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Lamé Function and Multi-order Exact Solutions to Nonlinear Coupled Systems*

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Abstract *Based on the Lamé function and Jacobi elliptic function, the perturbation method is applied to some nonlinear coupled systems, and there many multi-order solutions are derived to these nonlinear coupled systems.*

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Key words: nonlinear coupled system, perturbation method, Lamé function, Jacobi elliptic function, multi-order solutions

1 Introduction

During the past three decades, the nonlinear wave researches have made great progress, among which a number of new methods have been proposed to get the exact solutions to nonlinear wave equations. Among these methods, the homogeneous balance method,[1−3] the hyperbolic tangent function expansion method,[4−6] the nonlinear transformation method,[7,8] the trial function method,[9,10] sine-cosine method,[11] the Jacobi elliptic function expansion method,[12,13] and so on[14−16] are widely applied to solve nonlinear wave equations exactly and many solutions are obtained, from which the richness of structures is shown to exist in the different nonlinear wave equations. Furthermore, it deserves to discuss the stability of these solutions, there perturbation method is often applied. In this paper, based on the Jacobi elliptic functions and Lamé function,[17,18] perturbation method[18,19] is applied to get the multi-order exact solutions to nonlinear coupled systems.

2 Lamé Equation and Lamé Functions

Usually, Lamé equation[17] in terms of $y(x)$ can be written as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+[\lambda-n(n+1)m^2\mathrm{sn}^2x]y=0\,, \tag{1}$$

where λ is an eigenvalue, n is a positive integer, $\mathrm{sn}\,x$ is the Jacobi elliptic sine function with its modulus m $(0<m<1)$.

Set

$$\eta=\mathrm{sn}^2x\,, \tag{2}$$

then the Lamé equation (1) becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}\eta^2}+\frac{1}{2}\Big(\frac{1}{\eta}+\frac{1}{\eta-1}+\frac{1}{\eta-h}\Big)\frac{\mathrm{d}y}{\mathrm{d}\eta} -\frac{\mu+n(n+1)\eta}{4\eta(\eta-1)(\eta-h)}y=0\,, \tag{3}$$

where

$$h=m^{-2}>1\,,\qquad \mu=-h\lambda\,. \tag{4}$$

Equation (3) is a kind of Fuchs-typed equation with four regular singular points $\eta=0,1,h$, and $\eta=\infty$, the solution to Lamé equation (3) is known as Lamé function.

For example, when $n=3$, $\lambda=4(1+m^2)$, i.e. $\mu=-4(1+m^{-2})$, the Lamé function is

$$L_3(x)=\eta^{1/2}(1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2}=\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x\,. \tag{5}$$

When $n=2$, $\lambda=1+m^2$, i.e. $\mu=-(1+m^{-2})$, the Lamé function is

$$L_2(x)=(1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2}=\mathrm{cn}\,x\,\mathrm{dn}\,x\,. \tag{6}$$

In the equations (5) and (6), $\mathrm{cn}\,x$ and $\mathrm{dn}\,x$ are the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind,[17,18] respectively.

3 Lamé Equation, Lamé Functions and Their Application to Nonlinear Coupled Systems

3.1 *Variant Boussinesq Equations*

Variant Boussinesq equations read

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\alpha\frac{\partial^3u}{\partial t\partial x^2}=0\,,$$
$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\beta\frac{\partial^3u}{\partial x^3}=0\,. \tag{7}$$

We seek their travelling wave solutions of the following form,

$$u=u(\xi)\,,\qquad v=v(\xi)\,,\qquad \xi=k(x-ct)\,, \tag{8}$$

where k and c are wave number and wave speed, respectively.

Substituting Eq. (8) into Eq. (7) yields

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi}+u\frac{\mathrm{d}u}{\mathrm{d}\xi}+\frac{\mathrm{d}v}{\mathrm{d}\xi}-\alpha k^2c\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}=0\,,$$

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$$-c\frac{\mathrm{d}v}{\mathrm{d}\xi}+\frac{duv}{\mathrm{d}\xi}+\beta k^2\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}=0\,. \tag{9}$$

Integrating Eq. (9) once with respect to ξ and taking the integration constants as zero, we have

$$\alpha k^2c\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}+cu-\frac{1}{2}u^2-v=0\,,$$
$$\beta k^2\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}-cv+uv=0\,. \tag{10}$$

Applying the perturbation method and setting

$$u=u_0+\epsilon u_1+\epsilon^2u_2+\cdots\,,\quad v=v_0+\epsilon v_1+\epsilon^2v_2+\cdots\,, \tag{11}$$

where $\epsilon(0<\epsilon\ll 1)$ is a small parameter, u_0, u_1, u_2, and v_0, v_1, v_2 represent the zeroth-, the first- and the second-order solutions, respectively.

Substituting Eq. (11) into Eq. (10), we can obtain various order equations, for example, the zeroth-order equation (for ϵ^0) takes the following form,

$$\alpha k^2c\frac{\mathrm{d}^2u_0}{\mathrm{d}\xi^2}+cu_0-\frac{1}{2}u_0^2-v_0=0\,,$$
$$\beta k^2\frac{\mathrm{d}^2u_0}{\mathrm{d}\xi^2}-cv_0+u_0v_0=0\,, \tag{12}$$

and the first-order equation (for ϵ^1) is

$$\alpha k^2c\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2}+(c-u_0)u_1-v_1=0\,,$$
$$\beta k^2\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2}+(u_0-c)v_1+v_0u_1=0\,. \tag{13}$$

For the second-order equation (ϵ^2), it becomes

$$\alpha k^2c\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2}+(c-u_0)u_2-v_2=\frac{1}{2}u_1^2\,,$$
$$\beta k^2\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2}+(u_0-c)v_2+v_0u_2=-u_1v_1\,. \tag{14}$$

For the zeroth-order equation (12), the Jacobi elliptic sine function expansion method can be applied to solve it, i.e. the ansatz solution is supposed to take the following form,

$$u_0=a_0+a_1\mathrm{sn}\,\xi+a_2\mathrm{sn}^2\xi\,,\quad v_0=b_0+b_1\mathrm{sn}\,\xi+b_2\mathrm{sn}^2\xi\,, \tag{15}$$

where the expansion coefficients a_0, a_1, a_2, and b_0, b_1, b_2 can be determined by substituting Eq. (15) into Eq. (12). Then we have

$$a_0=c+\frac{\beta}{2\alpha c}-4(1+m^2)\alpha k^2c\,,\qquad a_1=0\,,\qquad a_2=12m^2c\alpha k^2\,,$$
$$b_0=\frac{\beta^2}{4\alpha^2c^2}+2(1+m^2)\beta k^2\,,\qquad b_1=0\,,\qquad b_2=-6m^2\beta k^2\,, \tag{16}$$

thus the zeroth-order solution for variant Boussinesq equations (7) is

$$u_0=c+\frac{\beta}{2\alpha c}-4(1+m^2)\alpha k^2c+12m^2c\alpha k^2\mathrm{sn}^2\xi\,,\qquad v_0=\frac{\beta^2}{4\alpha^2c^2}+2(1+m^2)\beta k^2-6m^2\beta k^2\mathrm{sn}^2\xi\,, \tag{17}$$

and there exists the relation between u_0 and v_0,

$$v_0-\frac{\beta^2}{4\alpha^2c^2}=-\frac{\beta}{2c\alpha}\Big(u_0-c-\frac{\beta}{2\alpha c}\Big)\,. \tag{18}$$

Substituting Eq. (17) into the first-order equation (13) yields

$$\alpha k^2c\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2}+\Big[-\frac{\beta}{2\alpha c}+4(1+m^2)\alpha k^2c-12m^2c\alpha k^2\mathrm{sn}^2\xi\Big]u_1-v_1=0\,,$$
$$\beta k^2\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2}+\Big[\frac{\beta}{2\alpha c}-4(1+m^2)\alpha k^2c+12m^2c\alpha k^2\mathrm{sn}^2\xi\Big]v_1+\Big[\frac{\beta^2}{4\alpha^2c^2}+2(1+m^2)\beta k^2-6m^2\beta k^2\mathrm{sn}^2\xi\Big]u_1=0\,, \tag{19}$$

i.e.,

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2}+\Big[-\frac{\beta}{2\alpha^2k^2c^2}+4(1+m^2)-12m^2\mathrm{sn}^2\xi\Big]u_1-\frac{1}{\alpha k^2c}v_1=0\,,$$
$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2}+\Big[\frac{1}{2\alpha k^2c}-4(1+m^2)\frac{\alpha c}{\beta}+12m^2\frac{c\alpha}{\beta}\mathrm{sn}^2\xi\Big]v_1+\Big[\frac{\beta}{4\alpha^2k^2c^2}+2(1+m^2)-6m^2\mathrm{sn}^2\xi\Big]u_1=0\,. \tag{20}$$

Here it is obvious that u_1 in Eqs. (20) takes the similar form as y in Eq. (1), so we can suppose that u_1 and v_1 take the following form,

$$u_1=AL_3(\xi)\,,\qquad v_1=BL_3(\xi)\,. \tag{21}$$

Substituting Eq. (21) into Eq. (20) yields

$$A=-\frac{2c\alpha}{\beta}B\,, \tag{22}$$

so the final first-order solution is

$$u_1=A\,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi\,,\qquad v_1=-\frac{\beta}{2c\alpha}A\,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi\,, \tag{23}$$

where A is an arbitrary constant. Obviously there exists the following relation

$$v_1 = -\frac{\beta}{2c\alpha}u_1\,. \tag{24}$$

In order to get the second-order solution of variant Boussinesq equations, we have to substitute the zeroth-order solution (17) and the first-order solution (23) into the second-order equation (14), so we have

$$\begin{aligned}
&\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2} + \Big[-\frac{\beta}{2\alpha^2k^2c^2} + 4(1+m^2) - 12m^2\mathrm{sn}^2\xi\Big]u_2 - \frac{1}{\alpha k^2c}v_2 = \frac{A^2}{2\alpha k^2c}\mathrm{sn}^2\xi\mathrm{cn}^2\xi\mathrm{dn}^2\xi\,,\\
&\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2} + \Big[\frac{1}{2\alpha k^2c} - 4(1+m^2)\frac{\alpha c}{\beta} + 12m^2\frac{c\alpha}{\beta}\mathrm{sn}^2\xi\Big]v_2\\
&\quad + \Big[\frac{\beta}{4\alpha^2k^2c^2} + 2(1+m^2) - 6m^2\mathrm{sn}^2\xi\Big]u_2 = \frac{1}{2\alpha k^2c}A^2\mathrm{sn}^2\xi\mathrm{cn}^2\xi\mathrm{dn}^2\xi\,.
\end{aligned} \tag{25}$$

Since $\mathrm{cn}^2\xi = 1-\mathrm{sn}^2\xi$, $\mathrm{dn}^2\xi = 1-m^2\mathrm{sn}^2\xi$, the special solution to Eq. (25) can be supposed to be

$$u_2 = A_0 + A_2\mathrm{sn}^2\xi + A_4\mathrm{sn}^4\xi\,,\qquad v_2 = B_0 + B_2\mathrm{sn}^2\xi + B_4\mathrm{sn}^4\xi\,. \tag{26}$$

Substituting Eq. (26) into Eq. (25) yields

$$\begin{aligned}
&A_0 = \frac{A^2}{48m^2\alpha ck^2}\,,\qquad A_2 = -\frac{(1+m^2)A^2}{24m^2\alpha ck^2}\,,\qquad A_4 = \frac{A^2}{16\alpha ck^2}\,,\\
&B_0 = -\frac{\beta A^2}{96m^2\alpha^2c^2k^2}\,,\qquad B_2 = \frac{\beta(1+m^2)A^2}{48m^2\alpha^2c^2k^2}\,,\qquad B_4 = -\frac{\beta A^2}{32\alpha^2c^2k^2}\,,
\end{aligned} \tag{27}$$

i.e., the second-order solution is

$$u_2 = \frac{A^2}{48m^2\alpha ck^2}[1-2(1+m^2)\mathrm{sn}^2\xi + 3m^2\mathrm{sn}^4\xi]\,,\qquad v_2 = -\frac{\beta A^2}{96m^2\alpha^2c^2k^2}[1-2(1+m^2)\mathrm{sn}^2\xi + 3m^2\mathrm{sn}^4\xi]\,. \tag{28}$$

Obviously there exists the following relation,

$$v_2 = -\frac{\beta}{2c\alpha}u_2\,. \tag{29}$$

3.2 *Coupled mKdV Equations*

In the above section, we applied Lamé equation under the condition of $n=3$ and $\lambda = 4(1+m^2)$ to solve variant Boussinesq equations and got its multi-order exact solutions. In this section, we will consider the Lamé equation under the condition of $n=2$ and $\lambda = 1+m^2$ and its application to obtain multi-order exact solution to coupled mKdV equations.

Here coupled mKdV equations read,

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \alpha u^2\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} + c_0\frac{\partial v}{\partial x} = 0\,,\\
&\frac{\partial v}{\partial t} + \gamma v\frac{\partial v}{\partial x} + \delta\frac{\partial uv}{\partial x} = 0\,.
\end{aligned} \tag{30}$$

We seek its travelling wave solutions in the frame of Eq. (8), then we have

$$\begin{aligned}
&\beta k^2\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} + \alpha u^2\frac{\mathrm{d}u}{\mathrm{d}\xi} - c\frac{\mathrm{d}u}{\mathrm{d}\xi} + c_0\frac{\mathrm{d}v}{\mathrm{d}\xi} = 0\,,\\
&-c\frac{\mathrm{d}v}{\mathrm{d}\xi} + \gamma v\frac{\mathrm{d}v}{\mathrm{d}\xi} + \delta\frac{\mathrm{d}uv}{\mathrm{d}\xi} = 0\,.
\end{aligned} \tag{31}$$

Integrating Eq. (31) once with respect to ξ and taking the integration constants as zero, we have

$$\begin{aligned}
&\beta k^2\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} - cu + \frac{\alpha}{3}u^3 + c_0v = 0\,,\\
&-cv + \frac{\gamma}{2}v^2 + \delta uv = 0\,.
\end{aligned} \tag{32}$$

Similarly, applying perturbation method and setting u and v to be expanded as Eq. (11), we can have the multi-order expansion equations, for example, the zeroth-order equation (for ϵ^0) is

$$\begin{aligned}
&\beta k^2\frac{\mathrm{d}^2u_0}{\mathrm{d}\xi^2} - cu_0 + \frac{\alpha}{3}u_0^3 + c_0v_0 = 0\,,\\
&-cv_0 + \frac{\gamma}{2}v_0^2 + \delta u_0v_0 = 0\,,
\end{aligned} \tag{33}$$

the first-order equation (for ϵ^1) is

$$\begin{aligned}
&\beta k^2\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2} + (\alpha u_0^2 - c)u_1 + c_0v_1 = 0\,,\\
&-cv_1 + (\gamma v_0 + \delta u_0)v_1 + \delta v_0u_1 = 0\,,
\end{aligned} \tag{34}$$

and the second-order equation (for ϵ^2) is

$$\begin{aligned}
&\beta k^2\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2} - cu + (\alpha u_0^2 - c)u_2 + c_0v_2 = -\alpha u_0u_1^2\,,\\
&-cv_2 + (\gamma v_0 + \delta u_0)v_2 + \delta v_0u_2 = -\frac{\gamma}{2}v_1^2 - \delta u_1v_1\,.
\end{aligned} \tag{35}$$

The zeroth-order equation (33) can be solved by the Jacobi elliptic sine function expansion method, where the ansatz solution

$$u_0 = a_0 + a_1\mathrm{sn}\,\xi\,,\qquad v_0 = b_0 + b_1\mathrm{sn}\,\xi \tag{36}$$

is chosen. Substituting Eq. (36) into Eq. (33) results in

$$u_0 = \pm\sqrt{-\frac{6\beta}{\alpha}}\,mk\,\mathrm{sn}\,\xi\,,\qquad v_0 = -\frac{4\delta}{\gamma^2}c_0 - \frac{2}{\gamma}(1+m^2)\beta k^2 \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta}{\alpha}}\,mk\,\mathrm{sn}\,\xi\,. \tag{37}$$

Then we can substitute Eq. (37) into the first-order equation (34) and get the rewritten first-order equation

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2}+\Big[\frac{2\delta}{\gamma\beta k^2}c_0+(1+m^2)-6m^2\mathrm{sn}^2\xi\Big]u_1+\frac{c_0}{\beta k^2}v_1=0\,, \tag{38a}$$

$$\Big[-\frac{4\delta^2}{\gamma^2}c_0-\frac{2\delta}{\gamma}(1+m^2)\beta k^2\mp\frac{2\delta^2}{\gamma}\sqrt{-\frac{6\beta}{\alpha}}\,mk\,\mathrm{sn}\,\xi\Big]u_1+\Big[-\frac{2\delta}{\gamma}c_0-(1+m^2)\beta k^2\mp\delta\sqrt{-\frac{6\beta}{\alpha}}\,mk\,\mathrm{sn}\,\xi\Big]v_1=0\,. \tag{38b}$$

It is obvious that u_1 in Eq. (38a) takes the similar form as y in Eq. (1) under the condition of $n=2$ and $\lambda=1+m^2$, so we can suppose that

$$u_1=AL_2(\xi)\,,\qquad v_1=BL_2(\xi)\,. \tag{39}$$

Then substituting Eq. (39) into Eq. (38) leads to

$$B=-\frac{2\delta}{\gamma}A\,, \tag{40}$$

so the first-order solution to coupled mKdV equations is

$$u_1=A\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi\,,\qquad v_1=-\frac{2\delta}{\gamma}A\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi\,. \tag{41}$$

In order to solve the second-order equation (35) of coupled mKdV equations, we have to substitute Eqs. (37) and (41) into the second-order (35) to get the rewritten form of the second-order equation,

$$\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2}+\Big[\frac{2\delta c_0}{\gamma\beta k^2}+(1+m^2)-6m^2\mathrm{sn}^2\xi\Big]u_2+\frac{c_0}{\beta k^2}=\mp\frac{\alpha}{\beta k}\sqrt{-\frac{6\beta}{\alpha}}\,mA^2\mathrm{sn}\,\xi[1-(1+m^2)\mathrm{sn}^2\xi+m^2\mathrm{sn}^4\xi]\,,$$

$$\Big[-\frac{4\delta^2}{\gamma^2}c_0-\frac{2\delta}{\gamma}(1+m^2)\beta k^2\mp\frac{2\delta^2}{\gamma}\sqrt{-\frac{6\beta}{\alpha}}\,mk\,\mathrm{sn}\xi\Big]u_2+\Big[-\frac{2\delta}{\gamma}c_0-(1+m^2)\beta k^2\mp\delta\sqrt{-\frac{6\beta}{\alpha}}\,mk\,\mathrm{sn}\,\xi\Big]v_2=0\,. \tag{42}$$

Similarly, its ansatz solution can be written as

$$u_2=A_1\mathrm{sn}\,\xi+A_3\,\mathrm{sn}^3\xi\,,\qquad v_2=B_1\mathrm{sn}\,\xi+B_3\,\mathrm{sn}^3\,\xi\,. \tag{43}$$

Combining Eqs. (42) and (43) leads to

$$u_2=\pm\frac{\alpha}{12\beta k}\sqrt{-\frac{6\beta}{\alpha}}\,\frac{1+m^2}{m}A^2\mathrm{sn}\,\xi(1-2m^2\mathrm{sn}^2\xi)\,,\qquad v_2=\mp\frac{\delta\alpha}{6\beta\gamma k}\sqrt{-\frac{6\beta}{\alpha}}\,\frac{1+m^2}{m}A^2\mathrm{sn}\,\xi\Big(1-\frac{2m^2}{1+m^2}\,\mathrm{sn}^2\xi\Big)\,, \tag{44}$$

which is the second-order exact solution to coupled mKdV equations.

4 Conclusion and Discussion

In this paper, the Lamé equation and Lamé functions are applied to solve nonlinear coupled systems. When perturbation method is considered, the multi-order solutions are obtained to these nonlinear coupled systems. The results got in this paper is very important for nonlinear instability of nonlinear coherent structures.

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