bai CL. Phys Lett A 2001;288:191.
[12] Fu ZT, Liu SK, Liu SD. Phys Lett A 2002;299:507.
[13] Fan EG. Phys Lett A 2000;277:212.
[14] Parkes EJ, Duffy BR. Phys Lett A 1997;229:217.
[15] Fu ZT, Liu SK, Liu SD, Zhao Q. Phys Lett A 2001;290:72.
[16] Liu SK, Fu ZT, Liu SD, Zhao Q. Phys Lett A 2001;289:69.
[17] Yan CT. Phys Lett A 1996;224:77.