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Breather solutions and breather lattice solutions to the sine-Gordon equation

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Abstract
In this paper, dependent and independent variable transformations are introduced to solve the sine–Gordon (SG) equation by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different kinds of solutions can be obtained for the (SG) equation, including breather solutions and breather lattice solutions.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Sine–Gordon (SG)-type equations, including the (SG) equation [1, 2]

$$u_{xt} = \sin u, \tag{1}$$

the double SG (DSG) equation

$$u_{xt} = \alpha \sin u + \beta \sin 2u, \tag{2}$$

and the triple SG (TSG) equation

$$u_{xt} = \alpha \sin u + \beta \sin 2u + \gamma \sin 3u, \tag{3}$$

are widely applied in physics and engineering. For example, the DSG equation is a frequent object of study in numerous physical applications, such as Josephson arrays, ferromagnetic materials, charge density waves and smectic liquid crystal dynamics [3–7]. Actually, SG and DSG equations also arise in nonlinear optics ^{3}He spin waves and other fields. For example, in the context of differential geometry, the SG equation's solutions correspond to surfaces of constant negative curvature [8], and these solutions are spatially periodic, they can be a soliton lattice and a breather lattice. Usually, breathers can directly affect electronic, optical, and transport properties of a material [9–12] due to the breather's spatial localization and temporally periodic characteristics. Specifically, they can enhance optical nonlinearities in polyenes and related low-dimensional electronic materials [9, 10]. In a resonant five-fold degenerate medium, the propagation and creation of ultra-short optical pulses, the SG and DSG models are usually used. However, in some cases, one has to consider other SG equations. For instance, the TSG equation is used to describe the propagation of strictly resonant sharp line optical pulses [13].

Due to the wide applications of SG-type equations, many solutions to them, such as $\tan^{-1}\coth s$, $\tan^{-1}\tanh s$, $\tan^{-1}\mathrm{sech}\, s$, $\tan^{-1}\mathrm{sn}\, s$ and so on, have been obtained in different functional forms by different methods [1, 2, 14–23]. Besides these solutions, there is a particularly interesting type of solution called the breather solution and the breather lattice solution. Typically analytical expressions for these breather-type solutions are unavailable and such solutions have to be traced by means of a numerical method [12, 24, 25]. In this paper, based on the introduced transformations, we will show systematical results about these breather-type solutions for the SG equation (1) by using the knowledge of elliptic equation and Jacobian elliptic functions [26–30].

2. The breather solution and breather lattice solutions to the SG equation

In order to solve the SG-type equations, certain dependent or independent variable transformations must be introduced. For example, the dependent variable transformation

$$u = 4\tan^{-1} v \quad \text{or} \quad v = \tan\frac{u}{4}, \tag{4}$$

has been introduced in [1, 2, 14] to solve the SG equation and the DSG equation.

So, in order to derive the breather solution and breather lattice solutions to the SG equation (1), first of all, we introduce independent variable transformation

$$s = ax + bt + s_0, \qquad r = cx + dt + r_0, \tag{5}$$

where s_0 and r_0 are two constants.

Considering the transformation (5), equation (1) can be rewritten as

$$abu_{ss} + (ad + bc)u_{sr} + cdu_{rr} = \sin u. \tag{6}$$

Compared to the transformation given in [1, 2], transformation (5) has less constraints, of course, this will let us have more different types of solutions to the SG equation (1).

Next, we choose dependent variable transformation

$$u = 4\tan^{-1}\left[\frac{U(s)}{V(r)}\right], \tag{7}$$

just as given in [1, 2].

Substituting (7) into (6) yields

$$ab\left[(U^2+V^2)\frac{U_{ss}}{U} - 2U_s^2\right] + (ad+bc)\left[(U^2-V^2)\frac{U_sV_r}{UV}\right] - cd\left[(U^2+V^2)\frac{V_{rr}}{V} - 2V_r^2\right] = V^2 - U^2. \tag{8}$$

Successive differentiation of this result with respect to both s and r results in

$$2abVV_r\left(\frac{U_{ss}}{U}\right)_s - 2cdUU_s\left(\frac{V_{rr}}{V}\right)_r + (ad+bc)\left[2U_s^2\left(\frac{V_r}{V}\right)_r + (U^2-V^2)\left(\frac{U_s}{U}\right)_s\left(\frac{V_r}{V}\right)_r - 2V_r^2\left(\frac{U_s}{U}\right)_s\right] = 0. \tag{9}$$

In order to separate variables, a, b, c and d must satisfy the condition

$$ad + bc = 0. \tag{10}$$

Thus we have

$$\frac{1}{UU_s}\left(\frac{U_{ss}}{U}\right)_s - \frac{cd}{ab}\frac{1}{VV_r}\left(\frac{V_{rr}}{V}\right)_r = 0. \tag{11}$$

Here we assume that $a^2 + b^2 + c^2 + d^2 \neq 0$, and from (11), one has

$$\frac{1}{UU_s}\left(\frac{U_{ss}}{U}\right)_s = \frac{cd}{ab}\frac{1}{VV_r}\left(\frac{V_{rr}}{V}\right)_r = -4n^2, \tag{12}$$

i.e.

$$U_s^2 = -n^2U^4 + \mu_1U^2 + \nu_1, \qquad V_r^2 = -\frac{ab}{cd}n^2V^4 + \mu_2V^2 + \nu_2. \tag{13}$$

Considering (8), (10) and (13), we have the separated variable relations

$$U_s^2 = -n^2U^4 + \mu_1U^2 + \nu_1, \qquad V_r^2 = \frac{a^2}{c^2}n^2V^4 + \mu_2V^2 + \nu_2, \tag{14}$$

and corresponding constraints

$$a^2\mu_1 - c^2\mu_2 = \frac{a}{b}, \qquad a^2\nu_1 + c^2\nu_2 = 0, \qquad d = -\frac{bc}{a}. \tag{15}$$

Obviously, only when $a = c$, $b = -d$ and $b = \frac{1}{a}$, is (15) the same as given in [1]; actually, we will see below that this will omit some important solutions. Furthermore, not all Jacobi elliptic functions satisfying (14) can satisfy the constraints (15). Only some combinations of these Jacobi elliptic functions are the solutions that the SG equation (1) can admit. Next we will show the details.

First of all, let us examine some special cases, where the solutions can be expressed in terms of elementary functions. For convenience, we set $\mu_1 = p^2$, $\nu_1 = q^2$, then we have

$$\begin{aligned} U_s^2 &= -n^2U^4 + p^2U^2 + q^2, \\ V_r^2 &= \frac{a^2}{c^2}n^2V^4 + \left(\frac{a^2}{c^2}p^2 - \frac{a}{bc^2}\right)V^2 - \frac{a^2}{c^2}q^2. \end{aligned} \tag{16}$$

Case 1. $n = 0$, $a^2p^2 - a/b > 0$, $q = 0$. Here (16) yields

$$u_1 = 4\tan^{-1}\left[\gamma\exp\left(\pm ps \pm \sqrt{\frac{a^2}{c^2}p^2 - \frac{a}{bc^2}}r\right)\right], \tag{17}$$

where γ is integration constant, and solution (17) is called the shelf-shaped solution [1].

Case 2. $n = 0$, $a^2p^2 - a/b > 0$, $q \neq 0$. From (16), one has

$$u_2 = \pm 4\tan^{-1}\left[\sqrt{\frac{a^2bp^2 - a}{a^2bp^2}} \times \frac{\sinh(ps + c_1)}{\cosh(\sqrt{(a^2/c^2)p^2 - a/(bc^2)}r + c_2)}\right], \tag{18}$$

where c_1 and c_2 are constants of integration, and solution (18) represents the collision of two solitons [1].

Case 3. $n \neq 0$, $q = 0$. Here three subcases are of interest.

Case 3a. $a^2p^2 - a/b > 0$. The result is similar to (18). One has

$$u_3 = \pm 4\tan^{-1}\left[\sqrt{\frac{a^2bp^2}{a^2bp^2 - a}} \times \frac{\sinh(\sqrt{(a^2/c^2)p^2 - a/(bc^2)}r + c_2)}{\cosh(ps + c_1)}\right], \tag{19}$$

where c_1 and c_2 are constants of integration.

Case 3b. $a^2p^2 - a/b = 0$. The analytical solution is

$$u_4 = \pm 4\tan^{-1}\left[\left(\frac{ap}{c}r + c_2\right)\operatorname{sech}(ps + c_1)\right], \tag{20}$$

where c_1 and c_2 are constants of integration.

Case 3c. $a^2p^2 - a/b < 0$. We obtain the breather solution

$$u_5 = \pm 4\tan^{-1}\left[\sqrt{\frac{a^2bp^2}{a-a^2bp^2}} \times \frac{\sin\left(\sqrt{a/(bc^2)-(a^2/c^2)p^2}r + c_2\right)}{\cosh(ps+c_1)}\right], \tag{21}$$

where c_1 and c_2 are constants of integration.

The solutions above, expressed in terms of elementary functions have been reported in [1], here we can recover these solutions by using the above mentioned transformations. Apart from these solutions, there are still some solutions expressed in terms of suitable combinations of only some Jacobi elliptic functions, but not all Jacobi elliptic functions. Next, we will show these suitable combinations of some Jacobi elliptic functions to satisfy the SG equation (1); there are 13 cases which need to be addressed.

Case 1. When $U = \mathrm{sn}(s,k)$ and $V = \mathrm{dn}(r,m)$, where $\mathrm{sn}(s,k)$ and $\mathrm{dn}(r,m)$ are the Jacobi sine elliptic function and the Jacobi elliptic function of the third kind, respectively, and k and m are their modulus [28–30]. And then from (14), we have

$$\begin{aligned} n^2 &= -k^2, \quad \mu_1 = -(1+k^2), \quad \nu_1 = 1, \\ \frac{a^2}{c^2}n^2 &= -1, \quad \mu_2 = 2-m^2, \quad \nu_2 = -(1-m^2). \end{aligned} \tag{22}$$

Substituting (22) into the constraints (15), the parameters can be determined as

$$k=1, \quad m=0, \quad a^2=c^2, \quad b=-\frac{1}{4a}, \quad d=\pm\frac{1}{4a}, \tag{23}$$

then the solution to the SG equation (1) is

$$u_6 = 4\tan^{-1}\left[\tanh\left(ax - \frac{1}{4a}t + s_0\right)\right]. \tag{24}$$

Case 2. When $U = \mathrm{ns}(s,k) = \frac{1}{\mathrm{sn}(s,k)}$ and $V = \mathrm{nd}(r,m) = \frac{1}{\mathrm{dn}(r,m)}$, and then from (14), we have

$$\begin{aligned} n^2 &= -1, \quad \mu_1 = -(1+k^2), \quad \nu_1 = k^2, \\ \frac{a^2}{c^2}n^2 &= -(1-m^2), \quad \mu_2 = 2-m^2, \quad \nu_2 = -1. \end{aligned} \tag{25}$$

Substituting (25) into the constraints (15), the parameters can be determined as

$$k=1, \quad m=0, \quad a^2=c^2, \quad b=-\frac{1}{4a}, \quad d=\pm\frac{1}{4a}, \tag{26}$$

then the solution to the SG equation (1) is

$$u_7 = 4\tan^{-1}\left[\coth\left(ax - \frac{1}{4a}t + s_0\right)\right]. \tag{27}$$

The above two solutions, u_6 and u_7 are still two solutions expressed in terms of elementary functions where the special functions can only take their limiting forms.

Case 3. When $U = \mathrm{sn}(s,k)$ and $V = \mathrm{nd}(r,m) = \frac{1}{\mathrm{dn}(r,m)}$, and then from (14), we have

$$\begin{aligned} n^2 &= -k^2, \quad \mu_1 = -(1+k^2), \quad \nu_1 = 1, \\ \frac{a^2}{c^2}n^2 &= -(1-m^2), \quad \mu_2 = 2-m^2, \quad \nu_2 = -1. \end{aligned} \tag{28}$$

Substituting (28) into the constraints (15), the parameters can be determined as

$$k^2 = 1-m^2, \quad a^2 = c^2, \quad b = -\frac{1}{2(2-m^2)a}, \quad d = \pm\frac{1}{2(2-m^2)a}, \tag{29}$$

then the solution to the SG equation (1) is

$$u_8 = 4\tan^{-1}[\mathrm{sn}(s,k)\mathrm{dn}(r,m)]. \tag{30}$$

This is the periodic breather lattice solution given and analysed by [8, 12], when $k \to 0$, i.e. $m \to 1$, $\mathrm{sn}(s,k) \to \sin(s)$, $\mathrm{dn}(r,m) \to \mathrm{sech}(r)$, the breather lattice solution (30) turns out to be a breather solution

$$u_9 = 4\tan^{-1}\left[\frac{\sin(s)}{\cosh(r)}\right]. \tag{31}$$

Figures 1 and 2 describe the space-time evolution of the periodic solution of equations (30) and (31), for different values of m and k, their behaviours are quite different. Figure 1 shows the evolution of the breather lattice solution with the periodic characteristics in both spatial and temporal directions, while figure 2 is just the normal breather solution which has the periodic characteristics just in a specific direction.

Besides the above breather lattice solution, which has been reported elsewhere before, there are still some which have not been reported, and these solutions will be addressed below.

Case 4. When $U = \mathrm{cd}(s,k) = \frac{\mathrm{cn}(s,k)}{\mathrm{dn}(s,k)}$ and $V = \mathrm{nd}(r,m) = \frac{1}{\mathrm{dn}(r,m)}$, where $\mathrm{cn}(s,k)$ is the Jacobi cosine elliptic function [28–30]. And then from (14) and the constraints (15), the parameters can be determined as

$$k^2 = 1-m^2, \quad a^2 = c^2, \quad b = -\frac{1}{2(2-m^2)a}, \quad d = \pm\frac{1}{2(2-m^2)a}, \tag{32}$$

then the solution to the SG equation (1) is

$$u_{10} = 4\tan^{-1}[\mathrm{cd}(s,k)\mathrm{dn}(r,m)], \tag{33}$$

when $k \to 0$, i.e. $m \to 1$, $\mathrm{cd}(s,k) \to \cos(s)$, the breather lattice solution (33) turns to be another breather solution

$$u_{11} = 4\tan^{-1}\left[\frac{\cos(s)}{\cosh(r)}\right]. \tag{34}$$

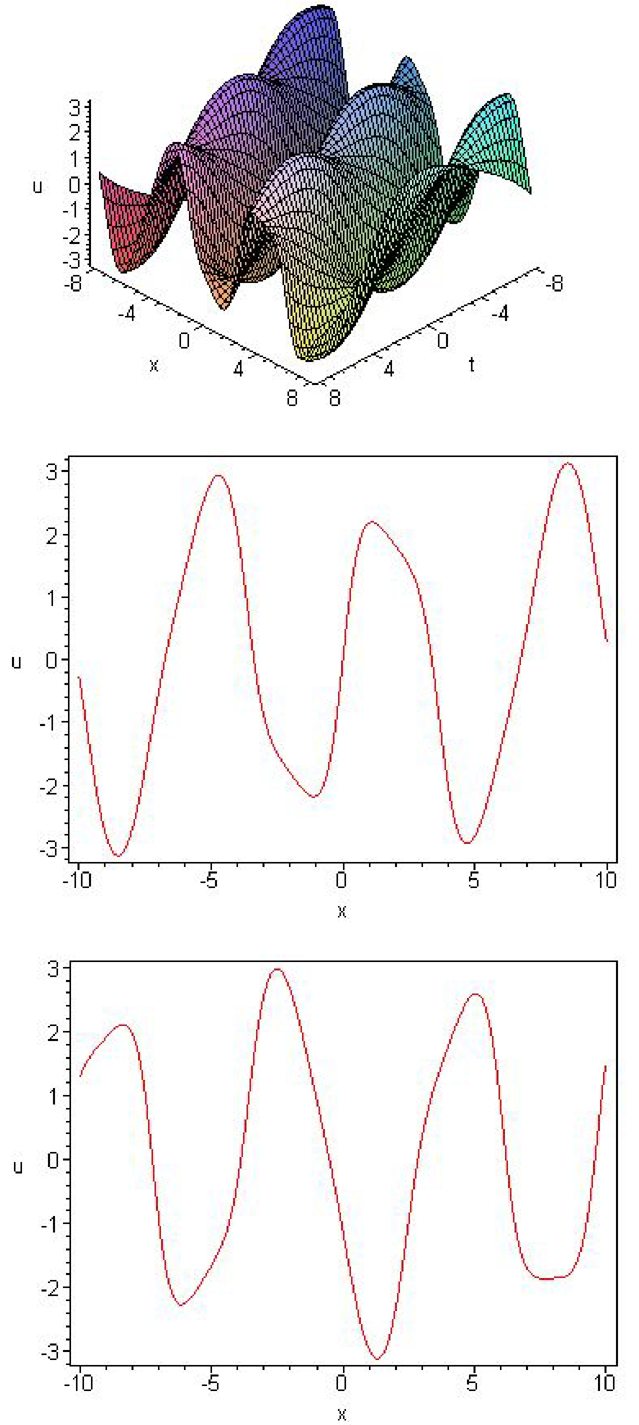

Figure 1. The upper panel shows the space-time evolution of the breather lattice solution of equations (29) and (30), where the parameters are chosen as $a = 1$, $c = 1$, $m = \frac{1}{2}$, $s_0 = r_0 = 0$, from which the other parameters can be determined as $b = -\frac{2}{7}$, $d = \frac{2}{7}$, $k = \frac{\sqrt{3}}{2}$. The middle panel shows the spatial profile at $t = 0$ and the bottom panel at $t = 10$.

Case 5. When $U = \mathrm{cn}(s, k)$ and $V = \mathrm{nc}(r, m) = \frac{1}{\mathrm{cn}(r,m)}$, and then from (14), we have

$$\begin{aligned} &n^2 = k^2, \quad \mu_1 = 2k^2 - 1, \quad \nu_1 = 1 - k^2, \\ &\frac{a^2}{c^2} n^2 = 1 - m^2, \quad \mu_2 = 2m^2 - 1, \quad \nu_2 = -m^2. \end{aligned} \tag{35}$$

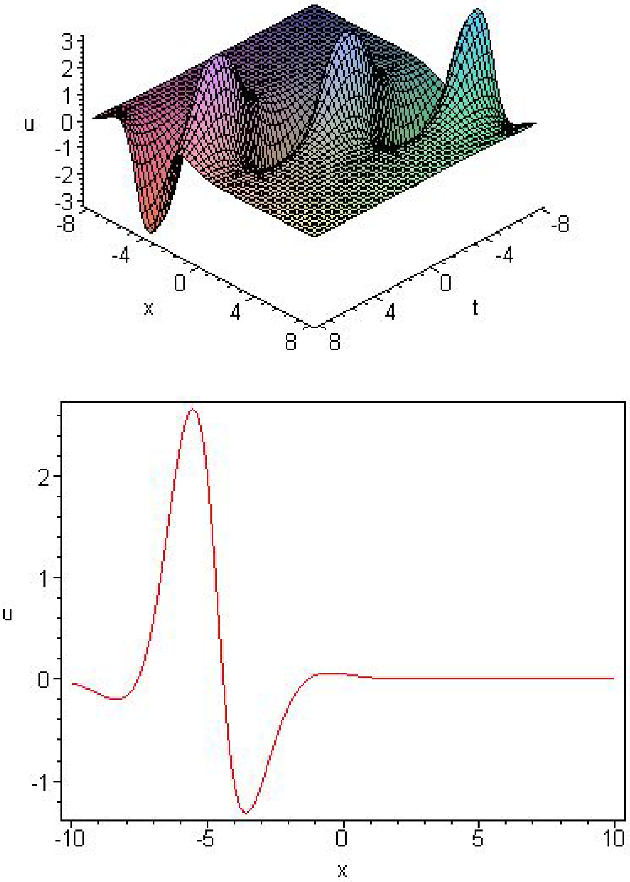

Figure 2. The upper panel shows the space-time evolution of the breather solution of equations (29) and (31), where the parameters are chosen as $a = 1$, $c = 1$, $m = 1$, $s_0 = r_0 = 0$, from which the other parameters can be determined as $b = -\frac{1}{2}$, $d = \frac{1}{2}$, $k = 0$. The bottom panel shows the spatial profile at $t = 10$.

Substituting (35) into the constraints (15), the parameters can be determined as

$$\begin{aligned} k^2 = 1 - m^2, \quad a^2 = c^2, \quad b = \frac{1}{2(1 - 2m^2)a}, \\ d = \pm \frac{1}{2(1 - 2m^2)a}, \end{aligned} \tag{36}$$

then the solution to the SG equation (1) is

$$u_{12} = 4 \tan^{-1}[\mathrm{cn}(s, k)\mathrm{cn}(r, m)]. \tag{37}$$

Case 6. When $U = \mathrm{cn}(s, k)$ and $V = \mathrm{ds}(r, m) = \frac{\mathrm{dn}(r,m)}{\mathrm{sn}(r,m)}$, and then from (14), we have

$$\begin{aligned} &n^2 = k^2, \quad \mu_1 = 2k^2 - 1, \quad \nu_1 = 1 - k^2, \\ &\frac{a^2}{c^2} n^2 = 1, \quad \mu_2 = 2m^2 - 1, \quad \nu_2 = -m^2(1 - m^2). \end{aligned} \tag{38}$$

Substituting (38) into the constraints (15), the parameters can be determined as

$$\begin{aligned} &1 - k^2 = k^2 m^2 (1 - m^2), \quad k^2 a^2 = c^2, \\ &b = \frac{1}{(3k^2 - 2k^2 m^2 - 1)a}, \quad d = -\frac{bc}{a}, \end{aligned} \tag{39}$$

then the solution to the SG equation (1) is

$$u_{13} = 4\tan^{-1}[\mathrm{cn}(s, k)\mathrm{sd}(r, m)]. \tag{40}$$

Case 7. When $U = \mathrm{sd}(s, k)$ and $V = \mathrm{nc}(r, m)$, and then from (14), we have

$$n^2 = k^2(1-k^2), \quad \mu_1 = 2k^2 - 1, \quad \nu_1 = 1,$$
$$\frac{a^2}{c^2}n^2 = 1 - m^2, \quad \mu_2 = 2m^2 - 1, \quad \nu_2 = -m^2. \tag{41}$$

Substituting (41) into the constraints (15), the parameters can be determined as

$$1 - m^2 = m^2k^2(1-k^2), \quad a^2 = m^2c^2,$$
$$b = \frac{m^2}{(2m^2k^2 - 3m^2 + 1)a}, \quad d = -\frac{bc}{a}, \tag{42}$$

then the solution to the SG equation (1) is

$$u_{14} = 4\tan^{-1}[\mathrm{sd}(s, k)\mathrm{cn}(r, m)]. \tag{43}$$

If the independent variable transformation given in [1] is adopted, the breather lattice solutions u_{13} and u_{14} can only take their limiting form (i.e. breather solution)

$$u_{15} = 4\tan^{-1}\left[\frac{\sin(r)}{\cosh(s)}\right], \tag{44}$$

where $k = 1$ and $m = 0$ is chosen in the breather lattice solution u_{13}, $k = 0$ and $m = 1$ in the breather lattice solution u_{14}.

Case 8. When $U = \mathrm{ns}(s, k)$ and $V = \mathrm{dn}(r, m)$, and then from (14), we have

$$n^2 = -1, \quad \mu_1 = -(1 + k^2), \quad \nu_1 = k^2,$$
$$\frac{a^2}{c^2}n^2 = -1, \quad \mu_2 = 2 - m^2, \quad \nu_2 = -(1 - m^2). \tag{45}$$

Substituting (45) into the constraints (15), the parameters can be determined as

$$k^2 = 1 - m^2, \quad a^2 = c^2,$$
$$b = -\frac{1}{2(2 - m^2)a}, \quad d = \pm\frac{1}{2(2 - m^2)a}, \tag{46}$$

then the solution to the SG equation (1) is

$$u_{16} = 4\tan^{-1}[\mathrm{ns}(s, k)\mathrm{nd}(r, m)], \tag{47}$$

when $k \to 0$, i.e. $m \to 1$, the breather lattice solution (47) turns out to be another kind of breather solution

$$u_{17} = 4\tan^{-1}\left[\frac{\cosh(r)}{\sin(s)}\right]. \tag{48}$$

Figure 3 describes the space-time evolution of another breather solution of equations (46) and (48), where its behaviour is quite different from that shown in figure 2.

Case 9. When $U = \mathrm{dc}(s, k)$ and $V = \mathrm{dn}(r, m)$, and then from (14), we have the same parameters determined as (46) and then the solution to the SG equation (1) is

$$u_{18} = 4\tan^{-1}[\mathrm{dc}(s, k)\mathrm{nd}(r, m)]. \tag{49}$$

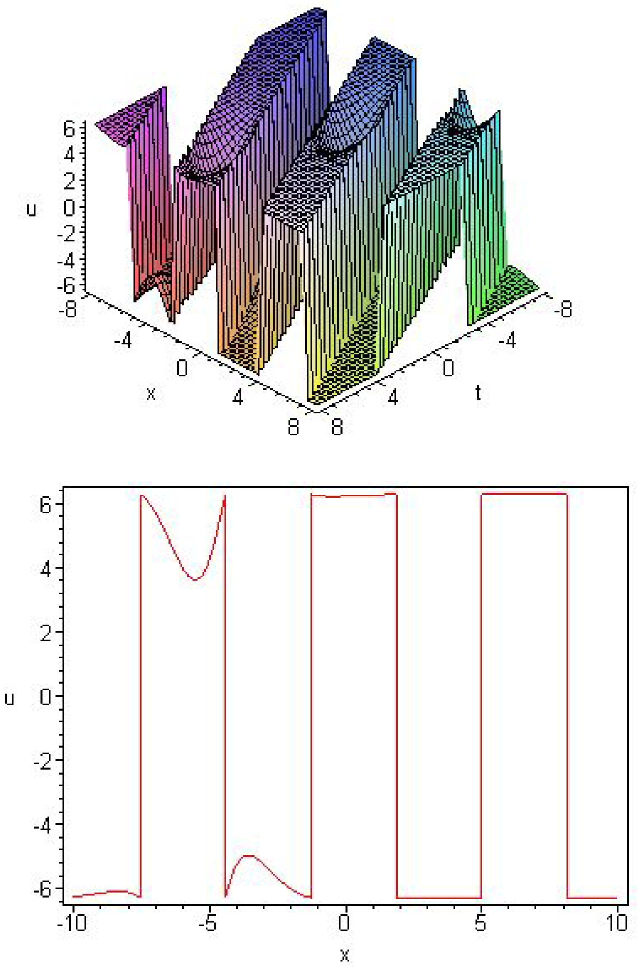

Figure 3. The upper panel shows the space-time evolution of the breather solution of equations (46) and (48), where the parameters are chosen as $a = 1$, $c = 1$, $m = 1$, $s_0 = r_0 = 0$, from which the other parameters can be determined as $b = -\frac{1}{2}$, $d = \frac{1}{2}$, $k = 0$. The bottom panel shows the spatial profile at $t = 10$.

Case 10. When $U = \mathrm{nc}(s, k)$ and $V = \mathrm{cn}(r, m)$, and then from (14), we have

$$n^2 = -(1-k^2), \quad \mu_1 = 2k^2 - 1, \quad \nu_1 = -k^2,$$
$$\frac{a^2}{c^2}n^2 = -m^2, \quad \mu_2 = 2m^2 - 1, \quad \nu_2 = 1 - m^2. \tag{50}$$

Substituting (50) into the constraints (15), the parameters can be determined as

$$k^2 = 1 - m^2, \quad a^2 = c^2, \quad b = \frac{1}{2(1 - 2m^2)a},$$
$$d = \pm\frac{1}{2(1 - 2m^2)a}, \tag{51}$$

then the solution to the SG equation (1) is

$$u_{19} = 4\tan^{-1}[\mathrm{nc}(s, k)\mathrm{nc}(r, m)], \tag{52}$$

when $k \to 0$, i.e. $m \to 1$, the breather lattice solution (52) turns out to be another kind of breather solution

$$u_{20} = 4\tan^{-1}\left[\frac{\cosh(r)}{\cos(s)}\right]. \tag{53}$$

Case 11. When $U = \mathrm{nc}(s, k)$ and $V = \mathrm{sd}(r, m)$, and then from (14), we have

$$n^2 = -(1-k^2), \quad \mu_1 = 2k^2 - 1, \quad \nu_1 = -k^2, \\ \frac{a^2}{c^2}n^2 = -m^2(1-m^2), \quad \mu_2 = 2m^2 - 1, \quad \nu_2 = 1. \tag{54}$$

Substituting (54) into the constraints (15), the parameters can be determined as

$$1-k^2 = k^2m^2(1-m^2), \quad k^2a^2 = c^2, \\ b = \frac{1}{(3k^2 - 2k^2m^2 - 1)a}, \quad d = -\frac{bc}{a}, \tag{55}$$

then the solution to the SG equation (1) is

$$u_{21} = 4\tan^{-1}[\mathrm{nc}(s, k)\mathrm{ds}(r, m)]. \tag{56}$$

Case 12. When $U = \mathrm{ds}(s, k)$ and $V = \mathrm{cn}(r, m)$, and then from (14), we have

$$n^2 = -1, \quad \mu_1 = 2k^2 - 1, \quad \nu_1 = -k^2(1-k^2), \\ \frac{a^2}{c^2}n^2 = -m^2, \quad \mu_2 = 2m^2 - 1, \quad \nu_2 = 1 - m^2. \tag{57}$$

Substituting (57) into the constraints (15), the parameters can be determined as

$$1-m^2 = m^2k^2(1-k^2), \quad a^2 = m^2c^2, \\ b = \frac{m^2}{(2m^2k^2 - 3m^2 + 1)a}, \quad d = -\frac{bc}{a}, \tag{58}$$

then the solution to the SG equation (1) is

$$u_{22} = 4\tan^{-1}[\mathrm{ds}(s, k)\mathrm{nc}(r, m)]. \tag{59}$$

Case 13. When $U = \mathrm{nd}(s, k)$ and $V = \mathrm{nc}(r, m)$, and then from (14), we have

$$n^2 = 1 - k^2, \quad \mu_1 = 2 - k^2, \quad \nu_1 = 1, \\ \frac{a^2}{c^2}n^2 = 1 - m^2, \quad \mu_2 = 2m^2 - 1, \quad \nu_2 = -m^2. \tag{60}$$

Substituting (60) into the constraints (15), the parameters can be determined as

$$k^2 = 2 - \frac{1}{m^2}, \quad a^2 = m^2c^2, \\ b = \frac{m^2}{2(1-m^2)a}, \quad d = -\frac{bc}{a}, \tag{61}$$

then the solution to the SG equation (1) is

$$u_{23} = 4\tan^{-1}[\mathrm{nd}(s, k)\mathrm{cn}(r, m)]. \tag{62}$$

Similarly, if the independent variable transformation given in [1] is adopted, the breather lattice solutions u_{21}, u_{22}, and u_{23} can also only take their limiting forms.

Figure 4 describes the space-time evolution of the breather lattice solution of equations (61) and (62), whose behaviour is different from what is shown in figure 1. Actually, all graphical presentations given in this paper are different, small or large. When the modulus m or k is set as different values, the breather lattice solutions given in this paper are also different.

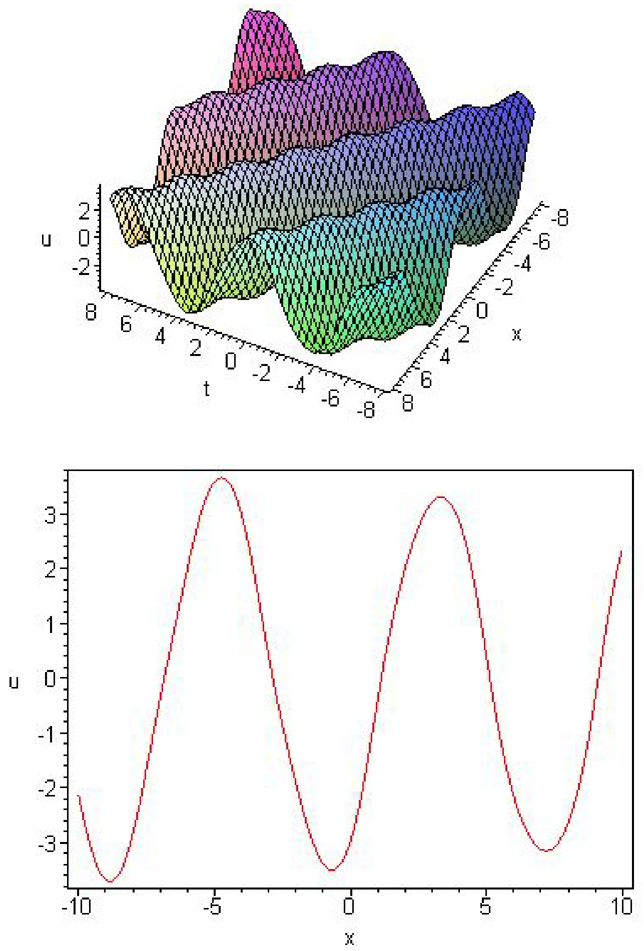

Figure 4. The upper panel shows the space-time evolution of the breather lattice solution of equations (61) and (62), where the parameters are chosen as $a = 0.8$, $c = 1$, $m = 0.8$, $s_0 = r_0 = 0$, from which the other parameters can be determined as $b = 0.889$, $d = -1.111$, $k = 0.66$. The bottom panel shows the spatial profile at $t = 10$.

3. Conclusion

In this paper, dependent and independent variable transformations are introduced to solve the SG equation by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different kinds of solutions, such as the breather solution and the breather lattice solution, can be obtained to the SG equation. We can see that besides the solutions expressed in terms of elementary functions, there are still solutions expressed in terms of the different combinations of Jacobi elliptic functions. However, not all the combinations of Jacobi elliptic functions are solutions to the SG equation (1), only those that can satisfy the constraints (15) can be solutions to the SG equation (1). Furthermore, when different independent variable transformations are adopted, there will be different results. For example, when we choose the independent variable transformation

$$s = ax + \frac{1}{a}t + s_0, \quad r = ax - \frac{1}{a}t + r_0, \tag{63}$$

which is given in [1], some breather lattice solutions such as u_{13}, u_{14}, u_{21}, u_{22} and u_{33} expressed in terms of Jacobi elliptic functions will be omitted. Under the independent variable transformation (5), all solutions can be expressed in terms of the 13 basic Jacobi elliptic functions listed in this paper, though there are only 11 combinations of Jacobi elliptic functions that can satisfy the constraints (15).

The objective of this paper is to obtain more kinds of breather lattice solutions, so we do not touch on the stability of those solutions which are nonsingular in the whole domain. Although we do not show the stability analysis to our solutions, from the results for the SG equation given by Kevrekidis *et al* [12] and for the (modified Korteweg-de Vries) mKdV the equation given by Kevrekidis *et al* [24, 25], we know that the solutions shown in this paper are usually unstable, but not all solutions are unstable. Even though the solutions are unstable, they can be stabilized by ac driving or damping, this has been reported by Kevrekidis *et al* [12, 24, 25].

Due to the wide applications of the SG equation, the analytical solutions given in this paper will be helpful in related research.

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