Commun. Theor. Phys. (Beijing, China) **41** (2004) pp. 189–194
Vol. 41, No. 2, February 15, 2004

New Exact Solutions to NLS Equation and Coupled NLS Equations*

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(Received May 16, 2003)

Abstract *A transformation is introduced on the basis of the projective Riccati equations, and it is applied as an intermediate in expansion method to solve nonlinear Schrödinger (NLS) equation and coupled NLS equations. Many kinds of envelope travelling wave solutions including envelope solitary wave solution are obtained, in which some are found for the first time.*

PACS numbers: 03.65.Ge
Key words: projective Riccati equations, NLS equation, envelope travelling wave solution, envelope solitary wave solution

1 Introduction

A number of problems are described in terms of suitable nonlinear models, such as nonlinear Schrödinger equations (NLS) in plasma physics,[1] KdV equation in shallow water model,[2] and so on, in branches of physics, mathematics, and other interdisciplinary sciences. It is an interesting topic to seek exact solutions to these nonlinear models. Here we will consider two nonlinear models, i.e. NLS equation and coupled NLS equations. The NLS equation reads

$$\mathrm{i}u_t + \alpha u_{xx} + \beta|u|^2 u = 0\,, \tag{1}$$

and the coupled NLS equations read

$$\begin{aligned} &\mathrm{i}u_t + \alpha u_{xx} + (\beta_1|u|^2 + \beta_2|v|^2)u = 0\,, \\ &\mathrm{i}v_t + \alpha v_{xx} + (\beta_1|v|^2 + \beta_2|u|^2)v = 0\,. \end{aligned} \tag{2}$$

In Ref. [3], on the base of Lamé equation and Lamé functions, we obtain the perturbed solutions to the NLS equation and coupled NLS equations. In this paper, we will reconsider these two nonlinear evolution systems. A transformation is obtained from the well-known projective Riccati equations,[4−6] and then this transformation is taken as an intermediate to solve these two systems, where many kinds of envelope travelling wave solutions including envelope solitary wave solutions are derived, among which some are found for the first time.

2 Exact Solutions to NLS Equation

In order to solve Eq. (1), we take the following transformation

$$u = \phi(\xi)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \qquad \xi = x - c_g t\,, \tag{3}$$

where k is the wave number, ω is the angular frequency, c_g is the group velocity, and $\phi(\xi)$ is a real function.

Substituting Eq. (3) into Eq. (1) results in the following ordinary differential equation,

$$\alpha\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + \mathrm{i}(2\alpha k - c_g)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + (\omega - \alpha k^2)\phi + \beta\phi^3 = 0\,. \tag{4}$$

Separating the real part and imaginary part yields

$$c_g = 2\alpha k\,, \tag{5}$$

and if we suppose

$$\omega - \alpha k^2 = -\gamma\,, \tag{6}$$

then equation (4) is rewritten as

$$\alpha\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} - \gamma\phi + \beta\phi^3 = 0\,. \tag{7}$$

In order to solve Eq. (7), we introduce the following crucial ansatz,

$$\begin{aligned} &\phi(\xi) = \sum_{i=1}^{n} f^{i-1}(\xi)[A_i f(\xi) + B_i g(\xi)] + A_0\,, \\ &A_n^2 + B_n^2 \neq 0\,, \end{aligned} \tag{8}$$

where n can be determined by balancing the highest order derivative term with the high degree nonlinear term in Eq. (7). And f and g are solutions to the well-known projective Riccati equations,[4−6]

$$\begin{aligned} &f'(\xi) = pf(\xi)g(\xi)\,, \\ &g'(\xi) = q + pg^2(\xi) - rf(\xi)\,, \end{aligned} \tag{9}$$

where $p \neq 0$ is a real constant, and q and r are two real constants. When $p = -1$ and $q = 1$, equations (9) reduce to the coupled equations given in Refs. [4] and [5], and when $p = \pm 1$ and $q \geq 0$, equations (9) reduce to the coupled equations given in Ref. [6]. There is a relation between f and g,

$$g^2 = -\frac{1}{p}\Big[q - 2rf + \frac{r^2+\delta}{q}f^2\Big]\,, \tag{10}$$

*The project supported by National Natural Science Foundation of China under Grant Nos. 40045016 and 40175016
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where $\delta = \pm 1$.

Applying the above expansion method, if we take the expansion order of ϕ as $O(\phi) = n$ and considering the relations (9), then $O(\mathrm{d}\phi/\mathrm{d}\xi) = n+1$, so partial balance between the highest degree nonlinear term and the highest order derivative term leads to $n = 1$. Obviously, the formal solution can be written as

$$\phi = A_0 + A_1 f(\xi) + B_1 g(\xi)\,, \qquad A_1^2 + B_1^2 \neq 0\,. \tag{11}$$

Considering the relation (10), from Eq. (11) one can have

$$\begin{aligned}\phi^3 = {} & \Big(A_0^3 - \frac{3q}{p}A_0B_1^2\Big) + \Big(3A_0^2B_1 - \frac{q}{p}B_1^3\Big)g + \Big(3A_0^2A_1 - \frac{3q}{p}A_1B_1^2 + \frac{6r}{p}A_0B_1^2\Big)f + \Big(6A_0A_1B_1 + \frac{2r}{p}B_1^3\Big)fg \\ & + \Big[3A_0A_1^2 + \frac{6r}{p}A_1B_1^2 - \frac{3(r^2+\delta)}{pq}A_0B_1^2\Big]f^2 + \Big[3A_1^2B_1 - \frac{(r^2+\delta)}{pq}B_1^3\Big]f^2g + \Big[A_1^3 - \frac{3(r^2+\delta)}{pq}A_1B_1^2\Big]f^3\end{aligned} \tag{12}$$

and

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} = -pqA_1f + prB_1fg + 3prA_1f^2 - \frac{2p(r^2+\delta)}{q}B_1f^2g - \frac{2p(r^2+\delta)}{q}A_1f^3\,. \tag{13}$$

Substituting Eqs (11), (12), and (13) into Eq. (7) results in the following algebraic equations,

$$\begin{aligned}& -\gamma A_0 + \beta\Big(A_0^3 - \frac{3q}{p}A_0B_1^2\Big) = 0\,, \\ & -\gamma A_1 + \beta\Big(-\frac{3q}{p}A_1B_1^2 + \frac{6r}{p}A_0B_1^2 + 3A_0^2A_1\Big) - \alpha pqA_1 = 0\,, \\ & -\gamma B_1 + \beta\Big(-\frac{q}{p}B_1^3 + 3A_0^2B_1\Big) = 0\,, \\ & \beta\Big(\frac{2r}{p}B_1^3 + 6A_0A_1B_1\Big) + \alpha prB_1 = 0\,, \\ & \beta\Big[\frac{6r}{p}A_1B_1^2 - \frac{3(r^2+\delta)}{pq}A_0B_1^2 + 3A_0A_1^2\Big] + 3\alpha prA_1 = 0\,, \\ & \beta\Big[3A_1^2B_1 - \frac{(r^2+\delta)}{pq}B_1^3\Big] - \frac{2\alpha p(r^2+\delta)}{q}B_1 = 0\,, \\ & \beta\Big[A_1^3 - \frac{3(r^2+\delta)}{pq}A_1B_1^2\Big] - \frac{2\alpha p(r^2+\delta)}{q}A_1 = 0\,,\end{aligned} \tag{14}$$

for the arbitrariness of the argument ξ, from which the parameters can be determined. For example, for $\delta = -1$, there are the following solutions:

Case 1 If $A_1 = 0$, $A_0 = 0$, $r = 0$, then

$$B_1 = \pm\sqrt{-\frac{2\alpha p^2}{\beta}}\,, \qquad pq = \frac{\gamma}{2\alpha}\,. \tag{15}$$

Obviously, there is the constraint $\alpha\beta < 0$.

Case 2 If $A_1 = 0$, $A_0 = 0$, $r \neq 0$, then

$$B_1 = \pm\sqrt{-\frac{\alpha p^2}{2\beta}}\,, \qquad pq = \frac{2\gamma}{\alpha}\,, \qquad r = \pm 1\,. \tag{16}$$

Obviously, there is the constraint $\alpha\beta < 0$, too.

Case 3 If $B_1 = 0$, $A_0 = 0$, then

$$A_1 = \pm\sqrt{\frac{2\alpha^2p^2}{\beta\gamma}}\,, \qquad pq = -\frac{\gamma}{\alpha}\,, \qquad r = 0\,. \tag{17}$$

Case 4 If $B_1 = 0$, $A_0 \neq 0$, then

$$A_1 = \pm\sqrt{\frac{\alpha^2p^2(r^2+2)}{\beta\gamma}}\,, \qquad A_0 = \pm\sqrt{\frac{\gamma r^2}{\beta(r^2+2)}}\,, \qquad pq = \frac{2(r^2-1)\gamma}{\alpha(r^2+2)}\,. \tag{18}$$

There is the constraint $r \neq 0$ and $r^2 \neq 1$.

Case 5 If $A_0 = 0$, $A_1 \neq 0$, $B_1 \neq 0$, then

$$A_1 = \pm\sqrt{\frac{\alpha^2 p^2 (r^2 - 1)}{4\beta\gamma}}\,, \qquad B_1 = \pm\sqrt{-\frac{\alpha p^2}{2\beta}}\,, \qquad pq = \frac{2\gamma}{\alpha} \tag{19}$$

with the constraint $r^2 \neq 1$.

For $\delta = 1$, there are the following solutions:

Case 1 If $A_1 = 0$, $A_0 = 0$, $r = 0$, then

$$B_1 = \pm\sqrt{-\frac{2\alpha p^2}{\beta}}\,, \qquad pq = \frac{\gamma}{2\alpha}\,. \tag{20}$$

Obviously, there is the constraint that $\alpha\beta < 0$.

Case 2 If $B_1 = 0$, $A_0 = 0$, then

$$A_1 = \pm\sqrt{-\frac{2\alpha^2 p^2}{\beta\gamma}}\,, \quad pq = -\frac{\gamma}{\alpha}\,, \quad r = 0\,. \tag{21}$$

Case 3 If $B_1 = 0$, $A_0 \neq 0$, then

$$A_1 = \pm\sqrt{\frac{\alpha^2 p^2 (r^2 - 2)}{\beta\gamma}}\,, \qquad A_0 = \pm\sqrt{\frac{\gamma r^2}{\beta (r^2 - 2)}}\,, \qquad pq = \frac{2(r^2+1)\gamma}{\alpha (r^2 - 2)}\,. \tag{22}$$

There is the constraint $r \neq 0$ and $r^2 \neq 2$.

Case 4 If $A_0 = 0$, $A_1 \neq 0$, $B_1 \neq 0$, then

$$A_1 = \pm\sqrt{\frac{\alpha^2 p^2 (r^2 + 1)}{4\beta\gamma}}\,, \qquad B_1 = \pm\sqrt{-\frac{\alpha p^2}{2\beta}}\,, \qquad pq = \frac{2\gamma}{\alpha}\,. \tag{23}$$

For the projective Riccati equations (9), when $pq < 0$ and $\delta = 1$, its solution is

$$f_1 = \frac{q}{r + \sinh(\sqrt{-pq}\xi)}\,, \tag{24}$$

$$g_1 = -\frac{\sqrt{-pq}}{p}\,\frac{\cosh(\sqrt{-pq}\,\xi)}{r + \sinh(\sqrt{-pq}\,\xi)}\,, \tag{25}$$

and when $pq < 0$ and $\delta = -1$, its solution is

$$f_2 = \frac{q}{r + \cosh(\sqrt{-pq}\,\xi)}\,, \tag{26}$$

$$g_2 = -\frac{\sqrt{-pq}}{p}\,\frac{\sinh(\sqrt{-pq}\,\xi)}{r + \cosh(\sqrt{-pq}\,\xi)}\,. \tag{27}$$

When $pq > 0$ and $\delta = -1$, its solutions are

$$f_3 = \frac{q}{r + \sin(\sqrt{pq}\,\xi)}\,, \tag{28}$$

$$g_3 = -\frac{\sqrt{pq}}{p}\,\frac{\cos(\sqrt{pq}\,\xi)}{r + \sin(\sqrt{pq}\,\xi)}\,, \tag{29}$$

and

$$f_4 = \frac{q}{r + \cos(\sqrt{pq}\,\xi)}\,, \tag{30}$$

$$g_4 = \frac{\sqrt{pq}}{p}\,\frac{\sin(\sqrt{pq}\,\xi)}{r + \cos(\sqrt{pq}\,\xi)}\,. \tag{31}$$

Combining Eqs. (3), (11), and the results from Eq. (15) $\sim$ (31), we can derive various envelope travelling solutions including envelope solitary wave solutions to NLS equation (1), for example,

Type 1 For $\delta = -1$, if $\alpha\beta < 0$ and $\alpha\gamma < 0$, then the solution to NLS equation (1) is

$$u_1 = B_1 g_2\, \mathrm{e}^{\mathrm{i}(kx - \omega t)} = \mp\sqrt{\frac{\gamma}{\beta}}\, \tanh\biggl(\sqrt{-\frac{\gamma}{2\alpha}}\,\xi\biggr)\, \mathrm{e}^{\mathrm{i}(kx - \omega t)}\,. \tag{32}$$

Type 2 For $\delta = -1$, if $\alpha\beta < 0$ and $\alpha\gamma > 0$, then the solution to NLS equation (1) is

$$u_2 = B_1 g_3 \,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \mp\sqrt{-\frac{\gamma}{\beta}}\ \cot\left(\sqrt{\frac{\gamma}{2\alpha}}\,\xi\right)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \tag{33}$$

and

$$u_3 = B_1 g_4 \,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \pm\sqrt{-\frac{\gamma}{\beta}}\ \tan\left(\sqrt{\frac{\gamma}{2\alpha}}\,\xi\right)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{34}$$

Type 3 For $\delta = -1$, if $\alpha\beta < 0$ and $\alpha\gamma < 0$, then the solution to NLS equation (1) is

$$u_4 = B_1 g_2 \,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \mp\sqrt{\frac{\gamma}{\beta}}\frac{\sinh(\sqrt{-2\gamma/\alpha}\,\xi)}{\cosh(\sqrt{-2\gamma/\alpha}\,\xi)\pm 1}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{35}$$

Type 4 For $\delta = -1$, if $\alpha\beta < 0$ and $\alpha\gamma > 0$, then the solution to NLS equation (1) is

$$u_5 = B_1 g_3 \,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \mp\sqrt{-\frac{\gamma}{\beta}}\frac{\cos(\sqrt{2\gamma/\alpha}\,\xi)}{\sin(\sqrt{2\gamma/\alpha}\,\xi)\pm 1}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \tag{36}$$

and

$$u_6 = B_1 g_4 \,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \pm\sqrt{-\frac{\gamma}{\beta}}\frac{\sin(\sqrt{2\gamma/\alpha}\,\xi)}{\cos(\sqrt{2\gamma/\alpha}\,\xi)\pm 1}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{37}$$

Type 5 For $\delta = -1$, if $\beta\gamma > 0$ and $\alpha\gamma > 0$, then the solution to NLS equation (1) is

$$u_7 = A_1 f_2 \,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \mp\sqrt{\frac{2\gamma}{\beta}}\ \mathrm{sech}\left(\sqrt{\frac{\gamma}{\alpha}}\,\xi\right)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{38}$$

Type 6 For $\delta = -1$, if $\beta\gamma > 0$ and $\alpha\gamma < 0$, then the solution to NLS equation (1) is

$$u_8 = A_1 f_3 \,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \pm\sqrt{\frac{2\gamma}{\beta}}\ \csc\left(\sqrt{-\frac{\gamma}{\alpha}}\,\xi\right)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \tag{39}$$

and

$$u_9 = A_1 f_4 \,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \pm\sqrt{\frac{2\gamma}{\beta}}\ \sec\left(\sqrt{-\frac{\gamma}{\alpha}}\,\xi\right)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{40}$$

Type 7 For $\delta = -1$, if $\beta\gamma > 0$ and $(r^2 - 1)\alpha\gamma < 0$, then the solution to NLS equation (1) is

$$\begin{aligned} u_{10} &= (A_0 + A_1 f_2)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &= \left[\pm\sqrt{\frac{\gamma r^2}{\beta(r^2+2)}} \pm \frac{2(r^2-1)\gamma}{p\alpha(r^2+2)}\sqrt{\frac{\alpha^2 p^2(r^2+2)}{\beta\gamma}}\frac{1}{\cosh(\sqrt{2(1-r^2)\gamma/\alpha(r^2+2)}\,\xi)+r}\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)} \end{aligned} \tag{41}$$

with the constraint that $r \neq 0$ and $r^2 \neq 1$.

Type 8 For $\delta = -1$, if $\beta\gamma > 0$ and $(r^2 - 1)\alpha\gamma > 0$, then the solution to NLS equation (1) is

$$\begin{aligned} u_{11} &= (A_0 + A_1 f_3)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &= \left[\pm\sqrt{\frac{\gamma r^2}{\beta(r^2+2)}} \pm \frac{2(r^2-1)\gamma}{p\alpha(r^2+2)}\sqrt{\frac{\alpha^2 p^2(r^2+2)}{\beta\gamma}}\frac{1}{\sin(\sqrt{2(r^2-1)\gamma/\alpha(r^2+2)}\,\xi)+r}\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \end{aligned} \tag{42}$$

and

$$\begin{aligned} u_{12} &= (A_0 + A_1 f_4)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &= \left[\pm\sqrt{\frac{\gamma r^2}{\beta(r^2+2)}} \pm \frac{2(r^2-1)\gamma}{p\alpha(r^2+2)}\sqrt{\frac{\alpha^2 p^2(r^2+2)}{\beta\gamma}}\frac{1}{\cos(\sqrt{2(r^2-1)\gamma/\alpha(r^2+2)}\,\xi)+r}\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)} \end{aligned} \tag{43}$$

with the constraint that $r \neq 0$ and $r^2 \neq 1$.

Type 9 For $\delta = -1$, if $\alpha\beta < 0$, $\alpha\gamma < 0$, and $r^2 > 1$, then the solution to NLS equation (1) is

$$\begin{aligned} u_{13} &= (A_1 f_2 + B_1 g_2)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &= \left[\pm\frac{2\gamma}{\alpha p}\sqrt{\frac{\alpha^2 p^2(r^2-1)}{4\beta\gamma}}\frac{1}{\cosh(\sqrt{-2\gamma/\alpha}\,\xi)+r} \mp \sqrt{\frac{\gamma}{\beta}}\frac{\sinh(\sqrt{-2\gamma/\alpha}\,\xi)}{\cosh(\sqrt{-2\gamma/\alpha}\,\xi)+r}\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)} \end{aligned} \tag{44}$$

with the constraint $r^2 \neq 1$.

Type 10 For $\delta = -1$, if $\alpha\beta < 0$, $\alpha\gamma > 0$ and $r^2 < 1$, then the solution to NLS equation (1) is

$$u_{14} = (A_1 f_3 + B_1 g_3)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \left[\pm\frac{2\gamma}{\alpha p}\sqrt{\frac{\alpha^2 p^2 (r^2-1)}{4\beta\gamma}}\frac{1}{\sin(\sqrt{2\gamma/\alpha}\,\xi)+r} \mp \sqrt{-\frac{\gamma}{\beta}}\frac{\cos(\sqrt{2\gamma/\alpha}\,\xi)}{\sin(\sqrt{2\gamma/\alpha}\,\xi)+r}\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{45}$$

and

$$u_{15} = (A_1 f_4 + B_1 g_4)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \left[\pm\frac{2\gamma}{\alpha p}\sqrt{\frac{\alpha^2 p^2 (r^2-1)}{4\beta\gamma}}\frac{1}{\cos(\sqrt{2\gamma/\alpha}\,\xi)+r} \pm \sqrt{-\frac{\gamma}{\beta}}\frac{\sin(\sqrt{2\gamma/\alpha}\,\xi)}{\cos(\sqrt{2\gamma/\alpha}\,\xi)+r}\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{46}$$

with the constraint $r^2 \neq 1$.

Type 11 For $\delta = 1$, if $\alpha\beta < 0$ and $\alpha\gamma < 0$, then the solution to NLS equation (1) is

$$u_{16} = B_1 g_1\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \mp\sqrt{\frac{\gamma}{\beta}}\coth\left(\sqrt{-\frac{\gamma}{2\alpha}}\,\xi\right)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{47}$$

Type 12 For $\delta = 1$, if $\beta\gamma < 0$ and $\alpha\gamma > 0$, then the solution to NLS equation (1) is

$$u_{17} = A_1 f_1\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = \mp\sqrt{-\frac{2\gamma}{\beta}}\operatorname{csch}\left(\sqrt{\frac{\gamma}{\alpha}}\,\xi\right)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{48}$$

Type 13 For $\delta = 1$, if $\beta\gamma(r^2-2) > 0$ and $(r^2-2)\alpha\gamma < 0$, then the solution to NLS equation (1) is

$$\begin{aligned} u_{18} &= (A_0 + A_1 f_1)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &= \left[\pm\sqrt{\frac{\gamma r^2}{\beta(r^2-2)}} \pm \frac{2(r^2+1)\gamma}{p\alpha(r^2-2)}\sqrt{\frac{\alpha^2 p^2 (r^2-2)}{\beta\gamma}}\frac{1}{\sinh(\sqrt{2(1+r^2)\gamma/\alpha(2-r^2)}\,\xi)+r}\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)} \end{aligned} \tag{49}$$

with the constraint that $r \neq 0$ and $r^2 \neq 2$.

Type 14 For $\delta = 1$, if $\alpha\beta < 0$ and $\alpha\gamma < 0$, then the solution to NLS equation (1) is

$$\begin{aligned} u_{19} &= (A_1 f_1 + B_1 g_1)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &= \left[\pm\frac{2\gamma}{\alpha p}\sqrt{\frac{\alpha^2 p^2 (r^2+1)}{4\beta\gamma}}\frac{1}{\sinh(\sqrt{-2\gamma/\alpha}\xi)+r} \mp \sqrt{\frac{\gamma}{\beta}}\frac{\cosh(\sqrt{-2\gamma/\alpha}\,\xi)}{\sinh(\sqrt{-2\gamma/\alpha}\,\xi)+r}\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \end{aligned} \tag{50}$$

Obviously, the solutions u_1, u_2, u_3, u_7, u_8, u_9, u_{16}, and u_{17} are general envelope solitary wave solutions and periodic solutions expressed by sine-cosine functions, which can be found in the usual expansion methods, such as the function transformation method,[7,8] the hyperbolic function expansion method,[9,10] the Jacobi elliptic function expansion method,[11,12] and the sine-cosine method.[13] But the solutions u_4, u_5, u_6, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{18}, and u_{19} cannot be obtained in these expansion methods. These solutions are new type envelope solitary wave solutions or new type envelope periodic solutions expressed by sine-cosine functions, some of which have not been found before.

3 Exact Solutions to Coupled NLS Equation

In order to solve Eq. (2), we take the following transformation,

$$u = \phi(\xi)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \qquad v = \psi(\xi)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \qquad \xi = x - c_g t\,. \tag{51}$$

If equations (5) and (6) are considered, then equation (2) can be rewritten as

$$\begin{aligned} &\alpha\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} - \gamma\phi + (\beta_1\phi^3 + \beta_2\phi\psi^2) = 0\,, \\ &\alpha\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} - \gamma\psi + (\beta_1\psi^3 + \beta_2\phi^2\psi) = 0\,. \end{aligned} \tag{52}$$

If $\phi = \psi$ is taken, then

$$\alpha\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} - \gamma\phi + (\beta_1 + \beta_2)\phi^3 = 0 \tag{53}$$

can be obtained from Eqs. (52).

Comparing Eq. (53) with Eq. (7), one can see that the difference between these two equations is that β in Eq. (7) is replaced by $\beta_1 + \beta_2$ in Eq. (53), so the solutions to Eq. (2) can be easily obtained from the solutions to Eq. (1), here we omit these details.

4 Conclusion

In this paper, we introduce an intermediate transformation from solutions to the projective Riccati equations and apply it to solve the NLS equation and the coupled NLS equations. Many solutions are obtained for these nonlinear systems, such as envelope solitary wave solutions constructed in terms of hyperbolic functions and envelope periodic solutions expressed in terms of sine and cosine functions. Some of which are not given in literatures to our knowledge. Of course, this transformation can be also applied to other nonlinear wave equations.

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