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Structures of equatorial envelope Rossby wave under the influence of new type of diabatic heating

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Abstract

The cubic nonlinear Schrödinger (NLS for short) equation with new type of external heating source is derived for large amplitude equatorial envelope Rossby wave in a shear flow. And then various periodic structures for these equatorial envelope Rossby waves are obtained with the help of Jacobi elliptic functions and elliptic equation. It is shown that different types of phase-locked diabatic heating play different roles in structures of equatorial envelope Rossby wave.

1. Introduction

In the last decades, the theory of equatorial waves has attracted much more attention on equatorial atmospheric dynamics and nonlinear dynamics. It provides a dynamical frame to analyze the slowly evolving large-scale phenomena in low latitudes and underlining dynamics. These theories of equatorial waves have been used for various purposes, especially in explaining some fundamental features of tropical climate and global changes, such as Walker circulation [1], the low-frequency Madden–Julian oscillation [2] and ENSO [3]. Among the nonlinear theories for equatorial waves, many are related to nonlinear Rossby wave activity, for it can manifest some of the prime events of geophysical fluid flows, and this activity often leads to a large-scale localized coherent structures that have remarkable permanence and stability. When the zonal flow shear is taken to be nonuniform, one can derive Rossby solitary waves and envelope Rossby solitary waves. Benney [4], Yamagata [5] and Zhao [6] investigated envelope Rossby solitary waves in barotropic shear and uniform or nonuniform flows, independently. However, they all did not consider the effect of external sources, especially the influence of diabatic heating from oceans. In our last paper [7], we applied the method of multi-scale expansion to derive the NLS equation with an external heating source satisfied by the large-amplitude equatorial Rossby waves. It reads

$$\mathrm{i}\frac{\partial A}{\partial T}+\alpha\frac{\partial^2 A}{\partial X^2}+\delta|A|^2A=\eta Q_{11}(X,T) \tag{1}$$

with the following coordinates transformation defined by Jeffrey [8]

$$T=T_2,\quad X=\frac{1}{\varepsilon}(X_2-c_gT_2)=X_1-c_gT_1 \tag{2}$$

where $Q_{11}(X,T)$ is the slowly varying external heating source, η denotes its strength, ε is a small parameter.

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In Ref. [7], we just considered two cases of diabatic heating, the first one is

$$Q_{11}(X,T) = 0 \tag{3}$$

and then Eq. (1) reduces to the canonical NLS equation

$$\mathrm{i}\frac{\partial A}{\partial T} + \alpha\frac{\partial^2 A}{\partial X^2} + \delta|A|^2 A = 0 \tag{4}$$

The second case is that the external heating source is an external travelling wave source, i.e.

$$Q_{11}(X,T) = \mathrm{e}^{\mathrm{i}[kX-\omega T]} \tag{5}$$

then Eq. (1) reduces to

$$\mathrm{i}\frac{\partial A}{\partial T} + \alpha\frac{\partial^2 A}{\partial X^2} + \delta|A|^2 A = \eta \mathrm{e}^{\mathrm{i}[kX-\omega T]} \tag{6}$$

And there the basic structures of these two NLS equations without and with phase-locked source are obtained by using knowledge of Jacobi elliptic functions and elliptic equation. It is shown that phase-locked diabatic heating plays an important role in periodic structures of rational form.

In this paper, we will consider other types of external heating to discuss the influence of different external heating on the structures of the equatorial envelope Rossby wave.

2. Structures to NLS equation with a new type of external heating source

First of all, we suppose that Eq. (1) takes solution of the following form

$$A(X,T) = \phi(\xi)\mathrm{e}^{\mathrm{i}(kX-\omega T)}, \quad \xi = s(X - C_g T) \tag{7}$$

and the external heating is chosen as

$$Q_{11}(X,T) = \psi(\xi)\mathrm{e}^{\mathrm{i}[kX-\omega T]} \tag{8}$$

Then Eq. (1) is rewritten as

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} = \frac{\gamma}{\alpha s^2}\phi - \frac{\delta}{\alpha s^2}\phi^3 + \frac{\eta}{\alpha s^2}\psi \tag{9}$$

with

$$C_g = 2\alpha k, \quad -\gamma = \omega - \alpha k^2 \tag{10}$$

The two cases of external heating considered in Ref. [7] are

$$\psi(\xi) = 0 \tag{11}$$

and

$$\psi(\xi) = 1 \tag{12}$$

There are still more types of $\psi(\xi)$, here we consider the quadratic external heating, i.e.

$$\psi(\xi) = \phi^2 \tag{13}$$

which can be taken as a resonant forcing, then Eq. (9) reduces to

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} = \frac{\gamma}{\alpha s^2}\phi + \frac{\eta}{\alpha s^2}\phi^2 - \frac{\delta}{\alpha s^2}\phi^3 \tag{14}$$

Eq. (14) can be solved in terms of Jacobi elliptic functions [9–11], here ansatz solution to Eq. (14) is taken as

$$\phi(\xi) = \sum_{j=0}^{n} b_j z^j \tag{15}$$

with

$$z'^2 = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4, \quad \text{or} \quad z'' = \frac{a_1}{2} + a_2 z + \frac{3a_3}{2} z^2 + 2a_4 z^3 \tag{16}$$

Partial balance between the highest degree nonlinear term and the highest order derivative term yields $n = 1$, i.e.

$$\phi(\xi) = b_0 + b_1 z, \quad b_1 \neq 0 \tag{17}$$

Substituting Eq. (17) into Eq. (14) results in

$$\left[\frac{\alpha s^2 a_1 b_1}{2} - \gamma b_0 - \eta b_0^2 + \delta b_0^3\right] + [\alpha s^2 a_2 b_1 - \gamma b_1 - 2\eta b_0 b_1 + 3\delta b_0^2 b_1] z + \left[\frac{3\alpha s^2 a_3 b_1}{2} - \eta b_1^2 + 3\delta b_0 b_1^2\right] z^2 + [2\alpha s^2 a_4 b_1 + \delta b_1^3] z^3 = 0 \tag{18}$$

then we have

$$\frac{\alpha s^2 a_1 b_1}{2} - \gamma b_0 - \eta b_0^2 + \delta b_0^3 = 0 \tag{19}$$

$$\alpha s^2 a_2 b_1 - \gamma b_1 - 2\eta b_0 b_1 + 3\delta b_0^2 b_1 = 0 \tag{20}$$

$$\frac{3\alpha s^2 a_3 b_1}{2} - \eta b_1^2 + 3\delta b_0 b_1^2 = 0 \tag{21}$$

$$2\alpha s^2 a_4 b_1 + \delta b_1^3 = 0 \tag{22}$$

From above four equations, we have

$$\begin{aligned} &b_1 = \pm\sqrt{-\frac{2\alpha s^2 a_4}{\delta}}, \quad b_0 = \frac{\eta}{3\delta} - \frac{\alpha s^2 a_3}{2\delta b_1} \\ &a_2 = \frac{\gamma}{\alpha s^2} + \frac{2\eta}{\alpha s^2} b_0 - \frac{3\delta}{\alpha s^2} b_0^2, \quad a_1 = \frac{2}{\alpha s^2}\frac{b_0}{b_1}(\gamma + \eta b_0 - \delta b_0^2) \end{aligned} \tag{23}$$

When $\eta \neq 0$, there are two cases to be considered, the first one is $a_3 = 0$ and $a_1 = 0$, then we have

$$b_1 = \pm\sqrt{-\frac{2\alpha s^2 a_4}{\delta}}, \quad b_0 = \frac{\eta}{3\delta}, \quad a_2 = \frac{\eta^2}{\delta \alpha s^2} \tag{24}$$

with constraint

$$\gamma = \frac{2\eta^2}{3\delta} \tag{25}$$

and Eq. (16) is rewritten as

$$z'^2 = a_0 + a_2 z^2 + a_4 z^4, \quad \text{or} \quad z'' = a_2 z + 2a_4 z^3 \tag{26}$$

Eq. (26) has many more kinds of solutions, we will show some next expressed in terms of different Jacobi elliptic functions [12].

(1) If $a_0 = 1$, $a_2 = \frac{\eta^2}{\delta\alpha s^2} = -(1 + m^2)$ and $a_4 = m^2$, then the solution is

$$\phi_1 = \frac{\eta}{3\delta} \pm \sqrt{-\frac{2\alpha s^2 m^2}{\delta}} \operatorname{sn}(\xi, m) \tag{27}$$

where $0 \leqslant m \leqslant 1$, is called modulus of Jacobi elliptic functions, see [12–16], and $\operatorname{sn}(\xi, m)$ is Jacobi elliptic sine function, see [12–16].

(2) If $a_0 = 1 - m^2$, $a_2 = \frac{\eta^2}{\delta\alpha s^2} = 2m^2 - 1$ and $a_4 = -m^2$, then the solution is

$$\phi_2 = \frac{\eta}{3\delta} \pm \sqrt{\frac{2\alpha s^2 m^2}{\delta}} \operatorname{cn}(\xi, m), \quad m^2 > \frac{1}{2} \tag{28}$$

where $\operatorname{cn}(\xi, m)$ is Jacobi elliptic cosine function, see [12–16].

(3) If $a_0 = 1 - m^2$, $a_2 = \frac{\eta^2}{\delta\alpha s^2} = 2 - m^2$ and $a_4 = -1$, then the solution is

$$\phi_3 = \frac{\eta}{3\delta} \pm \sqrt{\frac{2\alpha s^2}{\delta}}\mathrm{dn}(\xi, m) \tag{29}$$

where $\mathrm{dn}(\xi, m)$ is Jacobi elliptic function of the third kind, see [12–16].

(4) If $a_0 = m^2$, $a_2 = \frac{\eta^2}{\delta\alpha s^2} = -(1 + m^2)$ and $a_4 = 1$, then the solution is

$$\phi_4 = \frac{\eta}{3\delta} \pm \sqrt{-\frac{2\alpha s^2}{\delta}}\mathrm{ns}(\xi, m), \quad \mathrm{ns}(\xi, m) \equiv \frac{1}{\mathrm{sn}(\xi, m)} \tag{30}$$

(5) If $a_0 = -m^2$, $a_2 = \frac{\eta^2}{\delta\alpha s^2} = 2m^2 - 1$ and $a_4 = 1 - m^2$, then the solution is

$$\phi_5 = \frac{\eta}{3\delta} \pm \sqrt{-\frac{2\alpha s^2(1 - m^2)}{\delta}}\mathrm{nc}(\xi, m), \quad \mathrm{nc}(\xi, m) \equiv \frac{1}{\mathrm{cn}(\xi, m)}, \quad m^2 < \frac{1}{2} \tag{31}$$

(6) If $a_0 = -1$, $a_2 = \frac{\eta^2}{\delta\alpha s^2} = 2 - m^2$ and $a_4 = m^2 - 1$, then the solution is

$$\phi_6 = \frac{\eta}{3\delta} \pm \sqrt{-\frac{2\alpha s^2(m^2 - 1)}{\delta}}\mathrm{nd}(\xi, m), \quad \mathrm{nd}(\xi, m) \equiv \frac{1}{\mathrm{dn}(\xi, m)} \tag{32}$$

(7) If $a_0 = 1$, $a_2 = \frac{\eta^2}{\delta\alpha s^2} = 2m^2 - 1$ and $a_4 = (m^2 - 1)m^2$, then the solution is

$$\phi_7 = \frac{\eta}{3\delta} \pm \sqrt{-\frac{2\alpha s^2(m^2 - 1)m^2}{\delta}}\mathrm{sd}(\xi, m), \quad \mathrm{sd}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}, \quad m^2 > \frac{1}{2} \tag{33}$$

(8) If $a_0 = 1$, $a_2 = \frac{\eta^2}{\delta\alpha s^2} = -(1 + m^2)$ and $a_4 = m^2$, then the solution is

$$\phi_8 = \frac{\eta}{3\delta} \pm \sqrt{-\frac{2\alpha s^2 m^2}{\delta}}\mathrm{cd}(\xi, m), \quad \mathrm{cd}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)} \tag{34}$$

(9) If $a_0 = m^2(m^2 - 1)$, $a_2 = \frac{\eta^2}{\delta\alpha s^2} = 2m^2 - 1$ and $a_4 = 1$, then the solution is

$$\phi_9 = \frac{\eta}{3\delta} \pm \sqrt{-\frac{2\alpha s^2}{\delta}}\mathrm{ds}(\xi, m), \quad \mathrm{ds}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}, \quad m^2 < \frac{1}{2} \tag{35}$$

(10) If $a_0 = m^2$, $a_2 = \frac{\eta^2}{\delta\alpha s^2} = -(1 + m^2)$ and $a_4 = 1$, then the solution is

$$\phi_{10} = \frac{\eta}{3\delta} \pm \sqrt{-\frac{2\alpha s^2}{\delta}}\mathrm{dc}(\xi, m), \quad \mathrm{dc}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)} \tag{36}$$

There still exist many other kinds of solutions in terms of Jacobi elliptic functions [9–11], we do not show here. It is known that when $m \to 1$, $\mathrm{sn}(\xi, m) \to \tanh\xi$, $\mathrm{cn}(\xi, m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}(\xi, m) \to \mathrm{sech}\,\xi$ and when $m \to 0$, $\mathrm{sn}(\xi, m) \to \sin\xi$, $\mathrm{cn}(\xi, m) \to \cos\xi$, so we also can derive solutions expressed in terms of hyperbolic functions and trigonometric functions.

The second case we consider is $a_0 = 0$ and $a_1 = 0$, then Eq. (16) is rewritten as

$$z'^2 = a_2 z^2 + a_3 z^3 + a_4 z^4, \quad \text{or} \quad z'' = a_2 z + \frac{3}{2}a_3 z^2 + 2a_4 z^3 \tag{37}$$

whose solutions are

$$z = -\frac{a_2 a_3 \,\mathrm{sech}^2 \frac{\sqrt{a_2}}{2}\xi}{a_3^2 - a_2 a_4\left(1 - \tanh\frac{\sqrt{a_2}}{2}\xi\right)^2}, \quad a_2 > 0 \tag{38}$$

and

$$z = \frac{2a_2 \,\mathrm{sech}\sqrt{a_2}\xi}{\sqrt{a_3^2 - 4a_2 a_4} - a_3 \,\mathrm{sech}\sqrt{a_2}\xi}, \quad a_2 > 0, \quad a_3^2 - 4a_2 a_4 > 0 \tag{39}$$

Actually, Eq. (37) is the same as Eq. (14) with

$$a_2 = \frac{\gamma}{\alpha s^2}, \quad a_3 = \frac{2\eta}{3\alpha s^2}, \quad a_4 = -\frac{\delta}{2\alpha s^2} \tag{40}$$

So when $\gamma\alpha > 0$, one solution to Eq. (14) is

$$\phi_{11} = -\frac{12\gamma\eta \operatorname{sech}^2 \sqrt{\frac{\gamma}{4\alpha s^2}}\xi}{8\eta^2 + 9\delta\gamma\left(1 - \tanh\sqrt{\frac{\gamma}{4\alpha s^2}}\xi\right)^2} \tag{41}$$

and the other solution is

$$\phi_{12} = \frac{3\gamma \operatorname{sech}\sqrt{\frac{\gamma}{\alpha s^2}}\xi}{\alpha\sqrt{\frac{\eta^2}{\alpha^2} + \frac{9\gamma\delta}{2\alpha^2}} - \eta \operatorname{sech}\sqrt{\frac{\gamma}{\alpha s^2}}\xi} \tag{42}$$

with the constraint

$$2\eta^2 + 9\gamma\delta > 0 \tag{43}$$

3. Conclusion and discussion

A simple shallow-water model with influence of diabatic heating on a β-plane is applied to investigate the nonlinear equatorial Rossby waves in a shear flow. By the asymptotic method of multiple scales, the cubic nonlinear Schrödinger equation with an external heating source is derived for large amplitude equatorial envelope Rossby wave in a shear flow [7]. And there various periodic structures for these equatorial envelope Rossby waves are obtained with the help of Jacobi elliptic functions and elliptic equation. It is shown that the results are different for equatorial envelope Rossby waves without a source and with a phase-locked diabatic heating source, they have different structures due to the phase-locked diabatic heating source, and the phase-locked diabatic heating source plays an important role in forming periodic structures of rational form. Of course, these periodic structures contain solitons, solitary waves, as also singular structures, and they also have their different practical applications in explaining atmospheric events. Moreover, in this paper, we only consider another special case of external heating and find some new exact results. Actually, when $\eta = 0$, i.e. there is no external heating, then from Eq. (23), we have

$$b_1 = \pm\sqrt{-\frac{2\alpha s^2 a_4}{\delta}}, \quad b_0 = -\frac{\alpha s^2 a_3}{2\delta b_1}, \quad a_2 = \frac{\gamma}{\alpha s^2} - \frac{3\delta}{\alpha s^2} b_0^2, \quad a_1 = \frac{2}{\alpha s^2}\frac{b_0}{b_1}(\gamma - \delta b_0^2) \tag{44}$$

If $a_3 = 0$, then $b_0 = 0$ and

$$b_1 = \pm\sqrt{-\frac{2\alpha s^2 a_4}{\delta}}, \quad a_2 = \frac{\gamma}{\alpha s^2}, \quad a_1 = 0 \tag{45}$$

If b_1 is set as 1, then the solutions from ϕ_1 to ϕ_{10} are just the same as we given in Ref. [7]. Here we can see the external forcing plays an important role in two aspects. The first one is the basic state $b_0 = \frac{\eta}{3\delta}$, which is in proportion to the external strength, this results in different structures for equatorial envelope Rossby wave. And the second one is the modulation of a_2 or the modulus of Jacobi elliptic function m, which also leads to different structures for equatorial envelope Rossby wave. Moreover, the external heating results in structures for equatorial envelope Rossby wave of rational form, for example, ϕ_{11} and ϕ_{12}. Different from the results obtained in Ref. [7], there the solutions of rational form are composed of Jacobi elliptic functions. These two solutions are of the rational forms in terms of hyperbolic functions, which are resulted from diabatic heating. So we can say that different types of external heating will lead to different structures for equatorial envelope Rossby wave.

There needs more further research for more various heating sources, for this effort provides a better starting point for the treatment of general external heating sources and their impacts on the equatorial Rossby waves and climate changes.

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