Commun. Theor. Phys. (Beijing, China) **44** (2005) pp. 23–30

Vol. 44, No. 1, July 15, 2005

Exact Jacobian Elliptic Function Solutions to sine-Gordon Equation*

FU Zun-Tao,[1,2,†] YAO Zhen-Hua,[1] LIU Shi-Kuo,[1] and LIU Shi-Da[1,2]

[1]School of Physics & Laboratory for Severe Storm and Flood Disaster, Peking University, Beijing 100871, China[‡]

[2]State Key Laboratory for Turbulence and Complex Systems, Peking University, Beijing 100871, China

(Received November 30, 2004)

Abstract *In this paper, four transformations are introduced to solve single sine-Gordon equation by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different transformations are required in order to obtain more kinds of solutions to the single sine-Gordon equation.*

PACS numbers: 03.65.Ge
Key words: Jacobian elliptic function, periodic wave solution, transformation, sine-Gordon equation

1 Introduction

The sine-Gordon-type equations, including single sine-Gordon (SSG for short) equation

$$u_{xt} = \alpha \sin u \,, \tag{1}$$

double sine-Gordon (DSG for short) equation

$$u_{xt} = \alpha \sin u + \beta \sin(2u) \,, \tag{2}$$

and triple sine-Gordon (TSG for short) equation

$$u_{xt} = \alpha \sin u + \beta \sin(2u) + \gamma \sin(3u) \tag{3}$$

are widely applied in physics and engineering. For example, DSG equation is a frequent object of study in numerous physical applications, such as Josephson arrays, ferromagnetic materials, charge density waves, smectic liquid crystal dynamics.[1−5] Actually, SSG equation and DSG equation also arise in nonlinear optics ^{3}He spin waves and other fields. In a resonant fivefold degenerate medium, the propagation and creation of ultra-short optical pulses, the SSG and DSG models are usually used. However, in some cases, one has to consider other sine-Gordon equations. For instance, TSG equation is used to describe the propagation of strictly resonant sharp line optical pulses.[6]

Due to the wide applications of sine-Gordon type equations, many solutions to them, such as $\tan^{-1}\coth\xi$, $\tan^{-1}\tanh\xi$, $\tan^{-1}\mathrm{sech}\,\xi$, $\tan^{-1}\mathrm{sn}\,\xi$ and so on, have been obtained in different functional forms by different methods.[7−12] Due to the special forms of the sine-Gordon-type equations, it is very difficult to solve them directly, so it needs some transformations. In this paper, based on the introduced transformations, we will show systematical results about solutions for SSG equation (1) by using the knowledge of elliptic equation and Jacobian elliptic functions.[13−19]

2 The First Kind of Transformation and Solutions to SSG Equation

In order to solve the sine-Gordon-type equations, certain transformations must be introduced. For example, the transformation

$$u = 2\tan^{-1}v \quad \text{or} \quad v = \tan\frac{u}{2} \,, \tag{4}$$

has been introduced in Refs. [7] and [9] to solve DSG equation.

When the transformation (4) is considered, the SSG equation (1) can be rewritten as

$$(1+v^2)v_{tx} - 2vv_tv_x - \alpha v - \alpha v^3 = 0 \,. \tag{5}$$

Equation (5) can be solved in the frame

$$v = v(\xi), \quad \xi = k(x - ct) \,, \tag{6}$$

where k and c are wave number and wave speed, respectively.

Substituting Eq. (6) into Eq. (5), we have

$$(1+v^2)\frac{\mathrm{d}^2v}{\mathrm{d}\xi^2} - 2v\Big(\frac{\mathrm{d}v}{\mathrm{d}\xi}\Big)^2 + \alpha_1 v + \alpha_1 v^3 = 0 \,, \quad \alpha_1 = \frac{\alpha}{k^2c} \,. \tag{7}$$

And then we suppose that equation (7) has the following solution:

$$v = v(y) = \sum_{j=0}^{j=n} b_j y^j, \quad b_n \neq 0, \quad y = y(\xi) \,, \tag{8}$$

where y satisfies elliptic equation,[13−20]

$$y'^2 = a_0 + a_2y^2 + a_4y^4, \quad a_4 \neq 0 \tag{9}$$

with $y' = \mathrm{d}y/\mathrm{d}\xi$, then

$$y'' = a_2y + 2a_4y^3 \,. \tag{10}$$

*The project supported by National Natural Science Foundation of China under Grant No. 40305006
†Correspondence author, E-mail: fuzt@pku.edu.cn
‡Correspondence address

There n in Eq. (8) can be determined by the partial balance between the highest order derivative terms and the highest degree nonlinear term in Eq. (7). Here we know that the degree of v is

$$O(v) = O(y^n) = n\,, \tag{11}$$

and from Eqs. (9) and (10), one has

$$O(y'^2) = O(y^4) = 4, \quad O(y'') = O(y^3) = 3\,, \tag{12}$$

and actually one can have

$$O(y^{(l)}) = l + 1\,. \tag{13}$$

So one has

$$O(v) = n, \qquad O(v') = n + 1\,,$$
$$O(v'') = n + 2, \quad O(v^{(l)}) = n + l\,. \tag{14}$$

For SSG equation (1), we have $n = 1$, so the ansatz solution of Eq. (7) can be rewritten as

$$v = b_0 + b_1 y, \quad b_1 \neq 0\,. \tag{15}$$

Substituting Eq. (15) into Eq. (7) results in an algebraic equation for y, which can be used to determine expansion coefficients in Eq. (15) and some constraints can also be obtained. Here we have

$$b_0 = 0, \quad b_1^2 = \frac{a_2 + \alpha_1}{2a_0} = \frac{2a_4}{a_2 - \alpha_1}\,, \tag{16}$$

from which the constraints can be determined as

$$\frac{a_2 + \alpha_1}{2a_0} > 0, \quad \frac{2a_4}{a_2 - \alpha_1} > 0, \quad a_2^2 > 4a_0 a_4\,. \tag{17}$$

Considering the constraints (17), the solutions to the elliptic equation (9) can be used to derive the final results, here eleven cases can be obtained.

Case 1 If $a_0 = 0$, $a_2 = 1$, $a_4 = 1$, then

$$y = \operatorname{csch}\xi, \quad b_0 = 0, \quad b_1 = \pm 1, \quad c = -\frac{\alpha}{k^2}\,, \tag{18}$$

where k is an arbitrary constant. So the solution to SSG Eq. (1) is

$$u_1 = 2\tan^{-1}(\pm \operatorname{csch}\xi)\,. \tag{19}$$

Case 2 If $a_0 = 1 - m^2$, $a_2 = 2m^2 - 1$, $a_4 = -m^2$, where $0 \leq m \leq 1$ is called modulus of Jacobian elliptic functions,[20−23] then

$$y = \operatorname{cn}\xi, \quad b_0 = 0, \quad b_1 = \pm\sqrt{\frac{m^2}{1 - m^2}}\,,$$
$$c = \frac{\alpha}{k^2}, \quad 0 < m < 1\,, \tag{20}$$

where k is an arbitrary constant, and $\operatorname{cn}\xi$ is Jacobian elliptic cosine function.[20−23] So the solution to SSG Eq. (1) is

$$u_2 = 2\tan^{-1}\left(\pm\sqrt{\frac{m^2}{1 - m^2}}\operatorname{cn}\xi\right)\,. \tag{21}$$

Case 3 If $a_0 = -m^2$, $a_2 = 2m^2 - 1$, $a_4 = 1 - m^2$, then

$$y = \operatorname{nc}\xi \equiv \frac{1}{\operatorname{cn}\xi}, \quad b_0 = 0, \quad b_1 = \pm\sqrt{\frac{1 - m^2}{m^2}}\,,$$
$$c = -\frac{\alpha}{k^2}, \quad 0 < m < 1\,, \tag{22}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_3 = 2\tan^{-1}\left(\pm\sqrt{\frac{1 - m^2}{m^2}}\operatorname{nc}\xi\right)\,. \tag{23}$$

Case 4 If $a_0 = 1$, $a_2 = 2m^2 - 1$, $a_4 = (m^2 - 1)m^2$, then

$$y = \operatorname{sd}\xi \equiv \frac{\operatorname{sn}\xi}{\operatorname{dn}\xi}, \quad b_0 = 0, \quad b_1 = \pm m$$
$$c = \frac{\alpha}{k^2}, \quad 0 < m < 1\,, \tag{24}$$

where k is an arbitrary constant, and $\operatorname{sn}\xi$ is Jacobian elliptic sine function and $\operatorname{dn}\xi$ is Jacobian elliptic function of the third kind.[20−23] So the solution to SSG Eq. (1) is

$$u_4 = 2\tan^{-1}(\pm m \operatorname{sd}\xi)\,. \tag{25}$$

Case 5 If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$, $a_4 = 1$, then

$$y = \operatorname{cs}\xi \equiv \frac{\operatorname{cn}\xi}{\operatorname{sn}\xi}, \quad b_0 = 0, \quad b_1 = \pm\sqrt{\frac{1}{1 - m^2}}\,,$$
$$c = \frac{\alpha}{m^2 k^2}, \quad 0 < m < 1\,, \tag{26}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_5 = 2\tan^{-1}\left(\pm\sqrt{\frac{1}{1 - m^2}}\operatorname{cs}\xi\right)\,. \tag{27}$$

Case 6 If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$, $a_4 = 1$, then

$$y = \operatorname{cs}\xi \equiv \frac{\operatorname{cn}\xi}{\operatorname{sn}\xi}, \quad b_0 = 0, \quad b_1 = \pm 1\,,$$
$$c = -\frac{\alpha}{m^2 k^2}, \quad 0 < m < 1\,, \tag{28}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_6 = 2\tan^{-1}(\pm \operatorname{cs}\xi)\,. \tag{29}$$

Case 7 If $a_0 = m^2(m^2 - 1)$, $a_2 = 2m^2 - 1$, $a_4 = 1$, then

$$y = \operatorname{ds}\xi \equiv \frac{\operatorname{dn}\xi}{\operatorname{sn}\xi}, \quad b_0 = 0, \quad b_1 = \pm\frac{1}{m}\,,$$
$$c = -\frac{\alpha}{k^2}, \quad 0 < m \leq 1\,, \tag{30}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_7 = 2\tan^{-1}\left(\pm\frac{1}{m}\operatorname{ds}\xi\right)\,. \tag{31}$$

Actually, when $m \to 1$, u_7 recovers u_1.

Case 8 If $a_0 = 1$, $a_2 = 2 - m^2$, $a_4 = 1 - m^2$, then

$$y = \mathrm{sc}\,\xi \equiv \frac{\mathrm{sn}\,\xi}{\mathrm{cn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm\sqrt{1-m^2}\,,$$
$$c = -\frac{\alpha}{m^2 k^2}, \quad 0 < m < 1\,, \tag{32}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_8 = 2\tan^{-1}\Big(\pm\sqrt{1-m^2}\,\mathrm{sc}\,\xi\Big)\,. \tag{33}$$

Case 9 If $a_0 = 1$, $a_2 = 2 - m^2$, $a_4 = 1 - m^2$, then

$$y = \mathrm{sc}\,\xi \equiv \frac{\mathrm{sn}\,\xi}{\mathrm{cn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm 1\,,$$
$$c = \frac{\alpha}{m^2 k^2}, \quad 0 < m < 1\,, \tag{34}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_9 = 2\tan^{-1}(\pm\,\mathrm{sc}\,\xi)\,. \tag{35}$$

Case 10 If $a_0 = (1-m^2)/4$, $a_2 = (1+m^2)/2$, $a_4 = (1-m^2)/4$, then

$$y = \frac{\mathrm{cn}\,\xi}{1 \pm \mathrm{sn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm\sqrt{\frac{1-m^2}{(1-m)^2}}\,,$$
$$c = \frac{\alpha}{m k^2}, \quad 0 < m < 1\,, \tag{36}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{10} = 2\tan^{-1}\left(\pm\sqrt{\frac{1-m^2}{(1-m)^2}}\frac{\mathrm{cn}\,\xi}{1\pm\mathrm{sn}\,\xi}\right)\,. \tag{37}$$

Case 11 If $a_0 = (1-m^2)/4$, $a_2 = (1+m^2)/2$, $a_4 = (1-m^2)/4$, then

$$y = \frac{\mathrm{cn}\,\xi}{1 \pm \mathrm{sn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm\sqrt{\frac{1-m^2}{(1+m)^2}}\,,$$
$$c = -\frac{\alpha}{m k^2}, \quad 0 < m < 1\,, \tag{38}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{11} = 2\tan^{-1}\left(\pm\sqrt{\frac{1-m^2}{(1+m)^2}}\frac{\mathrm{cn}\,\xi}{1\pm\mathrm{sn}\,\xi}\right)\,. \tag{39}$$

Remark 1 The solutions u_{10} and u_{11} in terms of rational functions have not been reported in the literature, they are new solutions to SSG Eq. (1).

Remark 2 The solutions from u_2 to u_{11} in terms of Jacobian elliptic functions have not been given in Ref. [11].

Remark 3 In Ref. [10], Peng solved DSG equation in form of

$$u_{xt} = \sin u + \lambda\,\sin(2u)\,, \tag{40}$$

and obtained some solutions in terms of Jacobian elliptic functions. He pointed out that he can obtain solutions to SSG Eq. (1) with $\alpha = 1$ when $\lambda = 0$. However, his conclusion is wrong, for the coefficients of solutions he obtained in terms of sn (Eq. (40) in Ref. [10], dn (Eq. (46) in Ref. [10], ns (Eq. (51) in Ref. [10]) and dc (Eq. (52) in Ref. [10]) are imaginary, but in fact they should be real. For example, from the constraint (41) in Ref. [10]

$$(1-m^2)^2 k^2\omega^2 - 4\lambda(1+m^2)k\omega + 4\lambda^2 - 1 = 0\,, \tag{41}$$

if $\lambda = 0$, then we have

$$k\omega = \pm\frac{1}{1-m^2}\,. \tag{42}$$

Substituting Eq. (42) into solution (40) in Ref. [10],

$$u = 2\arctan\left(\pm\sqrt{\frac{-(1+m^2)k\omega + 2\lambda + 1}{2k\omega}} \times \mathrm{sn}\,(kx - \omega t)\right)\,, \tag{43}$$

we can derive the coefficient $\sqrt{[-(1+m^2)k\omega + 1]/2k\omega}$ to be $\mathrm{i}\,m$ or i, with $\mathrm{i} \equiv \sqrt{-1}$.

So solutions given by Peng in terms of sn (Eq. (40) in Ref. [10]), dn (Eq. (46) in Ref. [10]), ns (Eq. (51) in Ref. [10]) and dc (Eq. (52) in Ref. [10]) are not real solutions, which is contrary to the origin of SSG Eq. (1).

Remark 4 Based on the above results, we can see that when the auxiliary equation, such as elliptic equation (9), is applied to solve nonlinear evolution equations, the constraints must be involved, otherwise, the obtained solutions may be trivial.

3 The Second Kind of Transformation and Solutions to SSG Equation

The second transformation under consideration is

$$u = 2\tan^{-1}\Big(\frac{1}{v}\Big) \quad \text{or} \quad \frac{1}{v} = \tan\frac{u}{2}\,, \tag{44}$$

which has been introduced in Ref. [7] to solve DSG equation.

When the transformation (44) is considered, the SSG Eq. (1) can be rewritten as

$$(1+v^2)v_{tx} - 2vv_tv_x + \alpha v + \alpha v^3 = 0\,. \tag{45}$$

We can see that the difference between Eq. (5) and Eq. (45) is that the $-\alpha$ in Eq. (5) is replaced by α in Eq. (45), so the solutions to Eq. (1) under the transformation (44) can be easily obtained by replacing α by $-\alpha$ and v by $1/v$ in solutions from u_1 to u_{11}.

Case 1 If $a_0 = 0$, $a_2 = 1$, $a_4 = 1$, then the solution to SSG Eq. (1) is

$$u_{12} = 2\tan^{-1}(\pm\sinh\xi)\,. \tag{46}$$

Case 2 If $a_0 = 1 - m^2$, $a_2 = 2m^2 - 1$, $a_4 = -m^2$, then the solution to SSG Eq. (1) is u_3.

Case 3 If $a_0 = -m^2$, $a_2 = 2m^2 - 1$, $a_4 = 1 - m^2$, then the solution to SSG Eq. (1) is u_2.

Case 4 If $a_0 = 1$, $a_2 = 2m^2 - 1$, $a_4 = (m^2 - 1)m^2$, then the solution to SSG Eq. (1) is u_7.

Case 5 If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$, $a_4 = 1$, then the solution to SSG Eq. (1) is u_8.

Case 6 If $a_0 = m^2(m^2 - 1)$, $a_2 = 2m^2 - 1$, $a_4 = 1$, then the solution to SSG Eq. (1) is u_4.

Case 7 If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$, $a_4 = 1$, then the solution to SSG Eq. (1) is u_9.

Case 8 If $a_0 = 1$, $a_2 = 2 - m^2$, $a_4 = 1 - m^2$, then the solution to SSG Eq. (1) is u_5.

Case 9 If $a_0 = 1$, $a_2 = 2 - m^2$, $a_4 = 1 - m^2$, then the solution to SSG Eq. (1) is u_6.

Case 10 If $a_0 = (1 - m^2)/4$, $a_2 = (1 + m^2)/2$, $a_4 = (1 - m^2)/4$, then the solution to SSG Eq. (1) is

$$u_{13} = 2\tan^{-1}\left(\pm\sqrt{\frac{(1-m)^2}{1-m^2}}\frac{1 \pm \operatorname{sn}\xi}{\operatorname{cn}\xi}\right). \tag{47}$$

Case 11 If $a_0 = (1 - m^2)/4$, $a_2 = (1 + m^2)/2$, $a_4 = (1 - m^2)/4$, then the solution to SSG Eq. (1) is

$$u_{14} = 2\tan^{-1}\left(\pm\sqrt{\frac{(1+m)^2}{1-m^2}}\frac{1 \pm \operatorname{sn}\xi}{\operatorname{cn}\xi}\right). \tag{48}$$

Remark Most of the solutions from u_{12} to u_{14} in terms of Jacobian elliptic functions have not been given in the literature.

4 The Third Kind of Transformation and Solutions to SSG Equation

The third transformation is introduced in the form

$$u = 2\sin^{-1} v \quad \text{or} \quad v = \sin\frac{u}{2}, \tag{49}$$

and then the SSG equation (1) can be rewritten as

$$(1 - v^2)v_{tx} + vv_t v_x - \alpha v(1 - v^2)^2 = 0. \tag{50}$$

In the travelling wave frame (6), the formal solution of Eq. (50) by the elliptic equation expansion method (8) can be written as Eq. (15). Substituting Eq. (15) into Eq. (50), we have

$$b_0 = 0, \quad b_1 = \pm\sqrt{\frac{a_4 k^2 c}{\alpha}},$$
$$\frac{\alpha}{k^2 c} = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_0 a_4}}{2} \tag{51}$$

with constraints

$$\frac{a_4 k^2 c}{\alpha} > 0, \quad a_2^2 - 4a_0 a_4 \geq 0. \tag{52}$$

Similarly, there are some cases to consider. For example,

Case 1 If $a_0 = 1$, $a_2 = -(1 + m^2)$, $a_4 = m^2$, then

$$y = \operatorname{sn}\xi, \quad b_0 = 0, \quad b_1 = \pm m,$$
$$c = \frac{\alpha}{k^2}, \quad 0 < m \leq 1, \tag{53}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{15} = 2\sin^{-1}(\pm m \operatorname{sn}\xi). \tag{54}$$

Case 2 If $a_0 = 1$, $a_2 = -(1 + m^2)$, $a_4 = m^2$, then

$$y = \operatorname{sn}\xi, \quad b_0 = 0, \quad b_1 = \pm 1,$$
$$c = \frac{\alpha}{k^2 m^2}, \quad 0 < m \leq 1, \tag{55}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{16} = 2\sin^{-1}(\pm \operatorname{sn}\xi). \tag{56}$$

Actually, when $m \to 1$, u_{15} and u_{16} all recover,

$$u_{17} = 2\sin^{-1}(\pm\tanh\xi), \quad c = \frac{\alpha}{k^2}. \tag{57}$$

Case 3 If $a_0 = 1 - m^2$, $a_2 = 2m^2 - 1$, $a_4 = -m^2$, then

$$y = \operatorname{cn}\xi, \quad b_0 = 0, \quad b_1 = \pm 1,$$
$$c = -\frac{\alpha}{k^2 m^2}, \quad 0 < m \leq 1, \tag{58}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{18} = 2\sin^{-1}(\pm \operatorname{cn}\xi). \tag{59}$$

Actually, when $m \to 1$, u_{18} recovers

$$u_{19} = 2\sin^{-1}(\pm \operatorname{sech}\xi), \quad c = -\frac{\alpha}{k^2}. \tag{60}$$

Case 4 If $a_0 = m^2 - 1$, $a_2 = 2 - m^2$, $a_4 = -1$, then

$$y = \operatorname{dn}\xi, \quad b_0 = 0, \quad b_1 = \pm 1,$$
$$c = -\frac{\alpha}{k^2}, \quad 0 < m \leq 1, \tag{61}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{20} = 2\sin^{-1}(\pm \operatorname{dn}\xi). \tag{62}$$

Actually, when $m \to 1$, u_{20} recovers u_{19}.

Case 5 If $a_0 = m^2 - 1$, $a_2 = 2 - m^2$, $a_4 = -1$, then

$$y = \operatorname{dn}\xi, \quad b_0 = 0, \quad b_1 = \pm\frac{1}{\sqrt{1 - m^2}},$$
$$c = \frac{\alpha}{(m^2 - 1)k^2}, \quad 0 < m < 1, \tag{63}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{21} = 2\sin^{-1}\left(\pm\frac{1}{\sqrt{1 - m^2}}\operatorname{dn}\xi\right). \tag{64}$$

Case 6 If $a_0 = m^2$, $a_2 = -(1+m^2)$, $a_4 = 1$, then

$$y = \mathrm{ns}\,\xi, \quad b_0 = 0, \quad b_1 = \pm\frac{1}{m}\,,$$
$$c = \frac{\alpha}{k^2 m^2}, \quad 0 < m \le 1\,, \tag{65}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{22} = 2\,\sin^{-1}\Big(\pm\frac{1}{m}\mathrm{ns}\,\xi\Big)\,. \tag{66}$$

Case 7 If $a_0 = m^2$, $a_2 = -(1+m^2)$, $a_4 = 1$, then

$$y = \mathrm{ns}\,\xi, \quad b_0 = 0, \quad b_1 = \pm 1\,,$$
$$c = \frac{\alpha}{k^2}, \quad 0 < m \le 1\,, \tag{67}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{23} = 2\,\sin^{-1}(\pm\,\mathrm{ns}\,\xi)\,. \tag{68}$$

Actually, when $m \to 1$, u_{22} and u_{23} all recover,

$$u_{24} = 2\,\sin^{-1}(\pm\coth\xi), \quad c = \frac{\alpha}{k^2}\,. \tag{69}$$

Case 8 If $a_0 = -m^2$, $a_2 = 2m^2 - 1$, $a_4 = 1 - m^2$, then

$$y = \mathrm{nc}\,\xi, \quad b_0 = 0, \quad b_1 = \pm 1\,,$$
$$c = \frac{\alpha}{(1-m^2)k^2}, \quad 0 < m < 1\,, \tag{70}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{25} = 2\,\sin^{-1}(\pm\,\mathrm{nc}\,\xi)\,. \tag{71}$$

Case 9 If $a_0 = -1$, $a_2 = 2 - m^2$, $a_4 = m^2 - 1$, then

$$y = \mathrm{nd}\,\xi, \quad b_0 = 0, \quad b_1 = \pm 1\,,$$
$$c = \frac{\alpha}{(m^2-1)k^2}, \quad 0 < m < 1\,, \tag{72}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{26} = 2\,\sin^{-1}(\pm\,\mathrm{nd}\,\xi)\,. \tag{73}$$

Case 10 If $a_0 = -1$, $a_2 = 2 - m^2$, $a_4 = m^2 - 1$, then

$$y = \mathrm{nd}\,\xi, \quad b_0 = 0, \quad b_1 = \pm\sqrt{1-m^2}\,,$$
$$c = -\frac{\alpha}{k^2}, \quad 0 < m < 1\,, \tag{74}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{27} = 2\,\sin^{-1}(\pm\sqrt{1-m^2}\mathrm{nd}\,\xi)\,. \tag{75}$$

Case 11 If $a_0 = 1$, $a_2 = 2m^2 - 1$, $a_4 = (m^2-1)m^2$, then

$$y = \mathrm{sd}\,\xi, \quad b_0 = 0, \quad b_1 = \pm\sqrt{1-m^2}\,,$$
$$c = -\frac{\alpha}{m^2 k^2}, \quad 0 < m < 1\,, \tag{76}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{28} = 2\,\sin^{-1}(\pm\sqrt{1-m^2}\mathrm{sd}\,\xi)\,. \tag{77}$$

Case 12 If $a_0 = 1$, $a_2 = -(1+m^2)$, $a_4 = m^2$, then

$$y = \mathrm{cd}\,\xi, \quad b_0 = 0, \quad b_1 = \pm m\,,$$
$$c = \frac{\alpha}{k^2}, \quad 0 < m \le 1\,, \tag{78}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{29} = 2\,\sin^{-1}(\pm\,m\,\mathrm{cd}\,\xi)\,. \tag{79}$$

Case 13 If $a_0 = 1$, $a_2 = -(1+m^2)$, $a_4 = m^2$, then

$$y = \mathrm{cd}\,\xi, \quad b_0 = 0, \quad b_1 = \pm 1\,,$$
$$c = \frac{\alpha}{m^2 k^2}, \quad 0 < m \le 1\,, \tag{80}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{30} = 2\,\sin^{-1}(\pm\,\mathrm{cd}\,\xi)\,. \tag{81}$$

Case 14 If $a_0 = m^2(m^2-1)$, $a_2 = 2m^2 - 1$, $a_4 = 1$, then

$$y = \mathrm{ds}\,\xi, \quad b_0 = 0, \quad b_1 = \pm\frac{1}{\sqrt{1-m^2}}\,,$$
$$c = \frac{\alpha}{(1-m^2)k^2}, \quad 0 < m < 1\,, \tag{82}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{31} = 2\,\sin^{-1}\left(\pm\frac{1}{\sqrt{1-m^2}}\mathrm{ds}\,\xi\right)\,. \tag{83}$$

Case 15 If $a_0 = m^2$, $a_2 = -(1+m^2)$, $a_4 = 1$, then

$$y = \mathrm{dc}\,\xi, \quad b_0 = 0, \quad b_1 = \pm\frac{1}{m}\,,$$
$$c = \frac{\alpha}{m^2 k^2}, \quad 0 < m \le 1\,, \tag{84}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{32} = 2\,\sin^{-1}\Big(\pm\frac{1}{m}\mathrm{dc}\,\xi\Big)\,. \tag{85}$$

Case 16 If $a_0 = m^2$, $a_2 = -(1+m^2)$, $a_4 = 1$, then

$$y = \mathrm{dc}\,\xi, \quad b_0 = 0, \quad b_1 = \pm 1\,,$$
$$c = \frac{\alpha}{k^2}, \quad 0 < m \le 1\,, \tag{86}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{33} = 2\,\sin^{-1}(\pm\mathrm{dc}\,\xi)\,. \tag{87}$$

Apart from these Jacobian elliptic solutions, y also has some rational solutions in terms of Jacobian elliptic functions. For example,

Case 17 If $a_0 = -(1-m^2)/4$, $a_2 = (1+m^2)/2$, $a_4 = -(1-m^2)/4$, then

$$y = \frac{\mathrm{dn}\,\xi}{1 \pm m\,\mathrm{sn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm\frac{\sqrt{1-m^2}}{1+m},$$
$$c = -\frac{4\alpha}{(1+m)^2k^2}, \quad 0 < m < 1, \tag{88}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{34} = 2\,\sin^{-1}\left(\pm\frac{\sqrt{1-m^2}}{1+m}\frac{\mathrm{dn}\,\xi}{1 \pm m\,\mathrm{sn}\,\xi}\right). \tag{89}$$

Case 18 If $a_0 = -(1-m^2)/4$, $a_2 = (1+m^2)/2$, $a_4 = -(1-m^2)/4$, then

$$y = \frac{\mathrm{dn}\,\xi}{1 \pm m\,\mathrm{sn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm\frac{\sqrt{1-m^2}}{1-m},$$
$$c = -\frac{4\alpha}{(1-m)^2k^2}, \quad 0 < m < 1, \tag{90}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{35} = 2\,\sin^{-1}\left(\pm\frac{\sqrt{1-m^2}}{1-m}\frac{\mathrm{dn}\,\xi}{1 \pm m\,\mathrm{sn}\,\xi}\right). \tag{91}$$

Case 19 If $a_0 = m^2/4$, $a_2 = -(2-m^2)/2$, $a_4 = m^2/4$, then

$$y = \frac{m\,\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm\frac{m}{\sqrt{2-m^2+2\sqrt{1-m^2}}},$$
$$c = \frac{4\alpha}{(\sqrt{2-m^2+2\sqrt{1-m^2}})k^2}, \quad 0 < m \le 1, \tag{92}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{36} = 2\,\sin^{-1}\left(\pm\frac{m}{\sqrt{2-m^2+2\sqrt{1-m^2}}}\frac{m\,\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\,\xi}\right). \tag{93}$$

Case 20 If $a_0 = m^2/4$, $a_2 = -(2-m^2)/2$, $a_4 = m^2/4$, then

$$y = \frac{m\,\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm\frac{m}{\sqrt{2-m^2-2\sqrt{1-m^2}}},$$
$$c = \frac{4\alpha}{(\sqrt{2-m^2-2\sqrt{1-m^2}})k^2}, \quad 0 < m \le 1, \tag{94}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{37} = 2\,\sin^{-1}\left(\pm\frac{m}{\sqrt{2-m^2-2\sqrt{1-m^2}}}\frac{m\,\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\,\xi}\right). \tag{95}$$

When $m \to 1$, u_{36} and u_{37} all recover

$$u_{38} = 2\,\sin^{-1}\left(\pm\frac{\tanh\xi}{1 \pm \mathrm{sech}\,\xi}\right), \quad c = \frac{4\alpha}{k^2}. \tag{96}$$

Case 21 If $a_0 = 1/4$, $a_2 = -(2-m^2)/2$, $a_4 = m^4/4$, then

$$y = \frac{\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm\frac{m^2}{\sqrt{2-m^2+2\sqrt{1-m^2}}},$$
$$c = \frac{4\alpha}{(\sqrt{2-m^2+2\sqrt{1-m^2}})k^2}, \quad 0 < m \le 1, \tag{97}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{39} = 2\,\sin^{-1}\left(\pm\frac{m^2}{\sqrt{2-m^2+2\sqrt{1-m^2}}}\frac{\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\,\xi}\right). \tag{98}$$

Case 22 If $a_0 = 1/4$, $a_2 = -(2-m^2)/2$, $a_4 = (m^4)/4$, then

$$y = \frac{\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\,\xi}, \quad b_0 = 0, \quad b_1 = \pm\frac{m^2}{\sqrt{2-m^2-2\sqrt{1-m^2}}},$$
$$c = \frac{4\alpha}{(\sqrt{2-m^2-2\sqrt{1-m^2}})k^2}, \quad 0 < m \le 1, \tag{99}$$

where k is an arbitrary constant. So the solution to SSG equation (1) is

$$u_{40} = 2\,\sin^{-1}\left(\pm\frac{m^2}{\sqrt{2-m^2-2\sqrt{1-m^2}}}\frac{\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\,\xi}\right). \tag{100}$$

When $m \to 1$, u_{39} and u_{40} all recover u_{38}.

Remark Transformation (49) and the solutions from u_{15} to u_{40} in terms of Jacobian elliptic functions have not been given in the literature.

5 The Fourth Kind of Transformation and Solutions to SSG Equation

The fourth transformation is introduced in the form

$$u = 2\,\cos^{-1} v \quad \text{or} \quad v = \cos\frac{u}{2}, \tag{101}$$

and then the SSG equation (1) can be rewritten as

$$(1-v^2)v_{tx} + vv_tv_x + \alpha v(1-v^2)^2 = 0. \tag{102}$$

We can see that the difference between Eq. (102) and Eq. (50) is that the $-\alpha$ in Eq. (50) is replaced by α in Eq. (102), so the solutions to Eq. (1) under the transformation (101) can be easily obtained by replacing α by $-\alpha$ and sin by cos in solutions from u_{15} to u_{40}. So we have

$$b_0 = 0, \quad b_1 = \pm\sqrt{-\frac{a_4k^2c}{\alpha}},$$
$$\frac{\alpha}{k^2c} = -\frac{-a_2 \pm \sqrt{a_2^2 - 4a_0a_4}}{2}, \tag{103}$$

with constraints

$$-\frac{a_4k^2c}{\alpha} > 0, \quad a_2^2 - 4a_0a_4 \ge 0. \tag{104}$$

Similarly, there are some cases to be considered. For example,

Case 1 If $a_0 = 1$, $a_2 = -(1+m^2)$, $a_4 = m^2$, then the solution to SSG equation (1) is

$$u_{41} = 2\cos^{-1}(\pm m \,\mathrm{sn}\,\xi)\,. \tag{105}$$

Case 2 If $a_0 = 1$, $a_2 = -(1+m^2)$, $a_4 = m^2$, then the solution to SSG equation (1) is

$$u_{42} = 2\cos^{-1}(\pm \mathrm{sn}\,\xi)\,. \tag{106}$$

Actually, when $m \to 1$, u_{41} and u_{42} all recover

$$u_{43} = 2\cos^{-1}(\pm \tanh \xi), \quad c = -\frac{\alpha}{k^2}\,. \tag{107}$$

with

Case 3 If $a_0 = 1 - m^2$, $a_2 = 2m^2 - 1$, $a_4 = -m^2$, then the solution to SSG equation (1) is

$$u_{44} = 2\cos^{-1}(\pm \mathrm{cn}\,\xi)\,. \tag{108}$$

Actually, when $m \to 1$, u_{44} recovers

$$u_{45} = 2\cos^{-1}(\pm \mathrm{sech}\,\xi), \quad c = -\frac{\alpha}{k^2}\,. \tag{109}$$

Case 4 If $a_0 = m^2 - 1$, $a_2 = 2 - m^2$, $a_4 = -1$, then the solution to SSG equation (1) is

$$u_{46} = 2\cos^{-1}(\pm \mathrm{dn}\,\xi)\,. \tag{110}$$

Actually, when $m \to 1$, u_{46} recovers u_{45}.

Case 5 If $a_0 = m^2 - 1$, $a_2 = 2 - m^2$, $a_4 = -1$, then the solution to SSG equation (1) is

$$u_{47} = 2\cos^{-1}\Big(\pm\frac{1}{\sqrt{1-m^2}}\mathrm{dn}\,\xi\Big)\,. \tag{111}$$

Case 6 If $a_0 = m^2$, $a_2 = -(1+m^2)$, $a_4 = 1$, then the solution to SSG equation (1) is

$$u_{48} = 2\cos^{-1}\Big(\pm\frac{1}{m}\mathrm{ns}\,\xi\Big)\,. \tag{112}$$

Case 7 If $a_0 = m^2$, $a_2 = -(1+m^2)$, $a_4 = 1$, then the solution to SSG equation (1) is

$$u_{49} = 2\cos^{-1}(\pm \mathrm{ns}\,\xi)\,. \tag{113}$$

Actually, when $m \to 1$, u_{48} and u_{49} all recover

$$u_{50} = 2\cos^{-1}(\pm \coth \xi), \quad c = -\frac{\alpha}{k^2}\,. \tag{114}$$

Case 8 If $a_0 = -m^2$, $a_2 = 2m^2 - 1$, $a_4 = 1 - m^2$, then the solution to SSG equation (1) is

$$u_{51} = 2\cos^{-1}(\pm \mathrm{nc}\,\xi)\,. \tag{115}$$

Case 9 If $a_0 = -1$, $a_2 = 2 - m^2$, $a_4 = m^2 - 1$, then the solution to SSG equation (1) is

$$u_{52} = 2\cos^{-1}(\pm \mathrm{nd}\,\xi)\,. \tag{116}$$

Case 10 If $a_0 = -1$, $a_2 = 2 - m^2$, $a_4 = m^2 - 1$, then the solution to SSG equation (1) is

$$u_{53} = 2\cos^{-1}\left(\pm\sqrt{1-m^2}\ \mathrm{nd}\,\xi\right)\,. \tag{117}$$

Case 11 If $a_0 = 1$, $a_2 = 2m^2 - 1$, $a_4 = (m^2 - 1)m^2$, then the solution to SSG equation (1) is

$$u_{54} = 2\cos^{-1}\left(\pm\sqrt{1-m^2}\ \mathrm{sd}\,\xi\right)\,. \tag{118}$$

Case 12 If $a_0 = 1$, $a_2 = -(1+m^2)$, $a_4 = m^2$, then the solution to SSG equation (1) is

$$u_{55} = 2\cos^{-1}(\pm m \,\mathrm{cd}\,\xi)\,. \tag{119}$$

Case 13 If $a_0 = 1$, $a_2 = -(1+m^2)$, $a_4 = m^2$, then the solution to SSG equation (1) is

$$u_{56} = 2\cos^{-1}(\pm \mathrm{cd}\,\xi)\,. \tag{120}$$

Case 14 If $a_0 = m^2(m^2 - 1)$, $a_2 = 2m^2 - 1$, $a_4 = 1$, then the solution to SSG equation (1) is

$$u_{57} = 2\cos^{-1}\left(\pm\frac{1}{\sqrt{1-m^2}}\mathrm{ds}\,\xi\right)\,. \tag{121}$$

Case 15 If $a_0 = m^2$, $a_2 = -(1+m^2)$, $a_4 = 1$, then the solution to SSG equation (1) is

$$u_{58} = 2\cos^{-1}\Big(\pm\frac{1}{m}\mathrm{dc}\,\xi)\Big)\,. \tag{122}$$

Case 16 If $a_0 = m^2$, $a_2 = -(1+m^2)$, $a_4 = 1$, then the solution to SSG equation (1) is

$$u_{59} = 2\cos^{-1}(\pm \mathrm{dc}\,\xi)\,. \tag{123}$$

Apart from these Jacobian elliptic solutions, y also has some rational solutions in terms of Jacobian elliptic functions, for example,

Case 17 If $a_0 = -(1-m^2)/4$, $a_2 = (1+m^2)/2$, $a_4 = -(1-m^2)/4$, then the solution to SSG equation (1) is

$$u_{60} = 2\cos^{-1}\left(\pm\frac{\sqrt{1-m^2}}{1+m}\frac{\mathrm{dn}\,\xi}{1\pm m\,\mathrm{sn}\,\xi}\right)\,. \tag{124}$$

Case 18 If $a_0 = -(1-m^2)/4$, $a_2 = (1+m^2)/2$, $a_4 = -(1-m^2)/4$, then the solution to SSG equation (1) is

$$u_{61} = 2\cos^{-1}\left(\pm\frac{\sqrt{1-m^2}}{1-m}\frac{\mathrm{dn}\,\xi}{1\pm m\,\mathrm{sn}\,\xi}\right)\,. \tag{125}$$

Case 19 If $a_0 = m^2/4$, $a_2 = -(2-m^2)/2$, $a_4 = m^2/4$, then the solution to SSG equation (1) is

$$u_{62} = 2\cos^{-1}\left(\pm\frac{m}{\sqrt{2-m^2+2\sqrt{1-m^2}}}\frac{m\,\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi}\right)\,. \tag{126}$$

Case 20 If $a_0 = m^2/4$, $a_2 = -(2-m^2)2$, $a_4 = m^2/4$, then the solution to SSG equation (1) is

$$u_{63} = 2\cos^{-1}\left(\pm\frac{m}{\sqrt{2-m^2-2\sqrt{1-m^2}}}\frac{m\,\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi}\right)\,. \tag{127}$$

When $m \to 1$, u_{64} and u_{65} all recover

$$u_{64} = 2\cos^{-1}\left(\pm\frac{\tanh\xi}{1\pm\mathrm{sech}\,\xi}\right), \quad c = \frac{4\alpha}{k^2}\,. \tag{128}$$

Case 21 If $a_0 = 1/4$, $a_2 = -(2-m^2)/2$, $a_4 = m^4/4$, then the solution to SSG equation (1) is

$$u_{65} = 2\cos^{-1}\left(\pm\frac{m^2}{\sqrt{2-m^2+2\sqrt{1-m^2}}}\frac{\operatorname{sn}\xi}{1\pm\operatorname{dn}\xi}\right). \quad (129)$$

Case 22 If $a_0 = 1/4$, $a_2 = -(2-m^2)/2$, $a_4 = m^4/4$, then the solution to SSG equation (1) is

$$u_{66} = 2\cos^{-1}\left(\pm\frac{m^2}{\sqrt{2-m^2-2\sqrt{1-m^2}}}\frac{\operatorname{sn}\xi}{1\pm\operatorname{dn}\xi}\right). \quad (130)$$

When $m \to 1$, u_{65} and u_{66} all recover u_{64}.

Remark Transformation (101) and the solutions from u_{41} to u_{66} in terms of Jacobian elliptic functions have not been given in the literature.

6 Conclusion

In this paper, four transformations are introduced to solve single sine-Gordon equation by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different transformations are required in order to obtain more kinds of solutions to the single sine-Gordon equation. Here some new solutions have not been reported in the literature. It is shown that different transformations play different roles in obtaining exact solutions, some transformations may not work for a specific parameter of SSG equation (1). Of course, there are still more efforts needed to explore what kinds of transformations are more suitable to solving sine-Gordon equation. Because different transformations result in different partial balances for sine-Gordon equation, which will lead to different expansion truncations in the elliptic equation expansion method. Finally, these will result in different solutions of the sine-Gordon equation.

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