On Some Classes of Breather Lattice Solutions to the sinh-Gordon Equation

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In this paper, dependent and independent variable transformations are introduced to solve the sinh-Gordon equation by using the knowledge of the elliptic equation and Jacobian elliptic functions. It is shown that different kinds of solutions can be obtained to the sinh-Gordon equation, including breather lattice solutions and periodic wave solutions.

Key words: Jacobian Elliptic Function; sinh-Gordon Equation; Periodic Wave Solution; Breather Lattice Solution.
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1. Introduction

The sinh-Gordon (ShG) equation [1 – 8]

$$u_{xt} = \gamma \sinh u \tag{1}$$

is widely applied in physics and engineering, for example in integrable quantum field theories [1], noncommutative field theories [2], fluid dynamics [3]. Due to the wide applications of the sinh-Gordon equation, many achievements have been obtained in different aspects [3 – 9]. For instance, the ShG equation is known to be completely integrable [4] because it possesses similarity reductions to the third Painlevé equation [5].

Due to the special form of the sinh-Gordon equation, it is difficult to solve it directly, so we need some transformations. In this paper, based on the introduced transformations, we will show systematical results of the breather lattice solutions [10 – 12] for the ShG equation (1) by using the knowledge of the elliptic equation and Jacobian elliptic functions [13 – 15].

2. The Breather Lattice Solutions to the ShG Equation

In order to obtain the breather-type solutions to the ShG equation, we introduce the dependent variable transformation

$$\frac{u}{4} = \tanh^{-1} w \quad \text{or} \quad w = \tanh\frac{u}{4}, \tag{2}$$

which will let us rewrite the ShG equation (1) as

$$(1-w^2)w_{xt} + 2ww_xw_t - \gamma w(1+w^2) = 0. \tag{3}$$

Here one point must be stressed that not all transformations can be applied to solve the ShG equation to derive the breather-type solutions. In [9], the transformations

$$\frac{u}{2} = \sinh^{-1} w \quad \text{or} \quad w = \sinh\frac{u}{2} \tag{4}$$

and

$$\frac{u}{2} = \cosh^{-1} w \quad \text{or} \quad w = \cosh\frac{u}{2} \tag{5}$$

have been introduced to derive the periodic wave solutions to the ShG equation, and many different kinds of the periodic solutions expressed in terms of the Jacobi elliptic functions have been listed [9]. However, it can easily be checked that these two transformations can not let us derive the breather-type solutions to the ShG equation. Similarly, the transformations

$$\frac{u}{2} = \tanh^{-1} w \quad \text{or} \quad w = \tanh\frac{u}{2} \tag{6}$$

and

$$\frac{u}{2} = \tanh^{-1}\frac{1}{w} \quad \text{or} \quad \frac{1}{w} = \tanh\frac{u}{2} \tag{7}$$

can not be applied to obtain the breather-type solutions to the ShG equation.

From the above discussion we know that the suitable transformation is really important for us to derive different kinds of solutions to the ShG equation (1).

Next, we will introduce the independent variable transformation

$$r = ax + bt + r_0, \quad s = cx + dt + s_0, \tag{8}$$

where r_0 and s_0 are two constants.

Considering the transformation (8), (3) can be rewritten as

$$\begin{aligned}&(1-w^2)[abw_{rr} + (ad+bc)w_{rs} + cdw_{ss}]\\&+ 2w(bw_r + dw_s)(aw_r + cw_s) - \gamma w(1+w^2)\\&= 0.\end{aligned} \tag{9}$$

Compared to the transformation given in [16, 17], transformation (8) has less constraints, of course, this will let us have different types of solutions to the ShG equation (3).

Inspired by the transformation given in [16] and the results in [11, 12], we choose the dependent variable transformation

$$w = \alpha U(r)V(s), \tag{10}$$

where α is a constant amplitude to be determined, U and V satisfy the following elliptic equations:

$$\begin{aligned}U_r^2 &= -nU^4 + p_1U^2 + q_1,\\V_s^2 &= -\beta nV^4 + p_2V^2 + q_2,\end{aligned} \tag{11}$$

where β, n, p_1, p_2, q_1 and q_2 are determined constants for the specific choice of U and V. Here one point which must be stressed is that the introduction of β will let us have more choices to obtain different kinds of solutions to the ShG equation, and the specific choice made in (11) is the result of variable separation; a similar result can be found in [16] for the sine-Gordon equation.

Substituting (10) and (11) into (9) yields the algebraic equations

$$p_1a^2 - p_2c^2 = \gamma\frac{a}{b}, \tag{12a}$$

$$\beta nc^2 + q_1\alpha^2a^2 = 0, \tag{12b}$$

$$q_2\alpha^2c^2 + na^2 = 0, \tag{12c}$$

$$ad + bc = 0, \tag{12d}$$

from which we can determine

$$\begin{aligned}&\alpha^4 = \frac{\beta n^2}{q_1q_2}, \quad \frac{a^2}{c^2} = -\frac{\beta n}{q_1\alpha^2} = -\frac{q_2\alpha^2}{n},\\&p_1a^2 - p_2c^2 = \gamma\frac{a}{b}, \quad d = -\frac{bc}{a}.\end{aligned} \tag{13}$$

From (13) it is obvious that the determined constants in (11) must satisfy the constraints

$$\frac{\beta}{q_1q_2} > 0, \quad \frac{q_2}{n} < 0, \quad \frac{\beta n}{q_1} < 0. \tag{14}$$

This implies that not all combinations of Jacobi elliptic functions are solutions to the ShG equation (1) under the above-mentioned transformations. Only the combination of a couple of the Jacobi elliptic functions satisfies the constraint (14); it can be a solution to the ShG equation (1). Actually, there exist only 28 of these kinds of combinations; we will address them in detail.

Case 1. When $U = \mathrm{dn}(r,k)$ and $V = \mathrm{dn}(s,m)$, where $\mathrm{dn}(r,k)$ and $\mathrm{dn}(s,m)$ are the Jacobi elliptic function of the third kind, and k and m are their modulus [18 – 20], then from (11), we have

$$\begin{aligned}&n = 1, \quad p_1 = 2-k^2, \quad q_1 = -(1-k^2),\\&\beta n = 1, \quad p_2 = 2-m^2, \quad q_2 = -(1-m^2).\end{aligned} \tag{15}$$

Substituting (15) into (13), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}}, \quad \frac{a}{b} = \frac{1}{\gamma}[a^2(2-k^2) - c^2(2-m^2)],$$

$$d = -\frac{bc}{a}, \quad \alpha = \pm\left[\frac{1}{(1-k^2)(1-m^2)}\right]^{\frac{1}{4}}. \tag{16}$$

Then the solution to the ShG equation (3) is

$$w_1 = \pm\left[\frac{1}{(1-k^2)(1-m^2)}\right]^{\frac{1}{4}}[\mathrm{dn}(r,k)\mathrm{dn}(s,m)], \tag{17}$$

which is a kind of the breather lattice solution.

When $k \to 0$ (or $m \to 0$), the breather lattice solution (17) turns to be a periodic wave solution

$$w_{1'} = \pm\left[\frac{1}{(1-m^2)}\right]^{\frac{1}{4}}[\mathrm{dn}(s,m)]. \tag{18}$$

Case 2. When $U = \mathrm{sn}(r,k)$ and $V = \mathrm{sn}(s,m)$, where $\mathrm{sn}(s,m)$ is the Jacobi sine elliptic function [18 – 20], then $n = -k^2$, $p_1 = -(1+k^2)$, $q_1 = 1$,

$\beta n = -m^2$, $p_2 = -(1+m^2)$, $q_2 = 1$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) - c^2(1+m^2)],$$

$$d = -\frac{bc}{a}, \quad \alpha = \pm\sqrt{km}. \tag{19}$$

Then the solution to the ShG equation (3) is

$$w_2 = \pm\sqrt{km}[\mathrm{sn}(r,k)\mathrm{sn}(s,m)], \tag{20}$$

which is another kind of the breather lattice solution.

Case 3. When $U = \mathrm{sn}(r,k)$ and $V = \mathrm{cd}(s,m) = \frac{\mathrm{cn}(s,m)}{\mathrm{dn}(s,m)}$, where $\mathrm{cn}(s,m)$ is the Jacobi cosine elliptic function [18–20], then $n = -k^2$, $p_1 = -(1+k^2)$, $q_1 = 1$, $\beta n = -m^2$, $p_2 = -(1+m^2)$, $q_2 = 1$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) - c^2(1+m^2)],$$

$$d = -\frac{bc}{a}, \quad \alpha = \pm\sqrt{km}. \tag{21}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_3 = \pm\sqrt{km}[\mathrm{sn}(r,k)\mathrm{cd}(s,m)]. \tag{22}$$

Figure 1A shows the space-time evolution of the breather lattice solution of (21) and (22), where the parameters are chosen as $a = 1$, $c = 2$, $m = \frac{1}{8}$, $\gamma = 1$, $s_0 = r_0 = 0$, from which the other parameters can be determined as $b = \frac{32}{45}$, $d = -\frac{64}{25}$, $k = \frac{1}{2}$, $\alpha = \frac{1}{4}$. Figures 1B and 1C show the spatial profile at $t = 0$ and $t = 10$, respectively.

Figure 2A shows the space-time evolution of the periodic solution of (21) and (22), where the parameters are chosen as $a = \frac{1}{2}$, $c = 1$, $m = \frac{1}{4}$, $\gamma = 1$, $s_0 = r_0 = 0$, from which the other parameters can be determined as $b = \frac{8}{9}$, $d = -\frac{16}{9}$, $k = 1$, $\alpha = \frac{1}{2}$. Figure 2B shows the spatial profile at $t = 10$.

Figures 1 and 2 describe the space-time evolution of the periodic solution of (21) and (22), for different values of m and k; their behaviours are quite different. Figure 1 shows the evolution of the breather lattice solution, while Fig. 2 is just the normal periodic solution.

Case 4. When $U = \mathrm{cd}(r,k)$ and $V = \mathrm{cd}(s,m)$, then $n = -k^2$, $p_1 = -(1+k^2)$, $q_1 = 1$, $\beta n = -m^2$,

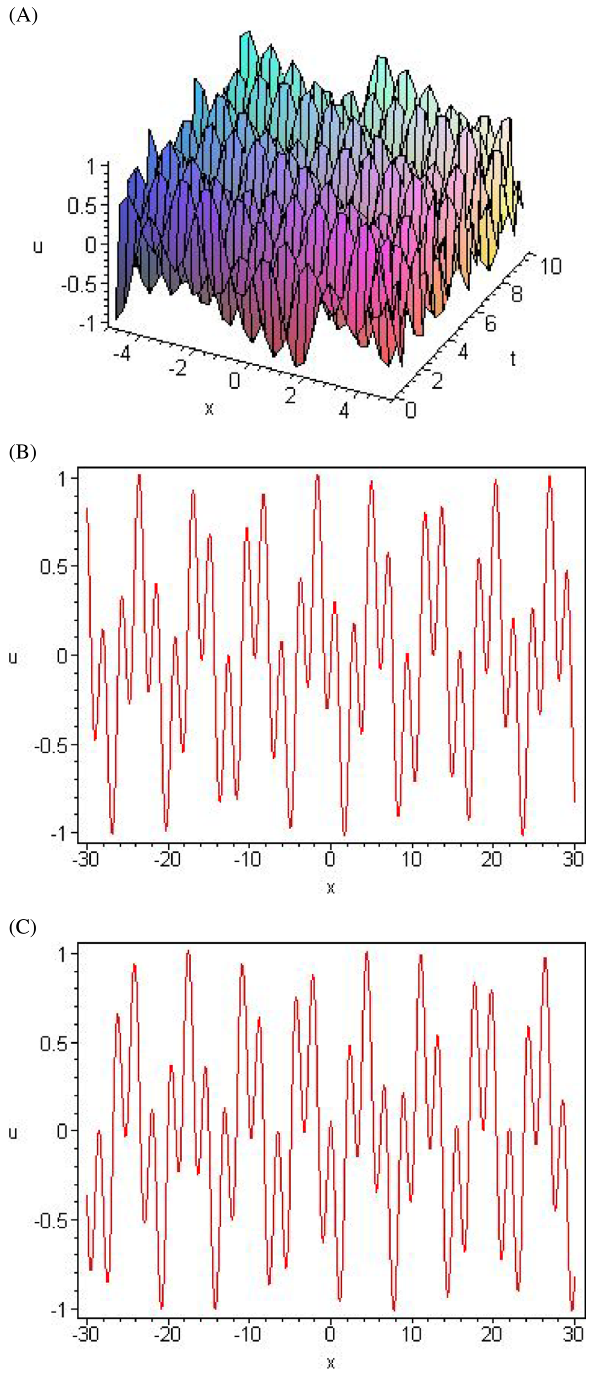
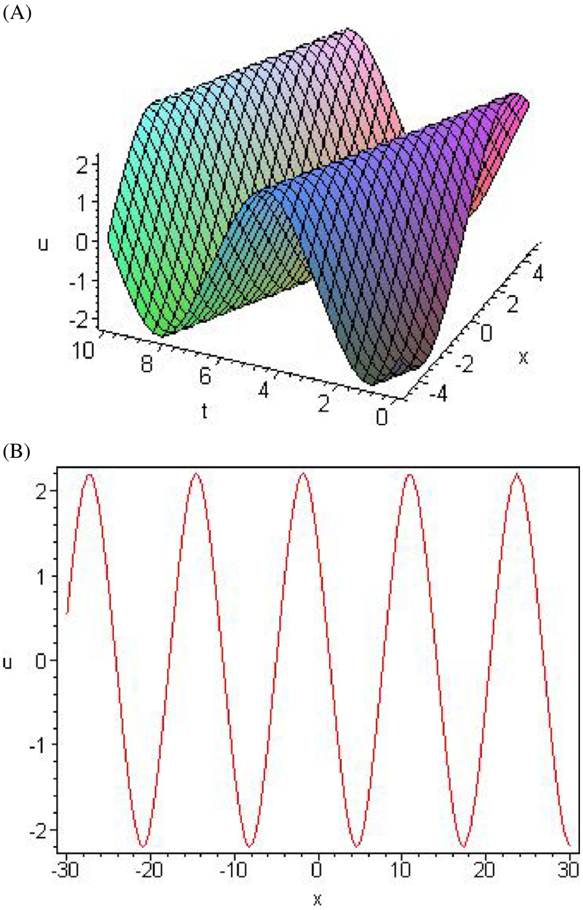


Fig. 1. (A) Space-time evolution of the breather lattice solution of (21) and (22), where the parameters are chosen as $a = 1$, $c = 2$, $m = 1/8$, $\gamma = 1$, $s_0 = r_0 = 0$, from which the other parameters can be determined as $b = 32/45$, $d = -64/25$, $k = 1/2$, $\alpha = 1/4$. (B) Spatial profile at $t = 0$. (C) Spatial profile at $t = 10$.

(A)



(B)

Fig. 2. (A) Space-time evolution of the periodic solution of equations (21) and (22), where the parameters are chosen as $a = 1/2$, $c = 1$, $m = 1/4$, $\gamma = 1$, $s_0 = r_0 = 0$, from which the other parameters can be determined as $b = 8/9$, $d = -16/9$, $k = 1$, $\alpha = 1/2$. (B) Spatial profile at $t = 10$.

$p_2 = -(1+m^2)$, $q_2 = 1$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) - c^2(1+m^2)],$$
$$d = -\frac{bc}{a}, \quad \alpha = \pm\sqrt{km}. \tag{23}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_4 = \pm\sqrt{km}[\mathrm{cd}(r,k)\mathrm{cd}(s,m)]. \tag{24}$$

Case 5. When $U = \mathrm{dn}(r,k)$ and $V = \mathrm{nd}(s,m) = \frac{1}{\mathrm{dn}(s,m)}$, then $n = 1$, $p_1 = 2-k^2$, $q_1 = -(1-k^2)$, $\beta n = 1-m^2$, $p_2 = 2-m^2$, $q_2 = -1$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}},$$
$$\frac{a}{b} = \frac{1}{\gamma}[a^2(2-k^2) - c^2(2-m^2)], \tag{25}$$
$$d = -\frac{bc}{a}, \quad \alpha = \pm\left[\frac{1-m^2}{1-k^2}\right]^{\frac{1}{4}}.$$

Then the breather lattice solution to the ShG equation (3) is

$$w_5 = \pm\left[\frac{1-m^2}{1-k^2}\right]^{\frac{1}{4}}[\mathrm{dn}(r,k)\mathrm{nd}(s,m)]. \tag{26}$$

Case 6. When $U = \mathrm{nd}(r,k)$ and $V = \mathrm{nd}(s,m)$, then $n = 1-k^2$, $p_1 = 2-k^2$, $q_1 = -1$, $\beta n = 1-m^2$, $p_2 = 2-m^2$, $q_2 = -1$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}},$$
$$\frac{a}{b} = \frac{1}{\gamma}[a^2(2-k^2) - c^2(2-m^2)], \tag{27}$$
$$d = -\frac{bc}{a}, \quad \alpha = \pm[(1-m^2)(1-k^2)]^{\frac{1}{4}}.$$

Then the breather lattice solution to the ShG equation (3) is

$$w_6 = \pm[(1-m^2)(1-k^2)]^{\frac{1}{4}}[\mathrm{nd}(r,k)\mathrm{nd}(s,m)]. \tag{28}$$

Apart from the nonsingular solutions given above, there are still some solutions which satisfy the constraint (14) but may blow up for specific values of independent variables. Although they may be rather unphysical, for reasons of mathematical tractability, they can be solutions to the ShG equation (1) in the mathematical form, too. We list them in the following parts.

Case 7. When $U = \mathrm{dc}(r,k)$ and $V = \mathrm{dc}(s,m)$, then $n = -1$, $p_1 = -(1+k^2)$, $q_1 = k^2$, $\beta n = -1$, $p_2 = -(1+m^2)$, $q_2 = m^2$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k},$$
$$\frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) - c^2(1+m^2)], \tag{29}$$
$$d = -\frac{bc}{a}, \quad \alpha = \pm\frac{1}{\sqrt{km}}.$$

Then the breather lattice solution to the ShG equation (3) is

$$w_7 = \pm\frac{1}{\sqrt{km}}[\mathrm{dc}(r,k)\mathrm{dc}(s,m)]. \tag{30}$$

Case 8. When $U = \mathrm{ns}(r,k) = \frac{1}{\mathrm{sn}(r,k)}$ and $V = \mathrm{ns}(s,m)$, then $n = -1$, $p_1 = -(1+k^2)$, $q_1 = k^2$, $\beta n = -1$, $p_2 = -(1+m^2)$, $q_2 = m^2$. From (13) the parameters can be determined as

$$\begin{aligned} &\frac{a^2}{c^2} = \frac{m}{k}, \\ &\frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) - c^2(1+m^2)], \\ &d = -\frac{bc}{a}, \quad \alpha = \pm\frac{1}{\sqrt{km}}. \end{aligned} \tag{31}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_8 = \pm\frac{1}{\sqrt{km}}[\mathrm{ns}(r,k)\mathrm{ns}(s,m)]. \tag{32}$$

Case 9. When $U = \mathrm{ns}(r,k)$ and $V = \mathrm{dc}(s,m) = \frac{\mathrm{dn}(s,m)}{\mathrm{cn}(s,m)}$, then $n = -1$, $p_1 = -(1+k^2)$, $q_1 = k^2$, $\beta n = -1$, $p_2 = -(1+m^2)$, $q_2 = m^2$. From (13) the parameters can be determined as

$$\begin{aligned} &\frac{a^2}{c^2} = \frac{m}{k}, \\ &\frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) - c^2(1+m^2)], \\ &d = -\frac{bc}{a}, \quad \alpha = \pm\frac{1}{\sqrt{km}}. \end{aligned} \tag{33}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_9 = \pm\frac{1}{\sqrt{km}}[\mathrm{ns}(r,k)\mathrm{dc}(s,m)]. \tag{34}$$

Case 10. When $U = \mathrm{sc}(r,k) = \frac{\mathrm{sn}(r,k)}{\mathrm{cn}(r,k)}$ and $V = \mathrm{sc}(s,m)$, then $n = -(1-k^2)$, $p_1 = 2-k^2$, $q_1 = 1$, $\beta n = -(1-m^2)$, $p_2 = 2-m^2$, $q_2 = 1$. From (13) the parameters can be determined as

$$\begin{aligned} &\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}}, \\ &\frac{a}{b} = \frac{1}{\gamma}[a^2(2-k^2) - c^2(2-m^2)], \\ &d = -\frac{bc}{a}, \quad \alpha = \pm[(1-m^2)(1-k^2)]^{\frac{1}{4}}. \end{aligned} \tag{35}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{10} = \pm[(1-m^2)(1-k^2)]^{\frac{1}{4}}[\mathrm{sc}(r,k)\mathrm{sc}(s,m)]. \tag{36}$$

Case 11. When $U = \mathrm{sc}(r,k)$ and $V = \mathrm{cs}(s,m) = \frac{\mathrm{cn}(s,m)}{\mathrm{sn}(s,m)}$, then $n = -(1-k^2)$, $p_1 = 2-k^2$, $q_1 = 1$, $\beta n = -1$, $p_2 = 2-m^2$, $q_2 = 1-m^2$. From (13) the parameters can be determined as

$$\begin{aligned} &\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}}, \\ &\frac{a}{b} = \frac{1}{\gamma}[a^2(2-k^2) - c^2(2-m^2)], \\ &d = -\frac{bc}{a}, \quad \alpha = \pm\left[\frac{1-k^2}{1-m^2}\right]^{\frac{1}{4}}. \end{aligned} \tag{37}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{11} = \pm\left[\frac{1-k^2}{1-m^2}\right]^{\frac{1}{4}}[\mathrm{sc}(r,k)\mathrm{cs}(s,m)]. \tag{38}$$

Case 12. When $U = \mathrm{cs}(r,k)$ and $V = \mathrm{cs}(s,m)$, then $n = -1$, $p_1 = 2-k^2$, $q_1 = 1-k^2$, $\beta n = -1$, $p_2 = 2-m^2$, $q_2 = 1-m^2$. From (13) the parameters can be determined as

$$\begin{aligned} &\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}}, \\ &\frac{a}{b} = \frac{1}{\gamma}[a^2(2-k^2) - c^2(2-m^2)], \\ &d = -\frac{bc}{a}, \quad \alpha = \pm\left[\frac{1}{(1-k^2)(1-m^2)}\right]^{\frac{1}{4}}. \end{aligned} \tag{39}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{12} = \pm\left[\frac{1}{(1-k^2)(1-m^2)}\right]^{\frac{1}{4}}[\mathrm{cs}(r,k)\mathrm{cs}(s,m)]. \tag{40}$$

Case 13. When $U = \mathrm{sn}(r,k)$ and $V = \mathrm{ns}(s,m)$, then $n = -k^2$, $p_1 = -(1+k^2)$, $q_1 = 1$, $\beta n = -1$, $p_2 = -(1+m^2)$, $q_2 = m^2$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) - c^2(1+m^2)], \quad d = -\frac{bc}{a}, \quad \alpha = \pm\sqrt{\frac{k}{m}}. \tag{41}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{13} = \pm\sqrt{\frac{k}{m}}[\mathrm{sn}(r,k)\mathrm{ns}(s,m)]. \tag{42}$$

Case 14. When $U = \mathrm{cd}(r,k) = \frac{\mathrm{cn}(r,k)}{\mathrm{dn}(r,k)}$ and $V = \mathrm{ns}(s,m)$, then $n = -k^2$, $p_1 = -(1+k^2)$, $q_1 = 1$, $\beta n = -1$, $p_2 = -(1+m^2)$, $q_2 = m^2$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) - c^2(1+m^2)], \quad d = -\frac{bc}{a}, \quad \alpha = \pm\sqrt{\frac{k}{m}}. \tag{43}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{14} = \pm\sqrt{\frac{k}{m}}[\mathrm{cd}(r,k)\mathrm{ns}(s,m)]. \tag{44}$$

Case 15. When $U = \mathrm{sn}(r,k)$ and $V = \mathrm{dc}(s,m) = \frac{\mathrm{dn}(s,m)}{\mathrm{cn}(s,m)}$, then $n = -k^2$, $p_1 = -(1+k^2)$, $q_1 = 1$, $\beta n = -1$, $p_2 = -(1+m^2)$, $q_2 = m^2$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) - c^2(1+m^2)], \quad d = -\frac{bc}{a}, \quad \alpha = \pm\sqrt{\frac{k}{m}}. \tag{45}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{15} = \pm\sqrt{\frac{k}{m}}[\mathrm{sn}(r,k)\mathrm{dc}(s,m)]. \tag{46}$$

Case 16. When $U = \mathrm{cd}(r,k)$ and $V = \mathrm{dc}(s,m)$, then $n = -k^2$, $p_1 = -(1+k^2)$, $q_1 = 1$, $\beta n = -1$, $p_2 = -(1+m^2)$, $q_2 = m^2$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) - c^2(1+m^2)], \quad d = -\frac{bc}{a}, \quad \alpha = \pm\sqrt{\frac{k}{m}}. \tag{47}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{16} = \pm\sqrt{\frac{k}{m}}[\mathrm{cd}(r,k)\mathrm{dc}(s,m)]. \tag{48}$$

Case 17. When $U = \mathrm{cn}(r,k)$ and $V = \mathrm{nc}(s,m) = \frac{1}{\mathrm{cn}(s,m)}$, then $n = k^2$, $p_1 = 2k^2-1$, $q_1 = 1-k^2$, $\beta n = -(1-m^2)$, $p_2 = 2m^2-1$, $q_2 = -m^2$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}}, \quad \frac{a}{b} = \frac{1}{\gamma}[a^2(2k^2-1) - c^2(2m^2-1)], \quad d = -\frac{bc}{a}, \quad \alpha = \pm\left[\frac{k^2(1-m^2)}{m^2(1-k^2)}\right]^{\frac{1}{4}}. \tag{49}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{17} = \pm\left[\frac{k^2(1-m^2)}{m^2(1-k^2)}\right]^{\frac{1}{4}}[\mathrm{cn}(r,k)\mathrm{nc}(s,m)]. \tag{50}$$

Case 18. When $U = \mathrm{sd}(r,k) = \frac{\mathrm{sn}(r,k)}{\mathrm{dn}(r,k)}$ and $V = \mathrm{ds}(s,m) = \frac{\mathrm{dn}(s,m)}{\mathrm{sn}(s,m)}$, then $n = k^2(1-k^2)$, $p_1 = 2k^2-1$, $q_1 = 1$, $\beta n = -1$, $p_2 = 2m^2-1$, $q_2 = -m^2(1-m^2)$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}}, \quad \frac{a}{b} = \frac{1}{\gamma}[a^2(2k^2-1) - c^2(2m^2-1)], \quad d = -\frac{bc}{a}, \quad \alpha = \pm\left[\frac{k^2(1-k^2)}{m^2(1-m^2)}\right]^{\frac{1}{4}}. \tag{51}$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{18} = \pm \left[\frac{k^2(1-k^2)}{m^2(1-m^2)} \right]^{\frac{1}{4}} [\mathrm{sd}(r,k)\mathrm{ds}(s,m)]. \quad (52)$$

Case 19. When $U = \mathrm{cn}(r,k)$ and $V = \mathrm{ds}(s,m)$, then $n = k^2$, $p_1 = 2k^2 - 1$, $q_1 = 1 - k^2$, $\beta n = -1$, $p_2 = 2m^2 - 1$, $q_2 = -m^2(1-m^2)$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}},$$
$$\frac{a}{b} = \frac{1}{\gamma}[a^2(2k^2-1) - c^2(2m^2-1)], \quad (53)$$
$$d = -\frac{bc}{a}, \quad \alpha = \pm \left[\frac{k^2}{m^2(1-m^2)(1-k^2)} \right]^{\frac{1}{4}}.$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{19} = \pm \left[\frac{k^2}{m^2(1-m^2)(1-k^2)} \right]^{\frac{1}{4}} [\mathrm{cn}(r,k)\mathrm{ds}(s,m)]. \quad (54)$$

Case 20. When $U = \mathrm{nc}(r,k)$ and $V = \mathrm{sd}(s,m)$, then $n = -(1-k^2)$, $p_1 = 2k^2 - 1$, $q_1 = -k^2$, $\beta n = m^2(1-m^2)$, $p_2 = 2m^2 - 1$, $q_2 = 1$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}},$$
$$\frac{a}{b} = \frac{1}{\gamma}[a^2(2k^2-1) - c^2(2m^2-1)], \quad (55)$$
$$d = -\frac{bc}{a}, \quad \alpha = \pm \left[\frac{m^2(1-m^2)(1-k^2)}{k^2} \right]^{\frac{1}{4}}.$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{20} = \pm \left[\frac{m^2(1-m^2)(1-k^2)}{k^2} \right]^{\frac{1}{4}} [\mathrm{nc}(r,k)\mathrm{sd}(s,m)]. \quad (56)$$

Case 21. When $U = \mathrm{sn}(r,k)$ and $V = \mathrm{cs}(s,m)$, then $n = -k^2$, $p_1 = -(1+k^2)$, $q_1 = 1$, $\beta n = -1$, $p_2 = 2 - m^2$, $q_2 = 1 - m^2$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}},$$
$$\frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) + c^2(2-m^2)], \quad (57)$$
$$d = -\frac{bc}{a}, \quad \alpha = \pm \left[\frac{k^2}{1-m^2} \right]^{\frac{1}{4}}.$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{21} = \pm \left[\frac{k^2}{1-m^2} \right]^{\frac{1}{4}} [\mathrm{sn}(r,k)\mathrm{cs}(s,m)]. \quad (58)$$

When $k \to 1$ and $m \to 0$, the breather lattice solution (58) turns to the solution

$$w_{21'} = \pm[\tanh(r)\cot(s)], \quad (59)$$

with

$$a^2 = c^2, \quad b = -\frac{\gamma}{4a}, \quad d = \pm\frac{\gamma}{4a}. \quad (60)$$

Case 22. When $U = \mathrm{cd}(r,k)$ and $V = \mathrm{cs}(s,m)$, then $n = -k^2$, $p_1 = -(1+k^2)$, $q_1 = 1$, $\beta n = -1$, $p_2 = 2 - m^2$, $q_2 = 1 - m^2$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}},$$
$$\frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) + c^2(2-m^2)], \quad (61)$$
$$d = -\frac{bc}{a}, \quad \alpha = \pm \left[\frac{k^2}{1-m^2} \right]^{\frac{1}{4}}.$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{22} = \pm \left[\frac{k^2}{1-m^2} \right]^{\frac{1}{4}} [\mathrm{cd}(r,k)\mathrm{cs}(s,m)]. \quad (62)$$

Case 23. When $U = \mathrm{ns}(r,k)$ and $V = \mathrm{cs}(s,m)$, then $n = -1$, $p_1 = -(1+k^2)$, $q_1 = k^2$, $\beta n = -1$, $p_2 = 2 - m^2$, $q_2 = 1 - m^2$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}},$$
$$\frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2) + c^2(2-m^2)], \quad (63)$$
$$d = -\frac{bc}{a}, \quad \alpha = \pm \left[\frac{1}{k^2(1-m^2)} \right]^{\frac{1}{4}}.$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{23} = \pm\left[\frac{1}{k^2(1-m^2)}\right]^{\frac{1}{4}}[\mathrm{ns}(r,k)\mathrm{cs}(s,m)]. \quad (64)$$

Case 24. When $U = \mathrm{dc}(r,k)$ and $V = \mathrm{cs}(s,m)$, then $n = -1$, $p_1 = -(1+k^2)$, $q_1 = k^2$, $\beta n = -1$, $p_2 = 2-m^2$, $q_2 = 1-m^2$. From (13) the parameters can be determined as

$$\begin{aligned}&\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}},\\ &\frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2)+c^2(2-m^2)],\\ &d = -\frac{bc}{a}, \quad \alpha = \pm\left[\frac{1}{k^2(1-m^2)}\right]^{\frac{1}{4}}.\end{aligned} \quad (65)$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{24} = \pm\left[\frac{1}{k^2(1-m^2)}\right]^{\frac{1}{4}}[\mathrm{dc}(r,k)\mathrm{cs}(s,m)]. \quad (66)$$

Case 25. When $U = \mathrm{dc}(r,k)$ and $V = \mathrm{sc}(s,m)$, then $n = -1$, $p_1 = -(1+k^2)$, $q_1 = k^2$, $\beta n = -(1-m^2)$, $p_2 = 2-m^2$, $q_2 = 1$. From (13) the parameters can be determined as

$$\begin{aligned}&\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}},\\ &\frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2)+c^2(2-m^2)],\\ &d = -\frac{bc}{a}, \quad \alpha = \pm\left[\frac{1-m^2}{k^2}\right]^{\frac{1}{4}}.\end{aligned} \quad (67)$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{25} = \pm\left[\frac{1-m^2}{k^2}\right]^{\frac{1}{4}}[\mathrm{dc}(r,k)\mathrm{sc}(s,m)]. \quad (68)$$

Case 26. When $U = \mathrm{ns}(r,k)$ and $V = \mathrm{sc}(s,m)$, then $n = -1$, $p_1 = -(1+k^2)$, $q_1 = k^2$, $\beta n = -(1-m^2)$, $p_2 = 2-m^2$, $q_2 = 1$. From (13) the parameters can be determined as

$$\begin{aligned}&\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}},\\ &\frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2)+c^2(2-m^2)],\\ &d = -\frac{bc}{a}, \quad \alpha = \pm\left[\frac{1-m^2}{k^2}\right]^{\frac{1}{4}}.\end{aligned} \quad (69)$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{26} = \pm\left[\frac{1-m^2}{k^2}\right]^{\frac{1}{4}}[\mathrm{ns}(r,k)\mathrm{sc}(s,m)]. \quad (70)$$

Case 27. When $U = \mathrm{sn}(r,k)$ and $V = \mathrm{sc}(s,m)$, then $n = -k^2$, $p_1 = -(1+k^2)$, $q_1 = 1$, $\beta n = -(1-m^2)$, $p_2 = 2-m^2$, $q_2 = 1$. From (13) the parameters can be determined as

$$\begin{aligned}&\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}},\\ &\frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2)+c^2(2-m^2)],\\ &d = -\frac{bc}{a}, \quad \alpha = \pm[k^2(1-m^2)]^{\frac{1}{4}}.\end{aligned} \quad (71)$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{27} = \pm[k^2(1-m^2)]^{\frac{1}{4}}[\mathrm{sn}(r,k)\mathrm{sc}(s,m)]. \quad (72)$$

Case 28. When $U = \mathrm{cd}(r,k)$ and $V = \mathrm{sc}(s,m)$, then $n = -k^2$, $p_1 = -(1+k^2)$, $q_1 = 1$, $\beta n = -(1-m^2)$, $p_2 = 2-m^2$, $q_2 = 1$. From (13) the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}}, \quad \frac{a}{b} = -\frac{1}{\gamma}[a^2(1+k^2)+c^2(2-m^2)],$$

$$d = -\frac{bc}{a}, \quad \alpha = \pm[k^2(1-m^2)]^{\frac{1}{4}}. \quad (73)$$

Then the breather lattice solution to the ShG equation (3) is

$$w_{28} = \pm[k^2(1-m^2)]^{\frac{1}{4}}[\mathrm{cd}(r,k)\mathrm{sc}(s,m)]. \quad (74)$$

3. Conclusion

In this paper, the independent variable transformation and the dependent variable transformation were introduced to solve the sinh-Gordon equation by using the knowledge of the elliptic equation and Jacobian elliptic functions. It was shown that suitable transformations are required in order to obtain the breather-type solutions to the sinh-Gordon equation. Some solutions with quite different structures have not been reported in the literature, including breather lattice solutions.

The aim of this paper was to obtain more kinds of breather lattice solutions. We did not touch on the stability of those solutions which are non-singular in the whole domain. Although we did not give the stability analysis to our solutions, from the results given by Kevrekidis et al. [11, 12], we can say that not all solutions given in our manuscript are unstable. Even though the solutions are unstable, they can be stabilized by ac driving and damping; this has been reported by Kevrekidis et al. [11, 12].

Due to wide applications of the ShG equation, the analytical solutions given in this paper will be helpful in related research.

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