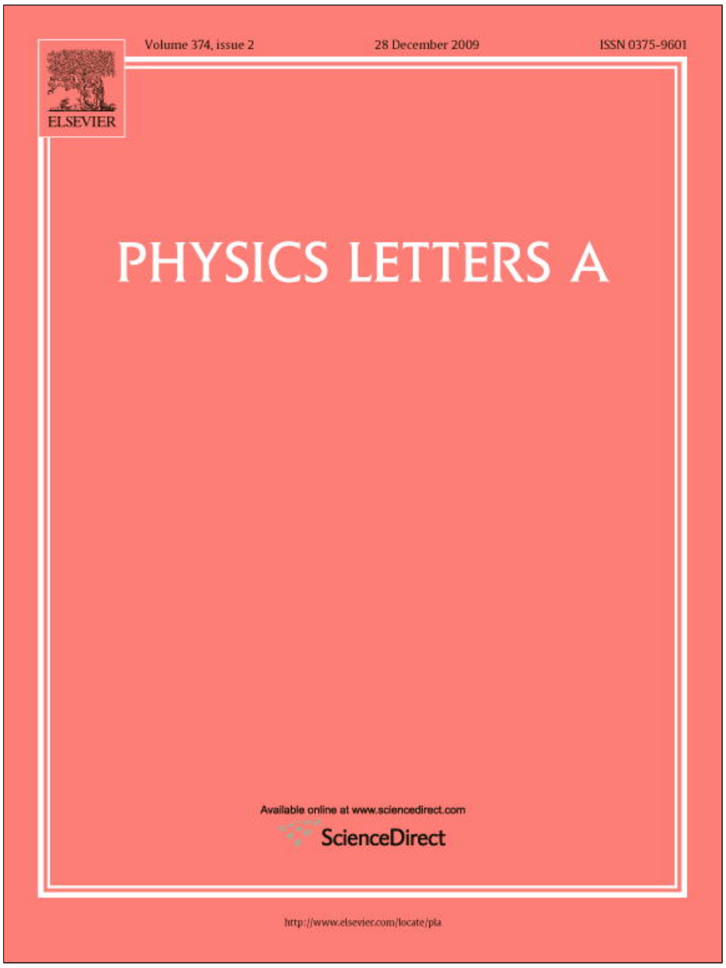
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New Lamé function and its application to nonlinear equations

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ABSTRACT

In this Letter, a new kind of Lamé functions are given. Based on the new Lamé functions and Jacobi elliptic function, the perturbation method is applied to the nonlinear equations, and many multi-order solutions of novel forms are derived. In addition, it is shown that different Lamé functions can exist in the first order solutions of nonlinear system.

1. Introduction

During the past three decades, the nonlinear wave researches have made great progress, among which a number of new methods have been proposed to get the exact solutions to nonlinear wave equations. Among these methods, the homogeneous balance method [1], the hyperbolic tangent function expansion method [2,3], the nonlinear transformation method [4,5], the trial function method [6,7], sine-cosine method [8], the Jacobi elliptic function expansion method [9–11], auxiliary equation and mapping method [12], Exp-function method [13] and so on are widely applied to solve nonlinear wave equations exactly. Furthermore, it deserves to discuss the stability of these solutions, there perturbation method is often applied to derive the multi-order exact solutions. In this Letter, based on the Jacobi elliptic functions [14,15] and a new kind of Lamé functions, perturbation method [15,16] is applied to get the multi-order exact solutions to some nonlinear equations.

2. Lamé equation and Lamé functions

Usually, Lamé equation [14,15] in terms of $y(x)$ can be written as

$$\frac{d^2 y}{dx^2} + \left[\lambda - n(n+1)m^2 \operatorname{sn}^2 x\right] y = 0, \tag{1}$$

where λ is an eigenvalue, n is a positive integer, $\operatorname{sn} x$ is the Jacobi elliptic sine function with its modulus m $(0 < m < 1)$.

Set

$$\eta = \operatorname{sn}^2 x \tag{2}$$

then the Lamé equation (1) becomes

$$\frac{d^2 y}{d\eta^2} + \frac{1}{2}\left(\frac{1}{\eta} + \frac{1}{\eta - 1} + \frac{1}{\eta - h}\right)\frac{dy}{d\eta} - \frac{\mu + n(n+1)\eta}{4\eta(\eta - 1)(\eta - h)} y = 0, \tag{3}$$

where

$$h = m^{-2} > 1, \qquad \mu = -h\lambda. \tag{4}$$

Eq. (3) is a kind of Fuchs-typed equations with four regular singular points $\eta = 0, 1, h$ and $\eta = \infty$, the solution to Lamé equation (3) is known as Lamé function.

There are different Lamé functions expressed in closed form, for example, when $n = 3$, $\lambda = 4(1 + m^2)$, i.e. $\mu = -4(1 + m^{-2})$, the Lamé function is

$$L_3^{\mathrm{sn}}(x) = \eta^{1/2}(1 - \eta)^{1/2}\left(1 - h^{-1}\eta\right)^{1/2} = \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x. \tag{5}$$

For $n = 2$, when $\lambda = 1 + m^2$, the Lamé functions is

$$L_2^{s}(x) = (1 - \eta)^{1/2}\left(1 - h^{-1}\eta\right)^{1/2} = \operatorname{cn} x \operatorname{dn} x, \tag{6}$$

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when $\lambda = 1 + 4m^2$, the Lamé functions is

$$L_2^c(x) = (1-\eta)^{1/2}\left(1-h^{-1}\eta\right)^{1/2} = \operatorname{sn} x \operatorname{dn} x \tag{7}$$

and when $\lambda = 4 + m^2$, the Lamé functions is

$$L_2^d(x) = (1-\eta)^{1/2}\left(1-h^{-1}\eta\right)^{1/2} = \operatorname{sn} x \operatorname{cn} x. \tag{8}$$

In Eqs. (5), (6), (7) and (8), $\operatorname{cn} x$ and $\operatorname{dn} x$ are the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind [14,15], respectively. Lamé functions given in this section have been applied to solve nonlinear equations to derive multi-order solutions [17]. Is there any other kind of Lamé functions and could they be applied to solve nonlinear equations, either? We will answer this question in the next sections.

3. New Lamé equation and new Lamé functions

In fact, Lamé equation in terms of $y(x)$ can also be written as

$$\frac{d^2y}{dx^2} + \left[\lambda - n(n+1)m^2 \operatorname{cd}^2 x\right] y = 0, \tag{9}$$

where $\operatorname{cd} x \equiv \frac{\operatorname{cn} x}{\operatorname{dn} x}$ is another kind of Jacobi elliptic function with its modulus m $(0 < m < 1)$, and λ is an eigenvalue, n is a positive integer, too.

Set

$$\zeta = \operatorname{cd}^2 x \tag{10}$$

then the Lamé equation (9) becomes

$$\frac{d^2y}{d\zeta^2} + \frac{1}{2}\left(\frac{1}{\zeta} + \frac{1}{\zeta-1} + \frac{1}{\zeta-h}\right)\frac{dy}{d\zeta} - \frac{\mu + n(n+1)\zeta}{4\zeta(\zeta-1)(\zeta-h)} y = 0, \tag{11}$$

where

$$h = m^{-2} > 1, \qquad \mu = -h\lambda. \tag{12}$$

Obviously, Eq. (11) takes the same form as Eq. (3), it is a kind of Fuchs-typed equations with four regular singular points $\zeta = 0, 1, h$ and $\zeta = \infty$, the solution to Lamé equation (11) is known as Lamé function.

For Eq. (11), there also exist different Lamé functions expressed in closed form, for example, when $n = 3$, $\lambda = 4(1+m^2)$, i.e. $\mu = -4(1+m^{-2})$, the Lamé function is

$$L_3^{\mathrm{cd}}(x) = \zeta^{1/2}(1-\zeta)^{1/2}\left(1-h^{-1}\zeta\right)^{1/2} = \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x, \tag{13}$$

this is another Lamé function different from that given in (5).

For $n = 2$, when $\lambda = 1 + m^2$, the Lamé functions is

$$L_2^{\mathrm{cd}}(x) = (1-\zeta)^{1/2}\left(1-h^{-1}\zeta\right)^{1/2} = \operatorname{sd} x \operatorname{nd} x, \tag{14}$$

when $\lambda = 1 + 4m^2$, the Lamé functions is

$$L_2^{\mathrm{sd}}(x) = (1-\zeta)^{1/2}\left(1-h^{-1}\zeta\right)^{1/2} = \operatorname{cd} x \operatorname{nd} x \tag{15}$$

and when $\lambda = 4 + m^2$, the Lamé functions is

$$L_2^{\mathrm{nd}}(x) = (1-\zeta)^{1/2}\left(1-h^{-1}\zeta\right)^{1/2} = \operatorname{sd} x \operatorname{cd} x. \tag{16}$$

In Eqs. (13), (14), (15) and (16), $\operatorname{sd} x \equiv \frac{\operatorname{sn} x}{\operatorname{dn} x}$, $\operatorname{nd} x \equiv \frac{1}{\operatorname{dn} x}$ are two new Jacobi elliptic functions. $L_2^{\mathrm{cd}}(x)$, $L_2^{\mathrm{sd}}(x)$ and $L_2^{\mathrm{nd}}(x)$ are three new Lamé functions different from those given in (6), (7) and (8). Could they be applied to solve nonlinear equations, either? We will answer this question in the next sections. Since we have reported the applications of Lamé functions $L_3^{\mathrm{sn}}(x)$ and/or $L_2^{s}(x)$ and/or $L_2^{c}(x)$ and/or $L_2^{c}(x)$ to KdV equation and mKdV equation in the Ref. [17], we will take these two nonlinear equations as examples to illustrate the applications of new Lamé functions to nonlinear equations to derive multi-order solutions of novel forms.

4. Applications of new Lamé functions to mKdV equation

mKdV equation reads

$$\frac{\partial u}{\partial t} + \alpha u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0. \tag{17}$$

We seek its travelling wave solutions of the following form

$$u = u(\xi), \quad \xi = k(x - ct), \tag{18}$$

where k and c are wave number and wave speed, respectively.

In the frame of (18), (17) can be written as

$$\beta k^2 \frac{d^2u}{d\xi^2} + \frac{\alpha}{3}u^3 - cu = C_0, \tag{19}$$

where integration with respect to ξ has been taken once and C_0 is the integration constant.

Here we consider perturbation method and setting

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots, \tag{20}$$

where ϵ $(0 < \epsilon \ll 1)$ is a small parameter, u_0, u_1 and u_2 represent the zeroth-order, first-order and second-order solutions, respectively.

Substituting (20) into (19), we can derive the zeroth-order, the first-order and the second-order equations as

$$\epsilon^0: \quad \beta k^2 \frac{d^2u_0}{d\xi^2} + \frac{\alpha}{3}u_0^3 - cu_0 = C_0, \tag{21}$$

$$\epsilon^1: \quad \beta k^2 \frac{d^2u_1}{d\xi^2} + \left(\alpha u_0^2 - c\right)u_1 = 0 \tag{22}$$

and

$$\epsilon^2: \quad \beta k^2 \frac{d^2u_2}{d\xi^2} + \left(\alpha u_0^2 - c\right)u_2 = -\alpha u_0 u_1^2. \tag{23}$$

From the zeroth-order Eq. (21) and the ansatz solution

$$u_0 = a_0 + a_1 \operatorname{cd} \xi \tag{24}$$

we can get the zeroth-order exact solution of mKdV equation

$$u_{01} = \pm mk\sqrt{-\frac{6\beta}{\alpha}} \operatorname{cd} \xi, \qquad c = -\left(1+m^2\right)\beta k^2. \tag{25}$$

Substituting the zeroth-order exact solution (25) into the first-order Eq. (22) leads to

$$\frac{d^2u_1}{d\xi^2} + \left[\left(1+m^2\right) - 6m^2 \operatorname{cd}^2 \xi\right]u_1 = 0 \tag{26}$$

which takes the same form as Lamé equation (9), so the first-order exact solution can be written as

$$u_{11} = A_1 L_2^{\mathrm{cd}}(\xi) = A_1 \operatorname{sd} \xi \operatorname{nd} \xi, \tag{27}$$

where A_1 is an arbitrary constant.

Substituting the zeroth-order exact solution (25) and the first-order exact solution (27) into the second-order Eq. (23) results in

$$\frac{d^2u_2}{d\xi^2} + \left[\left(1+m^2\right) - 6m^2 \operatorname{cd}^2 \xi\right]u_2 = \pm\sqrt{-\frac{6\alpha}{\beta}}\frac{mA_1^2}{k} \operatorname{cd} \xi \operatorname{sd}^2 \xi \operatorname{nd}^2 \xi \tag{28}$$

which is an inhomogeneous Lamé equation of the form (9), and it can be solved by introducing an ansatz solution

$$u_2 = b_1 \operatorname{cd} \xi + b_3 \operatorname{cd}^3 \xi. \tag{29}$$

Combining (28) with (29) reaches the second-order exact solution

$$u_{21} = \mp\sqrt{-\frac{6\alpha}{\beta}}\frac{(1+m^2)A_1^2}{12mk(1-m^2)^2}\operatorname{cd}\xi\left[1-\frac{2m^2}{1+m^2}\operatorname{cd}^2\xi\right]. \tag{30}$$

For the zeroth-order Eq. (21), the ansatz solution can also be written as

$$u_0 = a_0 + a_1\operatorname{sd}\xi \tag{31}$$

we can get the zeroth-order exact solution of mKdV equation

$$u_{02} = \pm mk\sqrt{\frac{6(1-m^2)\beta}{\alpha}}\operatorname{sd}\xi, \qquad c = \left(2m^2-1\right)\beta k^2. \tag{32}$$

Substituting the zeroth-order exact solution (32) into the first-order Eq. (22) leads to

$$\frac{d^2u_1}{d\xi^2} + \left[\left(2m^2-1\right)+6m^2\left(1-m^2\right)\operatorname{sd}^2\xi\right]u_1 = 0 \tag{33}$$

which takes the same form as Lamé equation (9), so the first-order exact solution can be written as

$$u_{12} = A_2L_2^{\mathrm{sd}}(\xi) = A_2\operatorname{cd}\xi\operatorname{nd}\xi, \tag{34}$$

where A_2 is an arbitrary constant.

Substituting the zeroth-order exact solution (32) and the first-order exact solution (34) into the second-order Eq. (23) results in

$$\frac{d^2u_2}{d\xi^2} + \left[\left(2m^2-1\right)+6m^2\left(1-m^2\right)\operatorname{sd}^2\xi\right]u_2 = \mp\sqrt{\frac{6(1-m^2)\alpha}{\beta}}\frac{mA_2^2}{k}\operatorname{sd}\xi\operatorname{cd}^2\xi\operatorname{nd}^2\xi \tag{35}$$

which is an inhomogeneous Lamé equation of the form (9), and it can be solved by introducing an ansatz solution

$$u_2 = b_1\operatorname{sd}\xi + b_3\operatorname{sd}^3\xi. \tag{36}$$

Combining (35) with (36) reaches the second-order exact solution

$$u_{22} = \mp\sqrt{\frac{6\alpha}{(1-m^2)\beta}}\frac{(1-2m^2)A_2^2}{12mk}\operatorname{sd}\xi \times\left[1+\frac{2m^2(1-m^2)}{1-2m^2}\operatorname{sd}^2\xi\right]. \tag{37}$$

For the zeroth-order Eq. (21), the ansatz solution can also be written as

$$u_0 = a_0 + a_1\operatorname{nd}\xi \tag{38}$$

we can get the zeroth-order exact solution of mKdV equation

$$u_{03} = \pm k\sqrt{\frac{6(1-m^2)\beta}{\alpha}}\operatorname{nd}\xi, \qquad c = \left(2-m^2\right)\beta k^2. \tag{39}$$

Substituting the zeroth-order exact solution (39) into the first-order Eq. (22) leads to

$$\frac{d^2u_1}{d\xi^2} + \left[\left(m^2-2\right)+6\left(1-m^2\right)\operatorname{nd}^2\xi\right]u_1 = 0 \tag{40}$$

which takes the same form as Lamé equation (9), so the first-order exact solution can be written as

$$u_{13} = A_3L_2^{\mathrm{nd}}(\xi) = A_3\operatorname{sd}\xi\operatorname{cd}\xi, \tag{41}$$

where A_3 is an arbitrary constant.

Substituting the zeroth-order exact solution (39) and the first-order exact solution (41) into the second-order Eq. (23) results in

$$\frac{d^2u_2}{d\xi^2} + \left[\left(m^2-2\right)+6\left(1-m^2\right)\operatorname{nd}^2\xi\right]u_2 = \mp\sqrt{\frac{6(1-m^2)\alpha}{\beta}}\frac{A_3^2}{k}\operatorname{nd}\xi\operatorname{sd}^2\xi\operatorname{cd}^2\xi \tag{42}$$

which is an inhomogeneous Lamé equation of the form (9), and it can be solved by introducing an ansatz solution

$$u_2 = b_1\operatorname{nd}\xi + b_3\operatorname{nd}^3\xi. \tag{43}$$

Combining (42) with (43) reaches the second-order exact solution

$$u_{23} = \pm\sqrt{\frac{6\alpha}{(1-m^2)\beta}}\frac{(2-m^2)A_3^2}{12m^4k}\operatorname{nd}\xi\left[1-\frac{2(1-m^2)}{2-m^2}\operatorname{nd}^2\xi\right]. \tag{44}$$

Obviously, the solutions given above to mKdV equations are different from those we have given in the Ref. [17], these solutions are solutions of novel forms have not been reported in the literature.

5. Applications of new Lamé functions to KdV equation

The second example we want to show is KdV equation, it reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0. \tag{45}$$

Substituting (18) into (45) yields

$$\beta k^2\frac{d^3u}{d\xi^3} + u\frac{du}{d\xi} - c\frac{du}{d\xi} = 0. \tag{46}$$

Integrating (46) once with respect to ξ and we have

$$\beta k^2\frac{d^2u}{d\xi^2} + \frac{1}{2}u^2 - cu = C_0, \tag{47}$$

where C_0 is an integration constant.

Substituting (20) into (47), we get the zeroth-order, the first-order and the second-order equations:

$$\epsilon^0:\quad \beta k^2\frac{d^2u_0}{d\xi^2} + \frac{1}{2}u_0^2 - cu_0 = C_0, \tag{48}$$

$$\epsilon^1:\quad \beta k^2\frac{d^2u_1}{d\xi^2} + (u_0-c)u_1 = 0 \tag{49}$$

and

$$\epsilon^2:\quad \beta k^2\frac{d^2u_2}{d\xi^2} + (u_0-c)u_2 = -\frac{1}{2}u_1^2. \tag{50}$$

The zeroth-order Eq. (48) can be solved by the Jacobi elliptic sine function expansion method, the ansatz solution

$$u_0 = a_0 + a_1\operatorname{cd}\xi + a_2\operatorname{cd}^2\xi \tag{51}$$

can be assumed.

Applying (51) to (48), the zeroth-order exact solution can be easily obtained

$$u_0 = c + 4\left(1+m^2\right)\beta k^2 - 12m^2\beta k^2\operatorname{cd}^2\xi. \tag{52}$$

Similarly, substituting (52) into the first-order equation (49) leads to

$$\frac{d^2u_1}{d\xi^2} + \left[4\left(1+m^2\right) - 12m^2\operatorname{cd}^2\xi\right]u_1 = 0 \tag{53}$$

obviously this is the Lamé equation (9), its solution is

$$u_1 = A \operatorname{cd}\xi \operatorname{sd}\xi \operatorname{nd}\xi \tag{54}$$

where A is an arbitrary constant.

Substituting the zeroth-order solution (52) and the first-order solution (54) into the second-order equation (50) results in

$$\frac{d^2u_2}{d\xi^2} + \left[4\left(1+m^2\right) - 12m^2 \operatorname{cd}^2\xi\right]u_2 = -\frac{A^2}{2\beta k^2} \operatorname{cd}^2\xi \operatorname{sd}^2\xi \operatorname{nd}^2\xi \tag{55}$$

it is obvious that this is an inhomogeneous Lamé equation with $n = 3$ and $\lambda = 4(1+m^2)$. Its solution of homogeneous equation is just the same one as (9) and its special solution of inhomogeneous terms can be assumed to be

$$u_2 = b_0 + b_2 \operatorname{cd}^2\xi + b_4 \operatorname{cd}^4\xi. \tag{56}$$

Then applying (56) to (55), the second-order exact solution of KdV equation (45) can be written as

$$u_2 = -\frac{A^2}{48m^2(1-m^2)^2\beta k^2}\left[1 - 2\left(1+m^2\right)\operatorname{cd}^2\xi + 3m^2 \operatorname{cd}^4\xi\right]. \tag{57}$$

The multi-order solutions to KdV equation given above are different from those given in the Ref. [17], these solutions are solutions of novel forms to KdV equation.

6. Conclusion and discussion

In this Letter, the new Lamé equation and Lamé functions are reported and applied to solve nonlinear equations, where mKdV equation and KdV equation are take as two examples to illustrate the applications of new Lamé functions to nonlinear equations to derive the multi-order solutions of novel forms when perturbation method is involved. The results got in this Letter is very important for nonlinear instability of nonlinear coherent structures of nonlinear equations. Additionally, the method and results given in this Letter can be easily applied to more nonlinear systems.

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