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Basic and Fluctuating Periodic Instantons in Quantum Tunneling*

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Abstract *Under condition of four potential fields, equations of motion and fluctuations in imaginary time are utilized to analytically derive the basic and fluctuating periodic instantons. It is shown that the basic instantons satisfy the elliptic or simple pendulum equations and their solutions are Jacobi elliptic functions, and fluctuating periodic instantons satisfy the Lamé equation and their solutions are Lamé functions. These results indicate that there exists the common solution family for different potential fields which are called the super-symmetry family.*

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Key words: instanton, quantum tunneling, Jacobi elliptic functions, Lamé functions

1 Introduction

The research fruits of quantum physics possess both important theoretical and experimental senses. Especially, the quantum tunneling where motion of particle can tunnel through the potential barrier has become the baseline of modern natural sciences and new technologies. In the eighties of the 20th century, the researches of quantum tunneling have been started[1−2] and Chinese scientists Liang *et al.*[3−12] have also made great progress in this field. In this paper, the analytical periodic intantons in quantum tunneling are derived normally.

Assuming that the potential and total energies of a particle are V and E, respectively. When $V < E$, there exists certain orbit for the classical particle. However, when $E \leq V$, the classical particle is unable to tunnel through the potential fields. For this reason, the imaginary time

$$\tau = \mathrm{i}t \quad (\mathrm{i} = \sqrt{-1}) \tag{1}$$

is introduced. Such that the classical particle becomes the pseudo-particle, which can tunnel through the potential fields and brings about the quantum tunneling.

Using the imaginary time, the equation of motion per unit mass is given by

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = V'(\phi)\,, \tag{2}$$

and then integrating equation (2) yields

$$\frac{1}{2}\Big(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\Big)^2 - V(\phi) = -E\,, \tag{3}$$

where ϕ is the displacement or wave function for the particle.

From Eq. (3), we see that $0 \leq E \leq V$ and then the potential field of pseudo-particle can be taken as $-V$. The solution of Eq. (3) is called the instanton configuration.

Considering that ϕ and ψ are ground and fluctuating states for the particle, respectively, then the fluctuating equation may be written as

$$\frac{1}{2}\Big[-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + V''(\phi)\Big]\psi = \omega^2\psi\,, \tag{4}$$

where ω is the circular frequency if fluctuation with $\omega^2 \geq 0$. If $\omega^2 < 0$, it implies that the ground state is unstable.

2 Basic and Fluctuating Periodic Instanton in Quantum Tunneling

Below, we will discuss two cases of potential field in details.

2.1 *Double-Well Potential Field*

The double-well potential field or ϕ^4 potential field is given by

$$V(\phi) = \frac{\omega_0^2}{2a^2}(\phi^2 - a^2)^2 = -\omega_0^2\Big(\phi^2 - \frac{\phi^4}{2a^2}\Big) + V_0\,,$$
$$\Big(V_0 = \frac{1}{2}\omega_0^2 a^2\Big)\,, \tag{5}$$

which is shown in Fig. 1 and ω_0 and a are two positive constants. Figure 1 shows that the potential field contains one potential barrier and two potential wells.

From Fig. 1, we can see that there are three equilibrium states which make $V'(\phi) = 0$ such that

$$\phi_0^* = 0\,, \quad \phi_1^* = -a\,, \quad \phi_2^* = a\,. \tag{6}$$

Since

$$V'(\phi) = -2\omega_0^2\Big(\phi - \frac{\phi^3}{a^2}\Big)\,, \quad V''(\phi) = -2\omega_0^2\Big(1 - \frac{3\phi^2}{a^2}\Big)\,. \tag{7}$$

So $\phi_0^* = 0$ is the maximum point which is unstable in the potential field, the maximum value is $V_0 = (1/2)\omega_0^2 a^2$, which is the top height of potential barrier and taken to be the potential barrier height. While $\phi_1^* = -a$ and $\phi_2^* = a$ are the minimum points, which are stable in the potential field, the minimum value is zero.

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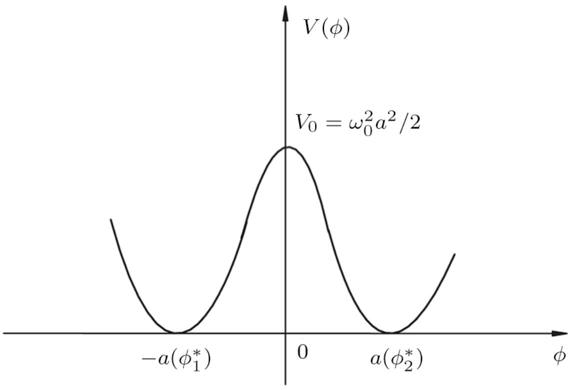

Fig. 1 Schematic plot of double-well potential field.

(i) ***Basic Periodic Instanton***

Substituting Eq. (5) into Eq. (3), we have

$$\Big(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\Big)^2 = (-2E+\omega_0^2a^2) - 2\omega_0^2\phi^2 + \frac{\omega_0^2}{a^2}\phi^4\,. \tag{8}$$

Notice that one solution to nonlinear elliptic equation[13]

$$\Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2 = k^2A^2 - (1+m^2)k^2y^2 + \frac{k^2m^2}{A^2}y^4 = \frac{k^2}{A^2}(A^2-y^2)(A^2-m^2y^2)\,, \tag{9}$$

is given by

$$y = A\mathrm{sn}(k(x-x_0),m)\,, \tag{10}$$

where $\mathrm{sn}\,(k(x-x_0),m)$ is the Jacobi elliptic sine function with its modulus m ($0\le m\le 1$), and x_0 is an arbitrary constant.

Comparing Eq. (8) with Eq. (9) leads to

$$k=\sqrt{\frac{2}{1+m^2}}\omega_0\,,\quad A=\pm\frac{kam}{\omega_0}\,, \qquad E=b^2V_0\,,\quad \Big(b=\frac{1-m^2}{1+m^2}\Big)\,, \tag{11}$$

and one solution to Eq. (8) is

$$\phi=\pm\sqrt{\frac{2}{1+m^2}}ma\mathrm{sn}\Big(\sqrt{\frac{2}{1+m^2}}\omega_0(\tau-\tau_0),m\Big)\,, \tag{12}$$

with an arbitrary constant τ_0. Equation (12) is the general periodic solution of quantum tunneling in double-well potential field. When the right hand of Eq. (12) takes the positive sign, then its figure is shown in Fig. 2 as the thin solid line. Equation (12) is called as the basic periodic instanton comparing to fluctuation. Since the period of $\mathrm{sn}(x,m)$ is $4K(m)$, then the period of basic periodic instanton is given by

$$T=4\sqrt{\frac{1+m^2}{2}}\frac{K(m)}{\omega_0}\,, \tag{13}$$

with the complete elliptic integral of the first kind

$$K(m)=\int_0^{\pi/2}\frac{1}{\sqrt{1-m^2\sin^2\varphi}}\mathrm{d}\varphi\,. \tag{14}$$

The solid thin line in Fig. 2 indicates $\phi=0$ at $\sqrt{2/(1+m^2)}\omega_0(\tau-\tau_0)=0$, and $\tau=\tau_0$ corresponds to ϕ_0^*. $\phi=-a$ at $\sqrt{2/(1+m^2)}\omega_0(\tau-\tau_0)=-K(m)$ and $\tau=\tau_1$ corresponds to $\phi_1^*=-a$. $\phi=a$ at $\sqrt{2/(1+m^2)}\omega_0(\tau-\tau_0)=K(m)$ and $\tau=\tau_2$ corresponds to $\phi_2^*=a$. Thus the pseudo-particle starts from τ_1 and tunnels through the potential barrier at τ_0, it will come close to the potential barrier height with increasing m and reaches the potential barrier height at $m=1$, at last it completes one tunneling at τ_2. In fact the pseudo-particle is reciprocating oscillation between τ_1, τ_0, and τ_2.

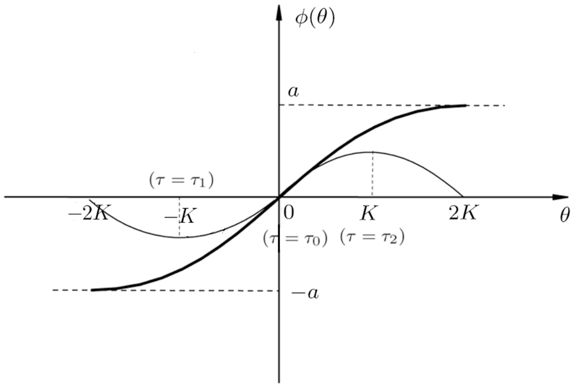

Fig. 2 Schematic plot of basic periodic instanton (thin solid line) and basic instanton (thick solid line) in double-well potential field.

Now, we illustrate two marginal cases for $m=0$ and $m=1$.

(a) When $m=0$, then $b=1$ and $E=V_0$, the total energy reaches the maximum, and the potential barrier height Eq. (12) is degenerated into

$$\phi=0\,. \tag{15}$$

This implies that ϕ is zero solution, and the pseudo-particle is located to the unstable potential barrier height where $\phi=0$ is known as the sphaleron.

(b) When $m=1$, then $b=0$ and $E=0$, the total energy reaches the minimum and Eq. (12) is degenerated into

$$\phi=\pm a\mathrm{tanh}\omega_0(\tau-\tau_0)\,, \tag{16}$$

which is the kink solution known as topological soliton. When the right hand of Eq. (16) takes the positive sign, then its figure is shown in Fig. 2 in thick solid line, where we can see that $\tau\to-\infty,\phi\to-a$; $\tau\to+\infty,\phi\to a$. Since $K(m)\to+\infty$ at $m=1$, then the topological soliton is the periodic instanton with infinite period, and it is also called the basic instanton.

As the physics is concerned, we must give the following two statements:

(a) Since $0\le E\le V$, $V=0$ at $\phi^*=\pm a$ certainly leads to $E=0$ at $\phi^*=\pm a$, which is known as the macroscopic quantum states. And for basic instanton represented by Eq. (16), there is $E=0$ as well, so there exists degenerated states to $E=0$.

(b) The topological soliton links up smoothly the two macroscopic quantum states $\phi_1^*=-a$ and $\phi_2^*=a$. Such that they interfere each other and split up the energy level.

In mathematics, the topological soliton is called also the heteroclinic orbit. However, in physics, it is called the connection of quantum tunneling, which is also called the tunnel effect.

(ii) ***Fluctuating Periodic Instanton***

In order to solve the fluctuating equation (4), it is convenient to introduce the dimensionless variable

$$\theta = \sqrt{\frac{2}{1+m^2}}\omega_0(\tau - \tau_0)\,, \tag{17}$$

which is the phase function. So the basic period instanton (12) can be rewritten as

$$\phi = \pm\sqrt{\frac{2}{1+m^2}}ma\mathrm{sn}(\theta, m)\,, \tag{18}$$

by means of Eq. (17), the fluctuating equation (4) in double-well potential field reduces to

$$\frac{1}{2}\Big[-\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \frac{1+m^2}{2\omega_0^2}V''(\phi)\Big]\psi = \frac{(1+m^2)\omega^2}{\omega_0^2}\psi\,. \tag{19}$$

Substituting $V''(\phi)$ in Eqs. (7) and (18) into Eq. (19) yields

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\theta^2} + (\lambda - 6m^2\mathrm{sn}^2\theta)\psi = 0\,, \tag{20}$$

with

$$\lambda = (1+m^2)\Big(1 + \frac{\omega^2}{\omega_0^2}\Big)\,. \tag{21}$$

Equation (20) is just a Lamé equation[14–16]

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (\lambda - l(1+l)\mathrm{sn}^2 x)y = 0\,, \tag{22}$$

for the case of $l = 2$. According to the theory of Lamé equation (see Appendix), we find that the modes and periodic solutions known as the fluctuating periodic instantons of Eq. (20) are given by

(a) $\lambda = 1 + m^2$, i.e. $(1+m^2)(1+\omega^2/\omega_0^2) = 1 + m^2$ and then

$$\omega^2 = 0\,, \quad (\text{zero mode})\,, \quad \psi_0 = C_0\mathrm{cn}\theta\mathrm{dn}\theta\,, \tag{23}$$

where $\mathrm{cn}\theta$ and $\mathrm{dn}\theta$ are the Jacobi elliptic cosine function and Jacobi elliptic function of the third kind, respectively, and C_0 is an arbitrary constant.

Taking $m = 0$ and $m = 1$, then Eq. (23) degenerates to

$$\omega^2 = 0\,, \quad \psi_0 = C_0\cos\theta\,, \tag{24}$$

$$\omega^2 = 0\,, \quad \psi_0 = C_0\mathrm{sech}^2\theta\,. \tag{25}$$

(b) $\lambda = 1 + 4m^2$, i.e. $(1+m^2)(1+\omega^2/\omega_0^2) = 1 + 4m^2$ and then

$$\omega^2 = \frac{3m^2}{1+m^2}\omega_0^2, \quad (\text{positive and zero modes})\,,$$
$$\psi_1 = C_1\mathrm{sn}\theta\mathrm{dn}\theta\,, \tag{26}$$

where C_1 is an arbitrary constant.

When $m = 0$ and $m = 1$, then Eq. (26) degenerates to

$$\omega^2 = 0\,, \quad \psi_1 = C_1\sin\theta\,, \tag{27}$$

$$\omega^2 = \frac{3}{2}\omega_0^2\,, \quad \psi_1 = C_1\tanh\theta\mathrm{sech}\theta\,. \tag{28}$$

(c) $\lambda = 4 + m^2$, i.e. $(1+m^2)(1+\omega^2/\omega_0^2) = 4 + m^2$ and then

$$\omega^2 = \frac{3}{1+m^2}\omega_0^2, \quad (\text{positive mode}), \quad \psi_2 = C_2\mathrm{sn}\theta\mathrm{cn}\theta\,, \tag{29}$$

where C_2 is an arbitrary constant.

When $m = 0$ and $m = 1$, then Eq. (29) degenerates to

$$\omega^2 = 3\omega_0^2\,, \quad \psi_2 = C_2\cos\theta\sin\theta\,, \quad (m = 0)\,,$$

$$\omega^2 = \frac{3}{2}\omega_0^2\,, \quad \psi_2 = C_2\tanh\theta\mathrm{sech}\theta\,, \quad (m = 1)\,. \tag{31}$$

(d) $\lambda = 2[(1+m^2) \pm \sqrt{1 - m^2 + m^4}]$, i.e. $(1+m^2)(1+\omega^2/\omega_0^2) = 2[(1+m^2) \pm \sqrt{1 - m^2 + m^4}]$ and then

$$\omega^2 = \omega_0^2\Big(1 \pm \frac{2M}{1+m^2}\Big), \quad (\text{all kinds of modes})\,,$$

$$\psi_3 = C_3\Big[\mathrm{sn}^2\theta - \frac{(1+m^2 \mp M)}{3m^2}\Big]\,, \tag{32}$$

with $M = \sqrt{1 - m^2 + m^4}$ and where C_3 is an arbitrary constant. When $m = 0$ and $m = 1$, then Eq. (32)

$$\omega^2 = (1 \pm 2)\omega_0^2\,, \quad \psi_3 = C_3\Big(\sin^2\theta - \frac{1}{3}\Big)\,, \quad (m = 0)\,, \tag{33}$$

$$\omega^2 = (1 \pm 1)\omega_0^2, \ \psi_3 = C_3\Big[\tanh^2\theta - \frac{1}{3}(2 \mp 1)\Big], \ (m = 1). \tag{34}$$

2.2 *Sine-Gordon Potential Field*

The sine-Gordon potential field is given by

$$V(\phi) = \frac{\omega_0^2}{\mu^2}(1 + \cos\mu\phi) = V_0\cos^2\frac{\mu\phi}{2}\,, \quad \Big(V_0 = \frac{2\omega_0^2}{\mu^2}\Big)\,, \tag{35}$$

which is plotted in Fig. 3, and ω_0 and μ are positive constants. Similar to Fig. 1, there are one potential barrier and two potential wells in $-(\pi + \varepsilon) \leq \mu\phi \leq (\pi + \varepsilon), (0 < \varepsilon \leq 1)$.

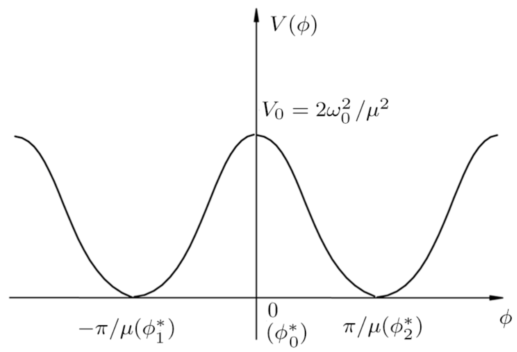

Fig. 3 Schematic plot of sine-Gordon potential field.

There are three equilibrium states which make

$$V'(\phi) = 0\,, \quad \phi_0^* = 0\,, \quad \phi_1^* = -\frac{\pi}{\mu}\,, \quad \phi_2^* = \frac{\pi}{\mu}\,. \tag{36}$$

Since

$$V'(\phi) = -\frac{\omega_0^2}{\mu^2}\sin\mu\phi\,, \quad V''(\phi) = -\omega_0^2\cos\mu\phi\,, \tag{37}$$

there $\phi_0^* = 0$ is the maximum point, which is unstable in the potential field, and the maximum value is

$V_0 = 2\omega_0^2/\mu^2$, which is the top height of potential barrier and is known as the potential barrier height. And $\phi_1^* = -\pi/\mu$ and $\phi_2^* = \pi/\mu$ are the minimum points, which are stable in the potential field, the minimum value is zero.

(i) ***Basic Periodic Instanton***

Substituting Eq. (35) in Eq. (3), we have Substituting Eq. (5) into Eq. (3), we have

$$\Big(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\Big)^2 = \frac{4\omega_0^2}{\mu^2}\Big[\Big(1-\frac{E}{V_0}\Big) - \sin^2\frac{\mu\phi}{2}\Big], \tag{38}$$

which is the equation of simple pendulum motion, its normal form[13] is given by

$$\Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2 = 4k^2\Big[m^2 - \sin^2\frac{y}{2}\Big], \tag{39}$$

one of its solution is

$$\sin\frac{y}{2} = \pm m\,\mathrm{sn}(k(x-x_0), m). \tag{40}$$

Comparing Eq. (38) with Eq. (39) yields

$$k = \frac{\omega_0}{\mu}, \quad E = (1-m^2)V_0, \tag{41}$$

and then the solution to Eq. (38) is

$$\sin\frac{\mu\phi}{2} = \pm m\,\mathrm{sn}(\omega_0(\tau-\tau_0), m), \tag{42}$$

where τ_0 is an arbitrary constant. Equation (42) is the general periodic solution known as the basic periodic instanton of quantum tunneling in sine-Gordon potential field. When the right hand of Eq. (42) takes the positive sign, then its figure is shown in Fig. 4 in thin solid line. The periodic of basic periodic instanton is given by

$$T = \frac{4K(m)}{\omega_0}. \tag{43}$$

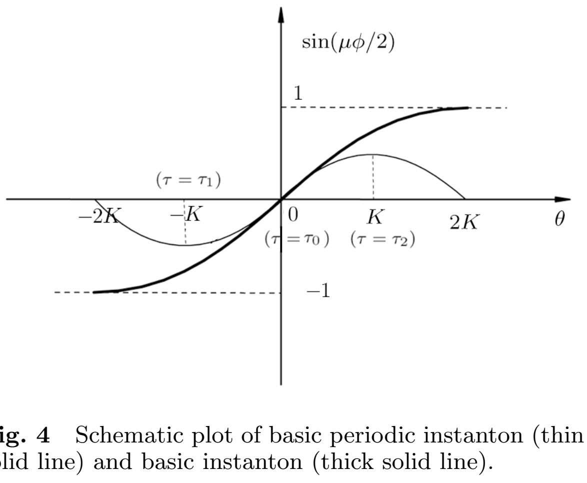

Fig. 4 Schematic plot of basic periodic instanton (thin solid line) and basic instanton (thick solid line).

Similar to the case for double-well potential field there exist also quantum tunneling and reciprocating oscillation for the pseudo-particle in sine-Gordon potential field. Now, we illustrate two marginal cases for $m = 0$ and $m = 1$.

(a) When $m = 0$ then $E = V_0$, the total energy reaches the maximum, Eq. (42) is degenerated into

$$\phi = 0, \quad (-(\pi+\varepsilon) \le \mu\phi \le (\pi+\varepsilon)), \tag{44}$$

which is just the sphaleron.

(b) When $m = 1$ then $E = 0$, the total energy reaches the minimum, Eq. (42) is degenerated into the following basic instanton

$$\sin\frac{\mu\phi}{2} = \pm\tanh\omega_0(\tau-\tau_0), \tag{45}$$

which is just the topological soliton and it links up smoothly $\phi_1^* = -\pi/\mu$ and $\phi_2^* = \pi/\mu$, so it is also called heteroclinic orbit. When the right hand of Eq. (45) takes the positive sign, detailed plot can be found in Fig. 4 in thick solid line.

Similar to the case for double-well potential field there also exist the degenerated states to $E = 0$ and the connection of quantum tunneling in the sine-Gordon potential field.

(ii) ***Fluctuating Periodic Instanton***

In order to solve the fluctuating equation (4), we introduce the following dimensionless variable

$$\theta = \omega_0(\tau-\tau_0), \tag{46}$$

so the basic periodic instanton (42) can be rewritten as

$$\sin\frac{\mu\phi}{2} = \pm m\,\mathrm{sn}(\theta, m). \tag{47}$$

By means of Eq. (46), the fluctuating equation (4) in sine-Gordon potential field reduces to

$$\frac{1}{2}\Big[-\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \frac{1}{\omega_0^2}V''(\phi)\Big]\psi = \frac{\omega^2}{\omega_0^2}\psi. \tag{48}$$

Substituting $V''(\phi)$ in Eqs. (37) and (47) into Eq. (48) yields

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\theta^2} + (\lambda - 2m^2\mathrm{sn}^2\theta)\psi = 0, \tag{49}$$

with

$$\lambda = 1 + \frac{2\omega^2}{\omega_0^2}. \tag{50}$$

Equation (49) is just a Lamé equation (22) for the case of $l = 1$. According to the theory of Lam*é* equation (see Appendix), we obtain the modes and fluctuating periodic instanton of Eq. (49) given by

(a) $\lambda = 1$ i.e. $1 + 2\omega^2/\omega_0^2 = 1$ and then

$$\omega^2 = 0, \quad (\text{zero mode}), \quad \psi_0 = C_0\mathrm{cn}\theta. \tag{51}$$

(b) $\lambda = 1 + m^2$ i.e. $1 + 2\omega^2/\omega_0^2 = 1 + m^2$ and then

$$\omega^2 = \frac{m^2}{2}\omega_0^2, \quad (\text{positive and zero modes}),$$
$$\psi_1 = C_1\mathrm{sn}\theta, \tag{52}$$

(c) $\lambda = m^2$ i.e. $1 + 2\omega^2/\omega_0^2 = m^2$ and then

$$\omega^2 = -\frac{1-m^2}{2}\omega_0^2, \quad (\text{negative and zero modes}),$$
$$\psi_2 = C_2\mathrm{dn}\theta. \tag{53}$$

3 Basic Periodic Bounce and Fluctuating Periodic Instanton in Quantum Tunneling

In this section, we will discuss still two cases of potential fields in details.

3.1 ***Double-Barrier Potential Field***

The double-barrier potential field is also a kind of ϕ^4 potential field, and it is given by

$$V(\phi)=\frac{\omega_0^2}{2a^2}[\phi^4-(\phi^2-a^2)^2]=\frac{2V_0}{a^2}\Big(\phi^2-\frac{1}{2a^2}\phi^4\Big)\,,\quad \Big(V_0=\frac{1}{2}\omega_0^2a^2\Big)\,, \tag{54}$$

which is plotted in Fig. 5, and ω_0 and a are positive constants. Figure 5 shows that there are two potential barriers and a potential well.

Comparing Fig. 5 with Fig. 1, we can see that Fig. 5 is the transposition of Fig. 1, and $V(\phi)\geq 0$ requires that $-\sqrt{2}a\leq\phi\leq\sqrt{2}a$.

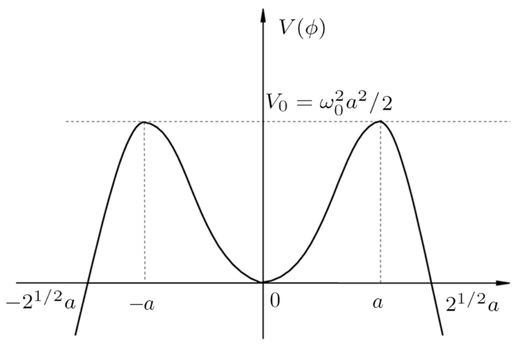

Fig. 5 Schematic plot of double-barrier potential field.

From Fig. 5, there are three equilibrium states

$$\phi_0^*=0\,,\quad \phi_1^*=-a\,,\quad \phi_2^*=a\,, \tag{55}$$

which is identical to Eq. (6) for double-well potential field. However, since

$$V'(\phi)=2\omega_0^2\Big(\phi-\frac{1}{a^2}\phi^3\Big)\,,\quad V''(\phi)=2\omega_0^2\Big(1-\frac{3}{a^2}\phi^2\Big)\,, \tag{56}$$

the sign is different from Eq. (7), so ϕ_0^* is stable and ϕ_1^* and ϕ_2^* are unstable. The maximum and minimum values are V_0 and zero, respectively. Besides, in Fig. 5 there are

$$\hat{\phi}_1^*=-\sqrt{2}a\,,\quad \hat{\phi}_2^*=\sqrt{2}a\,, \tag{57}$$

and their potential energies are zeros, which are known as the reverser, and its explanation will be presented next.

(i) ***Basic Periodic Instanton***

Substituting Eq. (54) into Eq. (3) yields

$$\Big(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\Big)^2=-2E+2\omega_0^2\phi^2-\frac{\omega_0^2}{a^2}\phi^4\,, \tag{58}$$

and notice that one solution of nonlinear elliptic equation[13]

$$\Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2=-k^2A^2n'^2+(1+n'^2)k^2y^2-\frac{k^2}{A^2}y^4=\frac{k^2}{A^2}(A^2-y^2)(A^2-n'^2y^2) \tag{59}$$

is given by

$$y=A\mathrm{dn}(k(x-x_0),n)\,. \tag{60}$$

Comparing Eq. (58) with Eq. (59) leads to

$$k=\sqrt{\frac{2}{1+n'^2}}\omega_0\,,\quad A=\pm\frac{ka}{\omega_0}\,,\quad E=b^2V_0\,,\quad \Big(b=\frac{1-m^2}{1+m^2}\Big) \tag{61}$$

and one solution to Eq. (58) is

$$\phi=\pm\sqrt{\frac{2}{1+n'^2}}a\mathrm{dn}\Big(\sqrt{\frac{2}{1+n'^2}}\omega_0(\tau-\tau_0),n\Big)\,, \tag{62}$$

where τ_0 is an arbitrary constant. Equation (62) is the general periodic solution of quantum tunneling in double-barrier potential field with modulus $n'(n'=\sqrt{1-n^2})$. Here n distinguishes from m in double-well potential field, and their relation is given by

$$n^2=\frac{4m}{(1+m)^2}\,,\quad n'^2=\Big(\frac{1-m}{1+m}\Big)^2\,. \tag{63}$$

Equation (62) is also the basic periodic instanton of quantum tunneling in double-barrier potential field with the period

$$T=2\sqrt{\frac{1+n'^2}{2}}\frac{K(n)}{\omega_0}\,. \tag{64}$$

When the right hand of Eq. (62) takes the positive sign, and then its corresponding figure is plotted in thin solid line in Fig. 6, from which we can see $\phi=\sqrt{2/(1+n'^2)}a$ and $\phi=\sqrt{2}a$ at $\phi=\sqrt{2/(1+n'^2)}\omega_0(\tau-\tau_0)=0$ when $n=1$, so $\tau=\tau_0$ corresponds to $\hat{\phi}_2^*=\sqrt{2}a$. $\phi=0$ at $\phi=\sqrt{2/(1+n'^2)}\omega_0(\tau-\tau_0)=-K(n)$, so $\tau=\tau_1$ corresponds to $\phi_0^*=0$. Thus the pseudo-particle starts from τ_1 and tunnels through the potential barrier at τ_0, it will come close to $\hat{\phi}_2^*=\sqrt{2}a$ with increasing n and reach $\hat{\phi}_2^*=\sqrt{2}a$ at $n=1$, later it returns to beginning point. So τ_0 and $\hat{\phi}_2^*=\sqrt{2}a$ known as the turning points and reverser, respectively, while the basic periodic instanton represented by Eq. (62) is known as the basic periodic bounce.

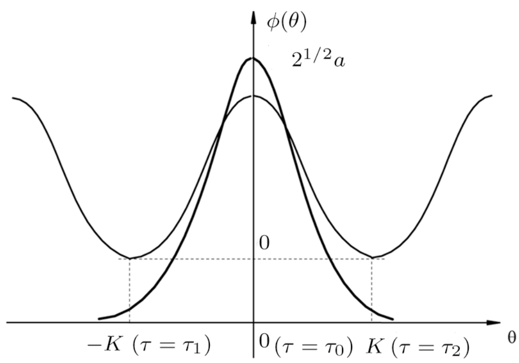

Fig. 6 Schematic plot of basic bounce(thin solid line) and basic bounce (thick solid line) in double-barrier potential field.

So far as the whole double-barrier potential field, the pseudo-particle begins from one turning point and tunnels through two potential barriers and finally reaches another turning point.

Now, we will illustrate the marginal cases for $n = 0$ and $n = 1$.

(a) when $n = 0$, then $n' = 1$ and $E = V_0$, the total energy reaches its maximum, the potential barrier height and Eq. (62) is degenerated into

$$\phi = \pm a\,, \tag{65}$$

which is the sphaleron.

(b) when $n = 1$, then $n' = 0$ and $E = 0$, the total energy reaches its minimum and Eq. (62) is degenerated into

$$\phi = \pm\sqrt{2}a\mathrm{sech}\sqrt{2}\omega_0(\tau - \tau_0)\,, \tag{66}$$

which is non-topological soliton known as the basic bounce with infinite period. Its figure is shown in thick solid line in Fig. 6, when the right hand of Eq. (66) takes the positive sign.

When physics is concerned, two points must be noted.

(a) Since $0 \le E \le V$, $V = 0$ at $\hat{\phi} = \pm\sqrt{2}a$ must lead to $E = 0$ at $\hat{\phi} = \pm\sqrt{2}a$, which may be known as the macroscopic quantum states. For basic bounce represented by Eq. (66), we also have $E = 0$. So there exist the degenerated states to $E = 0$, which is similar to the conclusion in the double-well potential field.

(b) For the non-topological soliton Eq. (66) the pseudo-particle begins from $\tau = \tau_0$ and tunnels through the potential barrier and returns to $\tau = \tau_0$. In mathematics, the non-topological soliton is also called the homoclinic orbit. Notice that in former two potential fields, the pseudo-particle enters into the potential well after it tunnels through the potential barrier. However, in the case of double-barrier potential field the pseudo-particle is able to enter into the region of $V < 0$ and it makes the decay of ground states and energy level take place, it is called the decay of quantum tunneling, which is different from the conclusion in double-well potential field.

(ii) ***Fluctuating Period Instanton***

In order to solve the fluctuating equation, it is convenient to introduce the dimensionless variable

$$\theta = \sqrt{\frac{2}{1+n'^2}}\omega_0(\tau - \tau_0)\,, \tag{67}$$

then the basic periodic bounce (62) can be rewritten as

$$\phi = \pm\sqrt{\frac{2}{1+n'^2}}a\mathrm{dn}(\theta, n)\,. \tag{68}$$

By means of Eq. (67), the fluctuating equation (4) in double-barrier potential field reduces to

$$\frac{1}{2}\Big[-\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \frac{1+n'^2}{\omega_0^2}V''(\phi)\Big]\psi = \frac{(1+n'^2)\omega^2}{\omega_0^2}\psi\,. \tag{69}$$

Substituting $V''(\phi)$ in Eqs. (56) and (62) into Eq. (69) yields

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\theta^2} + (\lambda - 6n^2\mathrm{sn}^2\theta)\psi = 0\,, \tag{70}$$

with

$$\lambda = (2-n^2)\frac{\omega^2}{\omega_0^2} + (4+n^2)\,. \tag{71}$$

The form of Lamé equation (70) is the same as Eq. (20) only is replaced m by n. Thus we obtain the modes and fluctuating periodic instantons as

(a) $\lambda = 4 + n^2$, i.e. $(2-n^2)\omega^2/\omega_0^2 + (4+n^2) = 4 + n^2$ and then

$$\omega^2 = 0\,, \quad (\text{zero mode})\,, \quad \psi_0 = C_0\mathrm{sn}\theta\mathrm{cn}\theta\,. \tag{72}$$

(b) $\lambda = 1 + n^2$, i.e. $(2-n^2)\omega^2/\omega_0^2 + (4+n^2) = 1 + n^2$ and then

$$\omega^2 = -\frac{3}{(2-n^2)}\omega_0^2\,, \quad (\text{negative mode})\,, \quad \psi_1 = C_1\mathrm{cn}\theta\mathrm{dn}\theta\,. \tag{73}$$

(c) $\lambda = 1 + 4n^2$, i.e. $(2-n^2)\omega^2/\omega_0^2 + (4+n^2) = 1 + 4n^2$ and then

$$\omega^2 = -\frac{3(1-n^2)}{2-n^2}\omega_0^2\,, \quad (\text{negative and zero modes})\,, \quad \psi_2 = C_2\mathrm{sn}\theta\mathrm{dn}\theta\,, \tag{74}$$

where C_2 is an arbitrary constant.

(d) $\lambda = 2[(1+n^2) \pm \sqrt{1-n^2+n^4}]$, i.e. $(2-n^2)\omega^2/\omega_0^2 + (4+n^2) = 2[(1+n^2) \pm \sqrt{1-n^2+n^4}]$ and then

$$\omega^2 = -\omega_0^2\Big(1 \mp \frac{2N}{2-n^2}\Big)\,, \quad (\text{all kinds of modes})\,, \quad \psi_3 = C_3\Big[\mathrm{sn}^2\theta - \frac{(1+n^2 \mp N)}{3n^2}\Big]\,, \tag{75}$$

with $N = \sqrt{1-n^2+n^4}$.

3.2 *Sub-Stationary State Potential Field*

The sub-stationary state potential field or ϕ^3 potential field is given by

$$V(\phi) = \frac{\omega_0^2}{2}\phi^2 - \frac{\beta}{3}\phi^3 = \frac{27V_0}{4a^3}\phi^2(a-\phi)\,,$$
$$\Big(a = \frac{3\omega_0^2}{2\beta}\,, \quad V_0 = \frac{\omega_0^6}{6\beta^2}\Big)\,, \tag{76}$$

which is plotted in Fig. 7 and ω_0 and β are two positive constants. $V(\phi) \ge 0$ requires that $-\infty < \phi \le a$. Figure 7 shows there are one potential barrier and one potential well.

From Fig. 7, we see that the equilibrium states have

$$\phi_0^* = 0\,, \quad \phi_1^* = \frac{2}{3}a\,. \tag{77}$$

Since

$$V'(\phi) = \omega_0^2\phi - \beta\phi^2\,, \quad V''(\phi) = \omega_0^2 - 2\beta\phi\,, \tag{78}$$

then ϕ_0^* is stable and the minimum value is zero and ϕ_1^* is unstable and the maximum value is V_0. Besides, in Fig. 7, we have

$$\hat{\phi} = a\,, \tag{79}$$

which is the reverser with the zero potential energy.

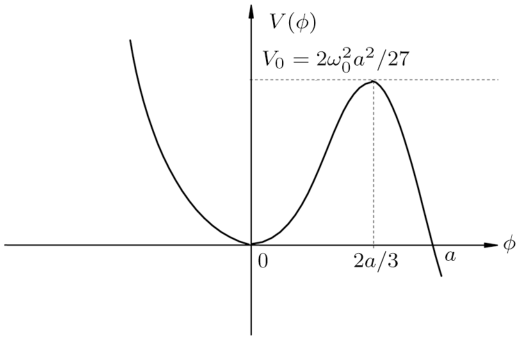

Fig. 7 Schematic plot of sub-stationary state potential field.

(i) ***Basic Periodic Instanton***

Substituting Eq. (76) into Eq. (3) leads to

$$\Big(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\Big)^2 = -2E + \omega_0^2\phi^2 - \frac{2\beta}{3}\phi^3 = -\frac{2\beta}{3}\Big(\phi^3 - \frac{3\omega_0^2}{2\beta}\phi^2 + \frac{3E}{\beta}\Big)\,. \tag{80}$$

Setting

$$\phi^3 - \frac{3\omega_0^2}{2\beta}\phi^2 + \frac{3E}{\beta} = 0\,, \tag{81}$$

to take three real roots: ϕ_1, ϕ_2 and ϕ_3 with $\phi_1 \geq \phi_2 \geq \phi_3$ and then we have

$$\phi_1 + \phi_2 + \phi_3 = \frac{3\omega_0^2}{2\beta} = a\,, \quad \phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1 = 0\,, \quad \phi_1\phi_2\phi_3 = -\frac{3E}{\beta}\,, \tag{82}$$

from which we have

$$\phi_1 = \frac{\omega_0^2}{2\beta} + \frac{\omega_0^2}{\beta}\cos\alpha\,, \quad \phi_2 = \frac{\omega_0^2}{2\beta} + \frac{\omega_0^2}{\beta}\cos\Big(\alpha + \frac{2\pi}{3}\Big)\,, \quad \phi_3 = \frac{\omega_0^2}{2\beta} + \frac{\omega_0^2}{\beta}\cos\Big(\alpha + \frac{4\pi}{3}\Big)\,, \tag{83}$$

where α satisfies

$$\cos 3\alpha = 1 - \frac{12\beta^2 E}{\omega_0^6} = 1 - \frac{2E}{V_0}\,. \tag{84}$$

So Eq. (80) can be rewritten as

$$\Big(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\Big)^2 = -\frac{2\beta}{3}(\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_3)\,. \tag{85}$$

Notice that one solution to nonlinear elliptic equation[13]

$$\Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2 = -B(y - y_1)(y - y_2)(y - y_3)\,, \quad (B > 0, y_1 \geq y_2 \geq y_3) \tag{86}$$

is given by

$$y = y_2 + (y_1 - y_2)\mathrm{cn}^2\Big(\sqrt{\frac{B(y_1 - y_3)}{4}}(x - x_0), m\Big)\,, \quad \Big(m^2 = \frac{y_1 - y_2}{y_1 - y_3}\Big)\,. \tag{87}$$

Comparing Eq. (85) with Eq. (86) leads to one solution to Eq. (85)

$$\phi = \phi_2 + (\phi_1 - \phi_2)\mathrm{cn}^2\Big(\sqrt{\frac{\beta(\phi_1 - \phi_3)}{6}}(\tau - \tau_0), m\Big)\,, \tag{88}$$

where τ_0 is an arbitrary constant, and

$$m^2 = \frac{\phi_1 - \phi_2}{\phi_1 - \phi_3}\,. \tag{89}$$

With the help of Eq. (83) and $\phi_1 - \phi_3 = (\omega_0^2/\beta)[\cos\alpha - \cos(\alpha + 4\pi/3)]$, we have

$$\sqrt{\frac{\beta(\phi_1 - \phi_3)}{6}} = \omega_1\,, \quad \omega_1 = \frac{\omega_0}{2}\sqrt{\cos\alpha - \frac{1}{\sqrt{3}}\sin\alpha}\,, \tag{90}$$

and then

$$\phi_1 - \phi_3 = \frac{6\omega_1^2}{\beta} = a_1\,, \quad \phi_1 - \phi_2 = m^2(\phi_1 - \phi_3) = m^2 a_1\,,$$

$$a_1 = a\Big(\cos\alpha - \frac{1}{\sqrt{3}}\sin\alpha\Big)\,, \tag{91}$$

so Eq. (88) can be written as

$$\phi = \phi_2 + m^2 a_1 \mathrm{cn}^2(\omega_1(\tau - \tau_0), m)\,, \tag{92}$$

which is the basic periodic bounce in sub-stationary state potential field with period

$$T = \frac{2K(m)}{\omega_1}\,. \tag{93}$$

Details of Eq. (92) is plotted in thin solid line in Fig. 8, where we can see that $\phi = \phi_1 = a$ at $\omega_1(\tau - \tau_0) = 0$, so $\tau = \tau_0$ corresponds to $\hat{\phi} = a$. $\phi = \phi_2 = 0$ at $\omega_1(\tau - \tau_0) = \pm K(m)$, so $\tau = \tau_1$ corresponds to $\phi_0^* = 0$. Then τ_0 is the turning point, $\hat{\phi} = a$ is the reverser. Hence, the pseudo-particle starts from $\tau = \tau_1$ and tunnels through the potential barrier at $\tau = \tau_0$, it is close to $\hat{\phi} = a$ as increasing of m and reaches $\hat{\phi} = a$ at $m = 1$, later it returns to $\tau = \tau_1$. That is to say, here there also exists a periodic bounce.

Now, we illustrate two marginal cases of $m = 0$ and $m = 1$.

(a) When $m = 0$, then $\phi_1 = \phi_2$ from Eq. (89) and by means of Eqs. (83), (84), (90), and (91) we have

$$\alpha = -\frac{\pi}{3}\,, \quad \cos\alpha = \frac{1}{2}\,, \quad \cos 3\alpha = -1\,, \quad \omega_1 = \frac{\omega_0}{2}\,, \quad a_1 = a\,, \tag{94}$$

$$\phi_1 = \phi_2 = \frac{\omega_0^2}{\beta} = \frac{2}{3}a\,, \quad \phi_3 = -\frac{\omega_0^2}{\beta} = -\frac{2}{3}a\,, \quad E = V_0\,, \tag{95}$$

which indicates that the total energy reaches the maximum, the potential barrier height, in $-\varepsilon < \phi \leq a$ $(0 < \varepsilon \leq 1)$ when $m = 0$. In this case, Eq. (92) is degenerated into

$$\phi_1 = \phi_2 = \phi_3 = \frac{\omega_0^2}{\beta} = \frac{2}{3}a\,, \tag{96}$$

which is the sphaleron.

(b) When $m=1$, then $\phi_2=\phi_3$ from Eq. (89) and by means of Eqs. (83), (84), (90), and (91) we have

$$\alpha=0,\quad \cos\alpha=1,\quad \cos3\alpha=1,\quad \omega_1=\frac{\omega_0}{2},\quad a_1=a, \tag{97}$$

$$\phi_1=\frac{2\omega_0^2}{3\beta}=a\,,\quad \phi_2=\phi_3=0\,,\quad E=0\,, \tag{98}$$

which indicates that the total energy reaches the minimum in $-\infty<\phi\le a-\varepsilon$ $(0<\varepsilon\le 1)$ when $m=1$. In this case, Eq. (92) is degenerated into

$$\phi=a\mathrm{sech}^2\frac{\omega_0}{2}(\tau-\tau_0)\,, \tag{99}$$

which is the basic bounce known also as the non-topological soliton or homoclinic orbit. Its details are plotted in thick solid line in Fig. 8.

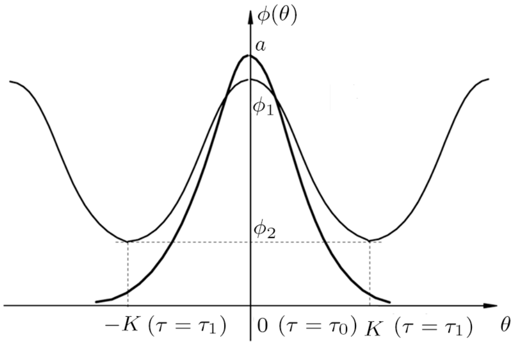

Fig. 8 Schematic plot of basic periodic bounce (thin solid line) and basic bounce (thick solid line) in sub-stationary state potential field.

Similar to the case of double-barrier potential field, there also exist the degenerated states to $E=0$ and the decay of quantum tunneling in the sub-stationary state potential field.

(ii) ***Fluctuating Periodic Instanton***

In order to solve the fluctuating equation (4), it is convenient to introduce the following dimensionless variable

$$\theta=\omega_1(\tau-\tau_0)\,, \tag{100}$$

and then the periodic bounce (92) can be rewritten as

$$\phi=\phi_1-m^2a_1\mathrm{sn}^2(\theta,m)\,. \tag{101}$$

By means of Eq. (100), the fluctuating equation (4) in sub-stationary state potential field reduces to

$$\frac{1}{2}\Big[-\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}+\frac{1}{\omega_1^2}V''(\phi)\Big]\psi=\frac{\omega^2}{\omega_1^2}\psi\,. \tag{102}$$

Substituting $V''(\phi)$ in Eq. (78) into Eq. (102) yields

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\theta^2}+(\lambda-12m^2\mathrm{sn}^2\theta)\psi=0\,, \tag{103}$$

with

$$\lambda=\frac{2\omega^2}{\omega_1^2}-\frac{\omega_0^2}{\omega_1^2}+\frac{12}{a_1}\phi_1\,. \tag{104}$$

Obviously, ϕ_1 is related to m. Considering that when $m=0$ then $\phi_1=\omega_0^2/\beta=2a/3$, $\cos\alpha=1/2$; when $m=1$ then $\phi_1=2\omega_0^2/3\beta=a$, $\cos\alpha=1$, we choose $\cos\alpha=(1+m^2)/2$. And notice that when both $m=0$ and $m=1$, there all exists $\omega_1=\omega_0^2/2$ and $a_1=a$. Hence

$$\phi_1=\frac{(2+m^2)\omega_0^2}{2\beta}=\frac{2+m^2}{3}a\,,$$

$$\lambda=\frac{8\omega^2}{\omega_0^2}+4(1+m^2)\,. \tag{105}$$

Equation (103) is just a Lamé equation (22) for the case of $l=3$. According to the theory of Lamé equation (see Appendix for details), we obtain the modes and fluctuating periodic instantons of Eq. (103) given by

(a) $\lambda=4(1+m^2)$ i.e. $8\omega^2/\omega_0^2+4(1+m^2)=4(1+m^2$ and then

$$\omega^2=0\,,\quad \text{(zero mode)}\,,\quad \psi_0=C_0\mathrm{sn}\theta\mathrm{cn}\theta\mathrm{dn}\theta\,. \tag{106}$$

(b) $\lambda=5(1+m^2)\pm2\sqrt{4-7m^2+4m^4}$ i.e. $8\omega^2/\omega_0^2+4(1+m^2)=5(1+m^2)\pm2\sqrt{4-7m^2+4m^4}$ and then

$$\omega^2=\frac{\omega_0^2}{8}[(1+m^2)\pm2D_1]\,,\quad \text{(all modes)}\,,$$

$$\psi_1=C_1\mathrm{sn}\theta\Big[1-\frac{2(1+m^2)\pm D_1}{3}\mathrm{sn}^2\theta\Big]\,, \tag{107}$$

with $D_1=\sqrt{4-7m^2+4m^4}$.

(c) $\lambda=2m^2+5\pm2\sqrt{4-m^2+m^4}$ i.e. $8\omega^2/\omega_0^2+4(1+m^2)=2m^2+5\pm2\sqrt{4-m^2+m^4}$ and then

$$\omega^2=\frac{\omega_0^2}{8}[(1-2m^2)\pm2D_2]\,,\quad \text{(all modes)}\,,\quad \psi_2=C_2\mathrm{cn}\theta[1-(2+m^2\pm2D_2)\mathrm{sn}^2\theta]\,, \tag{108}$$

with $D_2=\sqrt{4-m^2+m^4}$.

(d) $\lambda=5m^2+2\pm2\sqrt{1-m^2+4m^4}$ i.e. $8\omega^2/\omega_0^2+4(1+m^2)=5m^2+2\pm2\sqrt{1-m^2+4m^4}$ and then

$$\omega^2=\frac{\omega_0^2}{8}[(m^2-2)\pm2D_3]\,,\quad \text{(all modes)}\,,\quad \psi_3=C_3\mathrm{dn}\theta[1-(1+2m^2\pm D_3)\mathrm{sn}^2\theta]\,, \tag{109}$$

with $D_3=\sqrt{1-m^2+4m^4}$.

4 Conclusion and Discussion

There are many mathematical problems in specific physics fields, for the quantum tunneling, it is of great importance to solve analytically the related model under different potential fields. Here we show that the special functions are really helpful to reach this aim. The basic instantons satisfy the elliptic or simple pendulum equations and their solutions are Jacobi elliptic functions, and fluctuating periodic instantons satisfy the Lamé equation and their solutions are Lamé functions. These results indicate that there exists the common solution family for different potential fields, which are called the super-symmetry family. At the same time, these analytical solutions and their features tell us why and how the quantum tunneling has happened in specific models.

Appendix

The Lamé equation is defined as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+[\lambda-l(l+1)m^2\mathrm{sn}^2x]y=0\,, \tag{A1}$$

where l is a positive integer, m ($0\le m\le 1$) is the modulus, λ is the eigenvalue, which satisfies the periodicity boundary condition, and the corresponding eigenfunction is known as the Lamé function.

(i) In case of $l=1$, Lamé equation Eq. (A1) reduces to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(\lambda-2m^2\mathrm{sn}^2x)y=0\,, \tag{A2}$$

with its eigenvalue and eigenfunction given by

$$y=\mathrm{sn}(x,m)\,,\quad(\lambda=1+m^2)\,;\quad y=\mathrm{cn}(x,m)\,,\quad(\lambda=1)\,;\quad y=\mathrm{dn}(x,m),\quad(\lambda=m^2)\,. \tag{A3}$$

(ii) In case of $l=2$, Lamé equation (A1) reduces to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(\lambda-6m^2\mathrm{sn}^2x)y=0\,, \tag{A4}$$

with its eigenvalue and eigenfunction given by

$$y=\mathrm{cn}(x,m)\mathrm{dn}(x,m),\quad(\lambda=1+m^2)\,, \tag{A5}$$

$$y=\mathrm{sn}(x,m)\mathrm{dn}(x,m),\quad(\lambda=1+4m^2)\,, \tag{A6}$$

$$y=\mathrm{sn}(x,m)\mathrm{cn}(x,m),\quad(\lambda=4+m^2)\,, \tag{A7}$$

$$y=\mathrm{sn}^2(x,m)-\frac{1+m^2\mp\sqrt{1-m^2+m^4}}{3m^2}\,,\quad(\lambda=2[(1+m^2)\pm\sqrt{1-m^2+m^4}])\,. \tag{A8}$$

(iii) In case of $l=3$, Lam eequation (A1) reduces to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(\lambda-12m^2\mathrm{sn}^2x)y=0\,, \tag{A9}$$

with its eigenvalue and eigenfunction given by

$$y=\mathrm{sn}(x,m)\mathrm{cn}(x,m)\mathrm{dn}(x,m)\,,\quad(\lambda=4(1+m^2))\,, \tag{A10}$$

$$y=\mathrm{sn}(x,m)\Big[1-\frac{2(1+m^2)\pm\sqrt{4-7m^2+4m^4}}{3}\mathrm{sn}^2(x,m)\Big]\,, \tag{A11}$$

with $\lambda=5(1+m^2)\pm2\sqrt{4-7m^2+4m^4}$.

$$y=\mathrm{cn}(x,m)[1-(m^2+2\pm2\sqrt{4-m^2+m^4})\mathrm{sn}^2(x,m)]\,,\quad(\lambda=5+2m^2\pm2\sqrt{4-m^2+m^4})\,, \tag{A12}$$

$$y=\mathrm{dn}(x,m)[1-(2m^2+1\pm\sqrt{1-m^2+4m^4})\mathrm{sn}^2(x,m)]\,,\quad(\lambda=5m^2+2\pm2\sqrt{1-m^2+4m^4})\,. \tag{A13}$$

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