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Exact Solutions to Triple sine-Gordon Equation*

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Abstract *In this paper, a new transformation is introduced to solve triple sine-Gordon equation. It is shown that this intermediate transformation method is powerful to solve complex special type nonlinear evolution equation.*

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1 Introduction

Sine-Gordon-type equations, including single sine-Gordon (SSG for short) equation,

$$u_{xt} = \alpha \sin u \,, \tag{1}$$

double sine-Gordon (DSG for short) equation,

$$u_{xt} = \alpha \sin u + \beta \sin(2u) \tag{2}$$

and triple sine-Gordon (TSG for short) equation,

$$u_{xt} = \alpha \sin u + \beta \sin(2u) + \gamma \sin(3u) \tag{3}$$

are widely applied in physics and engineering. For example, DSG equation is a frequent object of study in numerous physical applications, such as Josephson arrays, ferromagnetic materials, charge density waves, smectic liquid crystal dynamics.[1−5] Actually, SSG equation and DSG equation also arise in nonlinear optics, ^{3}He spin waves, and other fields. In a resonant fivefold degenerate medium, the propagation and creation of ultra-short optical pulses, the SSG and DSG models are usually used. However, in some cases, one has to consider other sine-Gordon equations. For instance, TSG equation is used to describe the propagation of strictly resonant sharp line optical pulses.[6]

Due to the wide applications of sine-Gordon type equations, many solutions to them have been obtained in different functional forms, such as $\tan^{-1}\coth$, $\tan^{-1}\tanh$, $\tan^{-1}\text{sech}$, $\tan^{-1}\text{sn}$ and so on, by different methods.[7−12] Due to the special forms of the sine-Gordon-type equations, it is much difficult to solve them directly, so one needs some transformations. In this paper, based on the new introduced transformation, we will show some special results about solutions for TSG equation (3).

2 Intermediate Transformation and Solutions to TSG Equation

Due to the complexity of nonlinear evolution equations, it is usually much difficult to solve them directly. Some kinds of intermediate transformations are needed to simplify the original nonlinear evolution equations. For example, in Refs. [9], [10], and [13] ∼ [16], the intermediate transformation is elliptic equation,[17]

$$y'^2 = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 \tag{4}$$

with

$$u = u(y) \,, \tag{5}$$

where the prime denotes the derivative with respect to its argument.

In Ref. [11], Sirendaoreji introduced a new transformation

$$u' = F(u) \,, \tag{6}$$

where $F(u)$ is a suitable function of sine, cosine, hyperbolic sine, hyperbolic cosine, and so on. It has been shown that this transformation is much powerful in solving the sine-Gordon equation (1), the double sine-Gordon equation (2), the sinh-Gordon equation, and some other special types of nonlinear evolution equations. Usually, for different nonlinear evolution equations, the exact forms of the intermediate transformations are different. Next we will introduce a transformation $u' = F(u)$ of new form and apply it to solve TSG equation (3).

First of all, we suppose the solution of TSG equation (3) takes the form

$$u = u(\xi) \,, \qquad \xi = k(x - ct) \,, \tag{7}$$

where k and c are wave number and wave speed, respectively.

Under the frame of Eq. (7), TSG equation (3) can be rewritten as

$$u'' = \alpha_1 \sin u + \beta_1 \sin(2u) + \gamma_1 \sin(3u) \tag{8}$$

with

$$\alpha_1 = -\frac{\alpha}{k^2 c} \,, \quad \beta_1 = -\frac{\beta}{k^2 c} \,, \quad \gamma_1 = -\frac{\gamma}{k^2 c} \,. \tag{9}$$

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And then we suppose that equation (8) satisfies the transformation (6) with the following new form

$$u' = \frac{\mathrm{d}u}{\mathrm{d}\xi} = a\cos\frac{u}{2} + b\cos\frac{3u}{2}\,, \tag{10}$$

where a and b are real constants to be determined.

From the intermediate transformation (10), we have

$$u'' = -\frac{1}{4}(a^2+2ab)\sin u - ab\sin(2u) - \frac{3b^2}{4}\sin(3u)\,. \tag{11}$$

Combining Eq. (8) with Eq. (11) results in

$$\alpha_1 = -\frac{1}{4}(a^2+2ab)\,,\quad \beta_1 = -ab\,,\quad \gamma_1 = -\frac{3b^2}{4} \tag{12}$$

with constraints

$$\alpha_1 + \frac{\gamma_1}{3} \le 0\,,\quad \alpha_1 + \frac{\gamma_1}{3} \le \beta_1\,, \tag{13}$$

from which four cases can be obtained. For instance,

Case 1

$$a = \sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} + \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,,\qquad b = \sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} - \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,. \tag{14}$$

Case 2

$$a = \sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} - \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,,\qquad b = \sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} + \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,. \tag{15}$$

Case 3

$$a = -\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} + \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,,\qquad b = -\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} - \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,. \tag{16}$$

Case 4

$$a = -\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} - \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,,\qquad b = -\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} + \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,. \tag{17}$$

In order to obtain the exact solution, the intermediate transformation (10) must be solved. Actually it can be rewritten as

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = \cos\frac{u}{2}\Big(a - 3b + 4b\cos^2\frac{u}{2}\Big)\,, \tag{18}$$

where two cases need to be considered. If $a = 3b$, then we have

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = 4b\cos^3\frac{u}{2}\,, \tag{19}$$

from which the solution to u can be determined as

$$4b(\xi-\xi_0) = \frac{\sin(u/2)}{\cos^2(u/2)} + \frac{1}{2}\ln\frac{1+\sin(u/2)}{1-\sin(u/2)}\,, \tag{20}$$

where ξ_0 is an integration constant.

So we have four solutions to TSG equation (3) for $a = 3b$, the first one is

$$u_1 = 2\sin^{-1} v_1 \tag{21}$$

where v_1 is defined by

$$4\Big[\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} - \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,\Big](\xi-\xi_0) = \frac{v_1}{1-v_1^2} + \frac{1}{2}\ln\frac{1+v_1}{1-v_1} \tag{22}$$

with the constraint

$$4\beta = 3\alpha + \gamma\,. \tag{23}$$

The second one is

$$u_2 = 2\sin^{-1} v_2\,, \tag{24}$$

where v_2 is defined by

$$4\Big[\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} + \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,\Big](\xi-\xi_0) = \frac{v_2}{1-v_2^2} + \frac{1}{2}\ln\frac{1+v_2}{1-v_2} \tag{25}$$

with the constraint

$$4\beta = 3\alpha + \gamma\,. \tag{26}$$

The third one is

$$u_3 = 2\sin^{-1} v_3\,, \tag{27}$$

where v_3 is defined by

$$-4\Big[\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} + \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,\Big](\xi-\xi_0) = \frac{v_3}{1-v_3^2} + \frac{1}{2}\ln\frac{1+v_3}{1-v_3} \tag{28}$$

with the constraint

$$4\beta = 3\alpha + \gamma\,. \tag{29}$$

The fourth one is

$$u_4 = 2\sin^{-1} v_4\,, \tag{30}$$

where v_4 is defined by

$$4\Big[-\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)} + \sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}\,\Big](\xi-\xi_0) = \frac{v_4}{1-v_4^2} + \frac{1}{2}\ln\frac{1+v_4}{1-v_4} \tag{31}$$

with the constraint

$$4\beta = 3\alpha + \gamma\,. \tag{32}$$

If $a \neq 3b$, then from Eq. (18), we have

$$\Big[\frac{1}{\cos(u/2)} - \frac{4b\cos(u/2)}{a-3b+4b\cos^2(u/2)}\Big]\mathrm{d}u = (a-3b)\,\mathrm{d}\xi\,, \tag{33}$$

i.e.

$$\frac{\mathrm{d}u}{\cos(u/2)}+\frac{1}{2}\frac{\mathrm{d}\sin(u/2)}{-(a+b)/4b+\sin^2(u/2)}=(a-3b)\mathrm{d}\xi\,, \quad (34)$$

where three cases need to be considered.

If $-(a+b)/4b>0$, then we have

$$(a-3b)(\xi-\xi_0)=ln\frac{1+\sin(u/2)}{1-\sin(u/2)}+\frac{1}{2\sqrt{-(a+b)/4b}}\tan^{-1}\frac{\sin(u/2)}{\sqrt{-(a+b)/4b}}\,, \quad (35)$$

where ξ_0 is an integration constant.

And then we derive another two solutions to TSG equation (3), which are

$$u_5=2\sin^{-1}v_5\,, \quad (36)$$

where v_5 is defined by

$$2\Big[2\sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}-\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)}\,\Big](\xi-\xi_0)$$
$$=\ln\frac{1+v_5}{1-v_5}+\sqrt{\frac{\sqrt{\beta_1-\alpha_1-\gamma_1/3}-\sqrt{-(\alpha_1+\gamma_1/3)}}{2\sqrt{-(\alpha_1+\gamma_1/3)}}}\tan^{-1}\sqrt{\frac{2[\sqrt{\beta_1-\alpha_1-\gamma_1/3}-\sqrt{-(\alpha_1+\gamma_1/3)}]}{\sqrt{-(\alpha_1+\gamma_1/3)}}}v_5\,, \quad (37)$$

and

$$u_6=2\sin^{-1}v_6\,, \quad (38)$$

where v_6 is defined by

$$-2\Big[2\sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}-\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)}\,\Big](\xi-\xi_0)$$
$$=\ln\frac{1+v_6}{1-v_6}+\sqrt{\frac{\sqrt{\beta_1-\alpha_1-\gamma_1/3}-\sqrt{-(\alpha_1+\gamma_1/3)}}{2\sqrt{-(\alpha_1+\gamma_1/3)}}}\tan^{-1}\sqrt{\frac{2[\sqrt{\beta_1-\alpha_1-\gamma_1/3}-\sqrt{-(\alpha_1+\gamma_1/3)}]}{\sqrt{-(\alpha_1+\gamma_1/3)}}}v_6 \quad (39)$$

with the constraint

$$\beta_1>0\,. \quad (40)$$

If $-(a+b)/4b=0$, then we have

$$(a-3b)(\xi-\xi_0)=\ln\frac{1+\sin(u/2)}{1-\sin(u/2)}-\frac{1}{2\sin(u/2)}\,, \quad (41)$$

where ξ_0 is an integration constant.

And then we derive another two solutions to TSG equation (3), they are

$$u_7=2\,\sin^{-1}v_7\,, \quad (42)$$

where v_7 is defined by

$$4\sqrt{\beta_1}\,(\xi-\xi_0)=\ln\frac{1+v_7}{1-v_7}-\frac{1}{2v_7}\,, \quad (43)$$

and

$$u_8=2\sin^{-1}v_8\,, \quad (44)$$

where v_8 is defined by

$$-4\sqrt{\beta_1}\,(\xi-\xi_0)=\ln\frac{1+v_8}{1-v_8}-\frac{1}{2v_8} \quad (45)$$

with the constraint

$$\alpha_1+\frac{\gamma_1}{3}=0\,,\qquad \beta_1>0\,. \quad (46)$$

If $-(a+b)/4b<0$, then we have

$$(a-3b)(\xi-\xi_0)=\ln\frac{1+\sin(u/2)}{1-\sin(u/2)}+\frac{1}{2}\sqrt{b/(a+b)}\ln\frac{\sin(u/2)-\sqrt{(a+b)/4b}}{\sin(u/2)+\sqrt{(a+b)/4b}}\,, \quad (47)$$

where ξ_0 is an integration constant.

And then we derive another four solutions to TSG equation (3), which are

$$u_9=2\sin^{-1}v_9\,, \quad (48)$$

where v_9 is defined by

$$2\Big[2\sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}-\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)}\,\Big](\xi-\xi_0)=\ln\frac{1+v_9}{1-v_9}+\frac{1}{4d}\ln\frac{v_9-d}{v_9+d}\,,$$
$$d\equiv\sqrt{\frac{\sqrt{-(\alpha_1+\gamma_1/3)}}{2[\sqrt{-(\alpha_1+\gamma_1/3)}-\sqrt{\beta_1-\alpha_1-\gamma_1/3}]}}\,, \quad (49)$$

and

$$u_{10}=2\sin^{-1}v_{10}\,, \quad (50)$$

where v_{10} is defined by

$$-2\Big[2\sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}-\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)}\,\Big](\xi-\xi_0)=\ln\frac{1+v_{10}}{1-v_{10}}+\frac{1}{4d}\ln\frac{v_{10}-d}{v_{10}+d}\,,$$

$$d\equiv\sqrt{\frac{\sqrt{-(\alpha_1+\gamma_1/3)}}{2[\sqrt{-(\alpha_1+\gamma_1/3)}-\sqrt{\beta_1-\alpha_1-\gamma_1/3}]}} \tag{51}$$

with constraint

$$\beta_1<0\,, \tag{52}$$

$$u_{11}=2\sin^{-1}v_{11}\,, \tag{53}$$

where v_{11} is defined by

$$2\Big[2\sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}+\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)}\,\Big](\xi-\xi_0)=\ln\frac{1+v_{11}}{1-v_{11}}+\frac{1}{4d}\ln\frac{v_{11}-d}{v_{11}+d}\,,$$

$$d\equiv\sqrt{\frac{\sqrt{-(\alpha_1+\gamma_1/3)}}{2[\sqrt{-(\alpha_1+\gamma_1/3)}+\sqrt{\beta_1-\alpha_1-\gamma_1/3}]}}\,, \tag{54}$$

and

$$u_{12}=2\sin^{-1}v_{12}\,, \tag{55}$$

where v_{12} is defined by

$$-2\Big[2\sqrt{\beta_1-\alpha_1-\frac{\gamma_1}{3}}+\sqrt{-\Big(\alpha_1+\frac{\gamma_1}{3}\Big)}\,\Big](\xi-\xi_0)=\ln\frac{1+v_{12}}{1-v_{12}}+\frac{1}{4d}\ln\frac{v_{12}-d}{v_{12}+d}\,,$$

$$d\equiv\sqrt{\frac{\sqrt{-(\alpha_1+\gamma_1/3)}}{2[\sqrt{-(\alpha_1+\gamma_1/3)}+\sqrt{\beta_1-\alpha_1-\gamma_1/3}]}}\,. \tag{56}$$

Remark 1 The solutions from u_1 to u_{12} have not been reported in the literature, they are new solutions to TSG equation (3).

Remark 2 Compared with solutions given in Ref. [8], the solutions from u_1 to u_{12} are implicit.

3 Conclusion

In this paper, a new transformation is introduced to solve triple sine-Gordon equation. It is shown that this intermediate transformation method is powerful to solve complex special type nonlinear evolution equation. Here it deserves noting that all solutions reported in this paper are not explicit ones, there are still more efforts required in further research for explicit solutions of TSG equation (3).

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