Commun. Theor. Phys. (Beijing, China) **41** (2004) pp. 527–530
Vol. 41, No. 4, April 15, 2004

Notes on Solutions to Burgers-type Equations*

FU Zun-Tao,[1,2] LIU Shi-Da,[1,2] and LIU Shi-Kuo[1]

[1]School of Physics, Peking University, Beijing 100871, China

[2]State Key Laboratory for Turbulence and Complex System, Peking University, Beijing 100871, China

(Received July 1, 2003)

Abstract *A transformation is introduced and applied to solve Burgers-type equations, such as Burgers equation, Burgers–KdV equation and Burgers-KdV-Kuramoto equation. Many kinds of travelling wave solutions including solitary wave solution are obtained, and it is shown that this is a powerful method to solve nonlinear equations with odd-order and even-order derivatives simultaneously.*

PACS numbers: 03.65.Ge
Key words: Burgers-type equations, travelling wave solution, solitary wave solution

1 Introduction

A number of problems are described in terms of suitable nonlinear models in branches of physics, mathematics, and other interdisciplinary sciences, such as nonlinear Schrödinger equations in plasma physics,[1] KdV equation in shallow water model,[2] and so on. It is an interesting topic to seek exact solutions to these nonlinear models. Contrary to the integrable nonlinear equations, non-integrable nonlinear equations are more difficult to be solved. Especially, if odd-order derivatives and even-order derivatives exist simultaneously in a nonlinear equation, then it is much more difficult to derive solutions to them. So new methods are needed for this kind of equations, for many methods do not work, yet. In this paper, we will consider three equations of this kind. The first one is Burgers equation, which reads

$$u_t + uu_x - \nu u_{xx} = 0\,. \tag{1}$$

It was firstly proposed by Burgers[3] to study issues of turbulence.

The second one is Burgers-KdV equation of the following form,

$$u_t + uu_x - \nu u_{xx} + \beta u_{xxx} = 0\,, \tag{2}$$

which was applied by Liu[4,5] to model the inverse energy cascade and intermittent turbulence, where a dispersion effect is taken into account.

And the last one is Burgers-KdV–Kuramoto equation,[6]

$$u_t + uu_x - \nu u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0\,, \tag{3}$$

where instability is taken into account, for irregular behavior of turbulence results from this instability.[7,8] Because these three equations are all of the basic features of Burgers equation, we call them Burgers-type equations.

In the next section, we will introduce a transformation and then apply this transformation to simplify the above three Burgers-type nonlinear equations. Then we will derive many travelling wave solutions to these Burgers-type equations, among which are solitary wave solutions.

2 Transformation and Solutions to Burgers Equation

In order to solve the above Burgers-type equations, the travelling wave frame is chosen, i.e.

$$\xi = x - ct\,. \tag{4}$$

Substituting Eq. (4) into Burgers equation (1) yields

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi} + u\frac{\mathrm{d}u}{\mathrm{d}\xi} - \nu\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = 0\,, \tag{5}$$

which can be integrated once and rewritten as

$$-cu + \frac{u^2}{2} - \nu\frac{\mathrm{d}u}{\mathrm{d}\xi} = A\,, \tag{6}$$

where A is an integration constant.

In order to solve Eq. (6) easily, here we introduce a transformation of the following form,

$$u(\xi) = \mathrm{e}^{k\xi}v(\eta) + u_0(\xi)\,, \quad \eta = \mathrm{e}^{k_1\xi} + a_0\,, \tag{7}$$

where $v(\eta)$ and $u_0(\xi)$ are functions undetermined, k, k_1, and a_0 are constants to be determined later.

From formula (7), we have

$$u'(\xi) = k\,\mathrm{e}^{k\xi}v(\eta) + k_1\,\mathrm{e}^{(k+k_1)\xi}v'(\eta) + u_0'(\xi)\,. \tag{8}$$

Combining Eq. (8) with Eq. (6) results in

$$\begin{aligned}&\nu k_1\,\mathrm{e}^{(k+k_1)\xi}v'(\eta) - \frac{1}{2}\,\mathrm{e}^{2k\xi}v^2(\eta)\\&\quad + [\nu k + c - u_0(\xi)]\,\mathrm{e}^{k\xi}v(\eta)\\&\quad + \Big[\nu u_0'(\xi) + cu_0 - \frac{1}{2}u_0^2 + A\Big] = 0\,.\end{aligned} \tag{9}$$

*The project supported by National Natural Science Foundation of China under Grant Nos. 40045016 and 40175016
†Correspondence author, email: fuzt@pku.edu.cn

If $k \neq 0$, setting

$$\nu k + c - u_0(\xi) = 0\,, \tag{10}$$

we have

$$u_0(\xi) = \nu k + c\,, \tag{11}$$

so $u_0(\xi)$ is a constant.

Let

$$k + k_1 = 2k\,, \tag{12}$$

i.e.

$$k_1 = k\,, \tag{13}$$

and under condition

$$\nu u_0'(\xi) + c u_0 - \frac{1}{2}u_0^2 + A = 0\,, \tag{14}$$

i.e.

$$A = -c u_0 + \frac{1}{2}u_0^2 = -\frac{1}{2}c^2 + \frac{1}{2}\nu^2 k^2\,, \tag{15}$$

eqaution (9) reduces to

$$v'(\eta) = \frac{1}{2\nu k}v^2(\eta)\,. \tag{16}$$

One exact solution to Eq. (16) is

$$v = -\frac{2\nu k}{\eta}\,, \tag{17}$$

i.e.

$$v = -\frac{2\nu k}{\mathrm{e}^{k\xi} + a_0}\,, \tag{18}$$

where a_0 and k are two arbitrary constants.

So travelling solution to Burgers equation (1) is

$$u_1(\xi) = -\frac{2\nu k\,\mathrm{e}^{k\xi}}{\mathrm{e}^{k\xi} + a_0} + c + \nu k\,, \tag{19}$$

with three arbitrary constants a_0, c, and k.

Making use of the identity

$$\frac{\mathrm{e}^{2y}}{1 + \mathrm{e}^{2y}} = \frac{1}{2}(1 + \tanh y)\,, \tag{20}$$

and setting $a_0 = 1$ in Eq. (19), we obtain the solitary wave solution to Burgers equation (1),

$$u_2(\xi) = -\nu k \tanh\Big(\frac{k}{2}\xi\Big) + c\,. \tag{21}$$

If choosing $a_0 = -1$ in Eq. (19) and making use of

$$\frac{\mathrm{e}^{2y}}{-1 + \mathrm{e}^{2y}} = \frac{1}{2}[1 + \coth y]\,, \tag{22}$$

then we have singular travelling solution to Burgers equation (1),

$$u_3(\xi) = -\nu k \coth\Big(\frac{k}{2}\xi\Big) + c\,. \tag{23}$$

If $k = 0$, then equation (9) reduces to

$$\nu k_1\,\mathrm{e}^{k_1\xi}v'(\eta) - \frac{1}{2}v^2(\eta) + [c - u_0(\xi)]v(\eta) + \Big[\nu u_0'(\xi) + c u_0 - \frac{1}{2}u_0^2 + A\Big] = 0\,. \tag{24}$$

Setting

$$k_1 = 0\,, \tag{25}$$

then

$$\eta = a_0\,, \qquad v = b_0 = \text{constant}\,, \tag{26}$$

so equation (24) is rewritten as

$$\nu u_0'(\xi) - \frac{1}{2}u_0^2 + (c - b_0)u_0 + A - \frac{1}{2}b_0^2 + b_0 c = 0\,. \tag{27}$$

From Eq. (27), we can derive two solutions to Eq. (1). The first one is

$$u_4(\xi) = 2\nu k' \tan(k'\xi) + c\,, \tag{28}$$

and the second one is

$$u_5(\xi) = -2\nu k' \cot(k'\xi) + c\,, \tag{29}$$

with two arbitrary constants c and k'.

Obviously, the solutions u_2 and u_3 are solitary wave solutions expressed by hyperbolic functions, which can be found in the usual expansion methods, such as the function transformation method,[9,10] the hyperbolic function expansion method,[11,12] and the sine-cosine method,[13] but not the Jacobi elliptic function expansion method,[14,15] which is powerful in obtaining periodic solutions in terms of Jacobi elliptic functions. The solutions u_4 and u_5 can also be found by the extended hyperbolic function expansion method,[11] but u_1 cannot be obtained in these expansion methods, it is a general travelling wave solution.

3 Solutions to Burgers-KdV Equation

In the frame of (4), Burgers-KdV equation (2) reduces to

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi} + u\frac{\mathrm{d}u}{\mathrm{d}\xi} - \nu\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} + \beta\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} = 0\,. \tag{30}$$

Integrating Eq. (30) once yields

$$-cu + \frac{1}{2}u^2 - \nu\frac{\mathrm{d}u}{\mathrm{d}\xi} + \beta\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = A\,, \tag{31}$$

where A is an integration constant.

From transformation (7), we have

$$u''(\xi) = k^2\,\mathrm{e}^{k\xi}v(\eta) + k_1(2k + k_1)\,\mathrm{e}^{(k+k_1)\xi}v'(\eta) + k_1^2\,\mathrm{e}^{(k+2k_1)\xi}v''(\eta) + u_0''(\xi)\,. \tag{32}$$

Substituting Eqs. (7), (8), and (32) into Eq. (31) results in

$$\beta k_1^2\,\mathrm{e}^{(k+2k_1)\xi}v''(\eta) + [\beta k_1(2k + k_1) - \nu k_1]\,\mathrm{e}^{(k+k_1)\xi}v'(\eta) + \frac{1}{2}\,\mathrm{e}^{2k\xi}v^2(\eta) + [\beta k^2 - \nu k + u_0(\xi) - c]\,\mathrm{e}^{k\xi}v(\eta) + \Big[\beta u_0''(\xi) - \nu u_0'(\xi) + \frac{1}{2}u_0^2(\xi) - c u_0(\xi) - A\Big] = 0\,. \tag{33}$$

If $k \neq 0$, under the following conditions

$$k + 2k_1 = 2k\,,$$

$$\beta k_1(2k + k_1) - \nu k_1 = 0\,,$$

$$\beta k^2 - \nu k + u_0(\xi) - c = 0\,,$$

$$\beta u_0''(\xi) - \nu u_0'(\xi) + \frac{1}{2}u_0^2(\xi) - cu_0(\xi) - A = 0\,, \tag{34}$$

i.e.,

$$k = \frac{2\nu}{5\beta}\,, \qquad k_1 = \frac{k}{2}\,,$$

$$u_0 = c + \nu k - \beta k^2\,, \qquad A = \frac{1}{2}u_0^2 - cu_0\,, \tag{35}$$

equation (33) reduces to

$$v''(\eta) = -\frac{25\beta}{2\nu^2}v^2(\eta)\,. \tag{36}$$

One of the solutions to Eq. (36) is

$$v = -\frac{12\nu^2}{25\beta}\eta^{-2}\,, \tag{37}$$

so the solution to Burgers-KdV equation (2) is

$$u_1(\xi) = -\frac{12\nu^2}{25\beta}\frac{\mathrm{e}^{2\nu\xi/5\beta}}{(\,\mathrm{e}^{\nu\xi/5\beta} + a_0)^2} + c + \frac{6\nu^2}{25\beta} \tag{38}$$

with two arbitrary constants c and a_0.

Setting $a_0 = 1$ and making use of identity (20), the solitary wave solution to Burgers-KdV equation (2) is

$$u_2(\xi) = -\frac{3\nu^2}{25\beta}\Big(1 + \tanh\frac{\nu}{10\beta}\xi\Big)^2 + c + \frac{6\nu^2}{25\beta}\,. \tag{39}$$

Similarly, taking $a_0 = -1$ and making use of identity (22), the singular solution to Burgers-KdV equation (2) is

$$u_3(\xi) = -\frac{3\nu^2}{25\beta}\Big(1 + \coth\frac{\nu}{10\beta}\xi\Big)^2 + c + \frac{6\nu^2}{25\beta}\,. \tag{40}$$

4 Solutions to Burgers-KdV–Kuramoto Equation

Considering the Burgers-KdV–Kuramoto equation (3) in the travelling wave frame (4), we can rewrite it as

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi} + u\frac{\mathrm{d}u}{\mathrm{d}\xi} - \nu\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} + \beta\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} + \gamma\frac{\mathrm{d}^4u}{\mathrm{d}\xi^4} = 0\,. \tag{41}$$

Integrating (41) once yields

$$-cu + \frac{1}{2}u^2 - \nu\frac{\mathrm{d}u}{\mathrm{d}\xi} + \beta\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} + \gamma\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} = A\,, \tag{42}$$

where A is an integration constant.

Recalling the transformation (7), we have

$$\begin{aligned} u'''(\xi) &= k^3\,\mathrm{e}^{k\xi}v(\eta) + k_1(3k^2 + 3kk_1 + k_1^2)\,\mathrm{e}^{(k+k_1)\xi}v'(\eta) \\ &\quad + 3k_1^2(k + k_1)\,\mathrm{e}^{(k+2k_1)\xi}v''(\eta) + k_1^3\,\mathrm{e}^{(k+3k_1)\xi}v'''(\eta) + u_0'''(\xi)\,. \end{aligned} \tag{43}$$

Substituting Eqs. (7), (8), (32), and (43) into Eq. (42) leads to

$$\begin{aligned} &\gamma k_1^3\,\mathrm{e}^{(k+3k_1)\xi}v'''(\eta) + [3k_1^2(k + k_1)\gamma + \beta k_1^2]\,\mathrm{e}^{(k+2k_1)\xi}v''(\eta) + [\gamma k_1(3k^2 + 3kk_1 + k_1^2) \\ &\quad + \beta k_1(2k + k_1) - \nu k_1]\,\mathrm{e}^{(k+k_1)\xi}v'(\eta) + \frac{1}{2}\,\mathrm{e}^{2k\xi}v^2(\eta) + [\gamma k^3 + \beta k^2 - \nu k + u_0(\xi) - c]\,\mathrm{e}^{k\xi}v(\eta) \\ &\quad + \Big[\gamma u_0'''(\xi) + \beta u_0''(\xi) - \nu u_0'(\xi) + \frac{1}{2}u_0^2(\xi) - cu_0(\xi) - A\Big] = 0\,. \end{aligned} \tag{44}$$

So if $k \neq 0$, under conditions

$$k + 3k_1 = 2k\,, \qquad 3k_1^2(k + k_1)\gamma + \beta k_1^2 = 0\,, \qquad \gamma k_1(3k^2 + 3kk_1 + k_1^2) + \beta k_1(2k + k_1) - \nu k_1 = 0\,,$$

$$\gamma k^3 + \beta k^2 - \nu k + u_0(\xi) - c = 0\,, \qquad \gamma u_0'''(\xi) + \beta u_0''(\xi) - \nu u_0'(\xi) + \frac{1}{2}u_0^2(\xi) - cu_0(\xi) - A = 0\,, \tag{45}$$

i.e.,

$$k = -\frac{\beta}{4\gamma}\,, \quad k_1 = \frac{k}{3}\,, \quad \nu = -\frac{47\beta^2}{144\gamma}\,, \quad u_0 = c + \nu k - \beta k^2 - \gamma k^3\,, \quad A = \frac{1}{2}u_0^2 - cu_0\,, \tag{46}$$

equation (44) reduces to

$$v'''(\eta) = \frac{864\gamma^2}{\beta^3}v^2(\eta)\,. \tag{47}$$

One of the solutions to Eq. (47) is

$$v = -\frac{5\beta^3}{72\gamma^2}\eta^{-3}\,, \tag{48}$$

so the solution to Burgers-KdV–Kuramoto equation (3) is

$$u_1(\xi) = -\frac{5\beta^3}{72\gamma^2}\frac{\mathrm{e}^{-\beta\xi/4\gamma}}{(\,\mathrm{e}^{-\beta\xi/12\gamma} + a_0)^3} + c + \frac{5\beta^3}{144\gamma^2} \tag{49}$$

with two arbitrary constants c and a_0.

Setting $a_0 = 1$ and making use of identity (20), the solitary wave solution to Burgers-KdV–Kuramoto equation (3) is

$$u_2(\xi) = -\frac{5\beta^3}{576\gamma^2}\Big(1 - \tanh\frac{\beta}{8\gamma}\xi\Big)^3 + c + \frac{5\beta^3}{144\gamma^2}\,. \tag{50}$$

Similarly, taking $a_0 = -1$ and making use of identity (22), the singular solution to Burgers-KdV–Kuramoto equation (3) is

$$u_3(\xi) = -\frac{5\beta^3}{576\gamma^2}\Big(1 - \coth\frac{\beta}{8\gamma}\xi\Big)^3 + c + \frac{5\beta^3}{144\gamma^2}\,. \tag{51}$$

5 Conclusion

In this paper, we introduce an intermediate transformation to simplify Burgers-type nonlinear equations, and then derive the solutions to these simplified nonlinear equations. Based on the results from simplified nonlinear equations, solutions to the Burgers-type nonlinear equations, such as Burgers equation, Burgers-KdV equation and Burgers-KdV–Kuramoto equation, are obtained, there general travelling wave solutions and solitary wave solutions are given. Compared with other methods[12,16,17] used to solve this kind of nonlinear equations, which have odd-order and even-order derivatives simultaneously, method used in this paper is more simple and direct. Of course, this transformation can be also applied to other nonlinear wave equations.[18]

References

[1] P.K. Shukla, N.N. Rao, M.Y. Yu, *et al.*, Phys. Rep. **138** (1986) 1.

[2] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Tranforms*, Philadelphia, SIAM (1981).

[3] J.M. Burgers, Adv. Appl. Mech. **1** (1946) 171.

[4] S.D. Liu and S.K. Liu, Commun. Theor. Phys. (Beijing, China) **16** (1991) 497.

[5] S.D. Liu and S.K. Liu, Science in China **A35** (1992) 576.

[6] D.J. Benney, J. Math. Phys. **45** (1966) 52.

[7] T. Kawahara, Phys. Rev. Lett. **51** (1983) 381.

[8] S.D. Liu, S.K. Liu, Z.H. Huang, and Q. Zhao, Prog. Natur. Sci. **9** (1999) 912.

[9] C.L. Bai, Phys. Lett. **A288** (2001) 191.

[10] Z.T. Fu, S.K. Liu, and S.D. Liu, Phys. Lett. **A299** (2002) 507.

[11] E.G. Fan, Phys. Lett. **A277** (2000) 212.

[12] E.J. Parkes and B.R. Duffy, Phys. Lett. **A229** (1997) 217.

[13] C.T. Yan, Phys. Lett. **A224** (1996) 77.

[14] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Phys. Lett. **A289** (2001) 69.

[15] Z.T. Fu, S.K. Liu, S.D. Liu, and Q. Zhao, Phys. Lett. **A290** (2001) 72.

[16] M.L. Wang, Phys. Lett. **A213** (1996) 279.

[17] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Appl. Math. Mech. **22** (2001) 326.

[18] Z.S. Feng, Phys. Lett. **A312** (2003) 65.