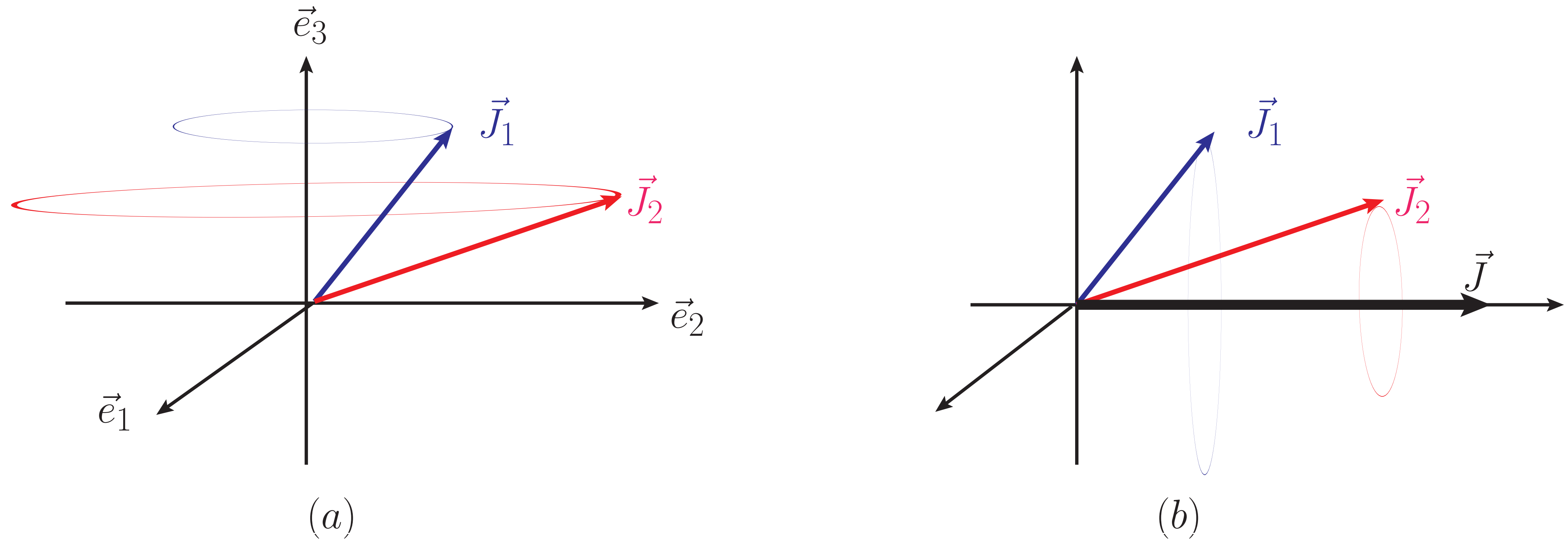
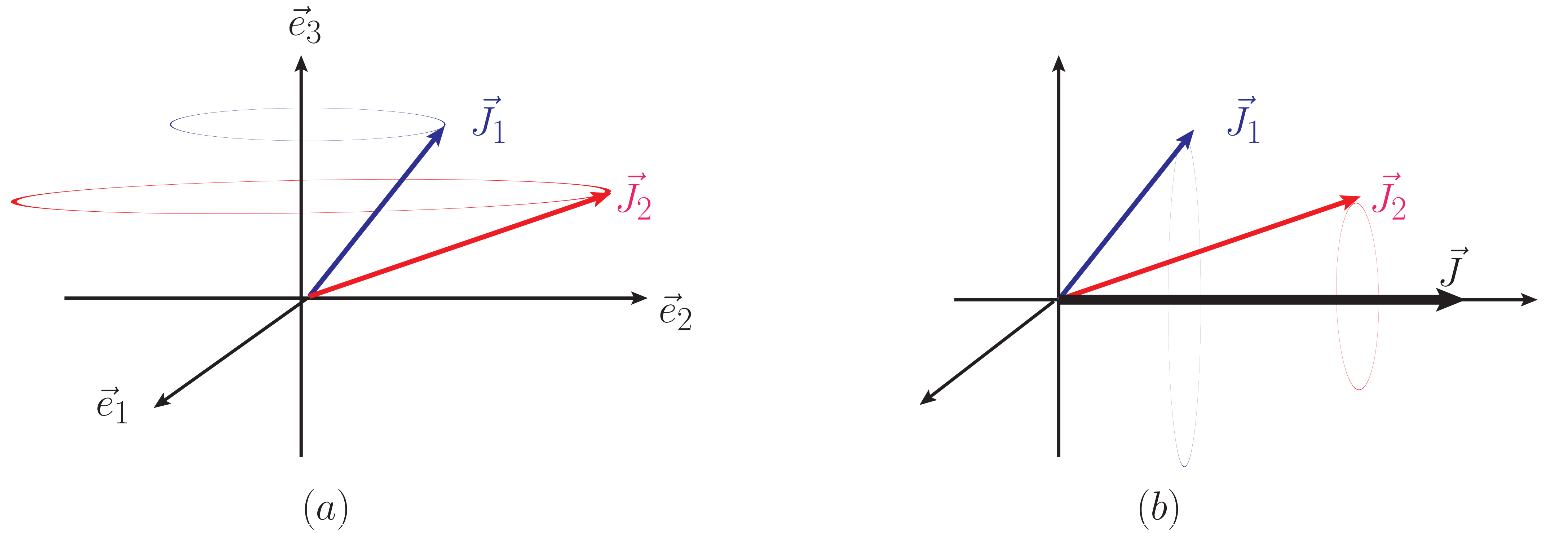
角动量耦合和
Clebsch-Gordon系数



经典物理中，两个物体的角动量是作用在同一空间中，因此总角动量等于各自角动量分量之和

量子力学中，两个物体的角动量作用在不同的希尔伯特空间

$$\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

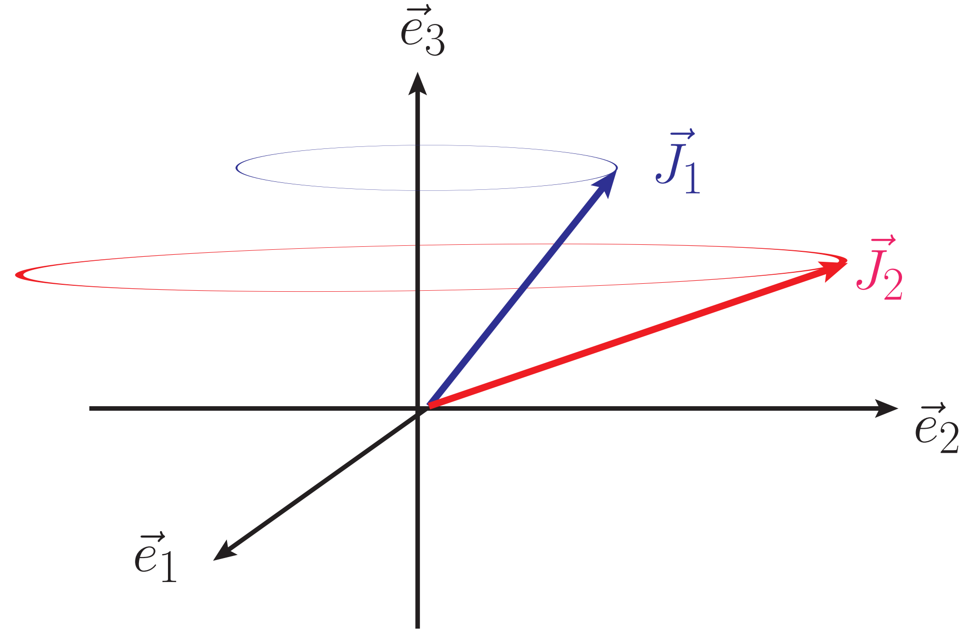


$$\vec{J} = \vec{J}_1 \otimes \hat{I}_2 + \hat{I}_1 \otimes \vec{J}_2 \equiv \vec{J}_1 + \vec{J}_2$$

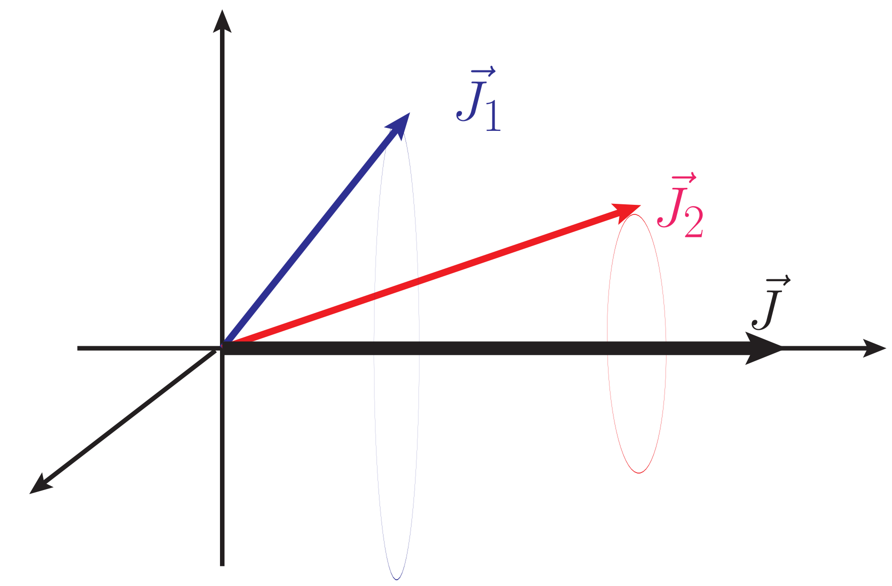
$$[\vec{J}_1, \vec{J}_2] = 0$$

$$\begin{aligned} [J_i, J_j] &= [J_{1i} + J_{2i}, J_{1j} + J_{2j}] = [J_{1i}, J_{1j}] + [J_{2i}, J_{2j}] \\ &= i\hbar\epsilon_{ijk} J_{1k} + i\hbar\epsilon_{ijk} J_{2k} = i\hbar\epsilon_{ijk} \left(J_{1k} + J_{2k}\right) \\ &= i\hbar\epsilon_{ijk} J_k \end{aligned}$$

$$[\vec{J}^2, J_z] = 0$$



(a)



(b)

$$\{\vec{J}_1^2, \hat{J}_{1z}, \vec{J}_2^2, \vec{J}_{2z}\}$$

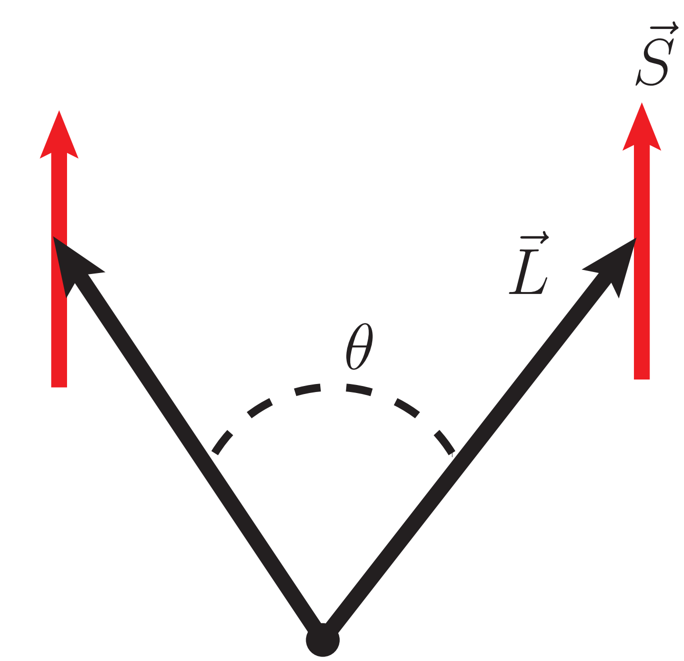
因子化基矢

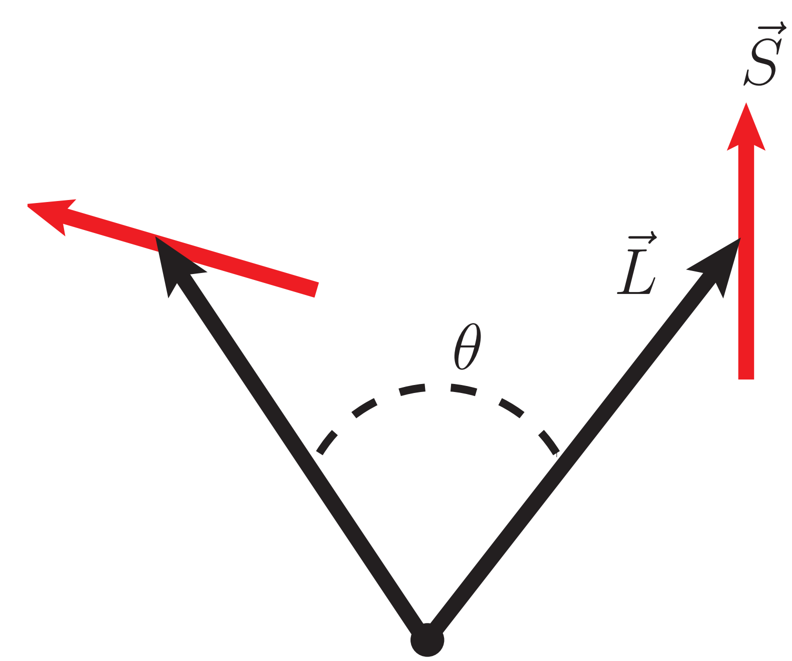
$$\{|j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv |j_1 m_1; j_2 m_2\rangle\}$$

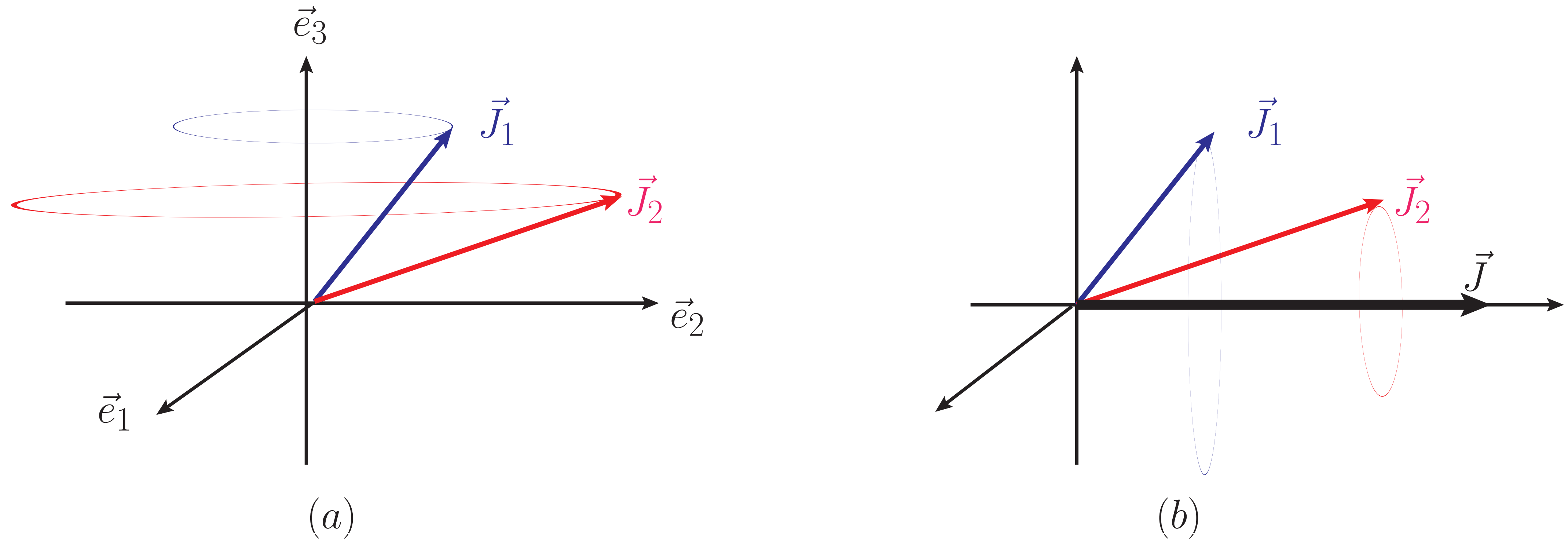
$$\{\vec{J}^2, \vec{J}_1^2, \vec{J}_2^2, J_z\}$$

耦合基矢

$$\{|j_1, j_2, j, m_j\rangle\}$$







总角动量为每个子系统角动量的量子化提供参考方向

$$\vec{J}_1 = \vec{J} - \vec{J}_2$$

$$\vec{J}_1^2 = (\vec{J} - \vec{J}_2)^2 = \vec{J}^2 + \vec{J}_2^2 - 2\vec{J}_2 \cdot \vec{J}$$

$$J_1(J_1+1) = J(J+1) + J_2(J_2+1) - \frac{2\vec{J}_2 \cdot \vec{J}}{\hbar^2}$$

$$\vec{J}_2 \cdot \vec{J} = \frac{J(J+1) + J_2(J_2+1) - J_1(J_1+1)}{2}\hbar^2$$

耦合基矢和因子化基矢完全描述相同的希尔伯特空间

——> 两者等价

维度: $(2j_1+1)(2j_2+1)$

下面我们讨论在因子化基矢张开的子空间中 $\vec{J}^2$ 和 $\hat{J}_z$ 的本征值和本征矢量形式，或者说，讨论两种基矢之间的转化关系。

$$|j_1, j_2, j, m_j\rangle = \sum_{m_1, m_2} C^{j,m}_{j_1,m_1,j_2,m_2} \, |j_1, m_1; j_2, m_2\rangle$$

Clebsch-Gordon系数：两套基矢之间的变换矩阵

$$C^{jm}_{j_1 m_1 j_2 m_2} = \langle j_1 m_1; j_2, m_2 \, | j_1 j_2 j m_j\rangle$$

C-G系数

因为耦合基矢是 J_z 的本征态，而且 $J_z \leq J$ ，所以角动量耦合的总角动量的最大值应该是 J_{1z} 和 J_{2z} 的最大值之和。

总角动量在z方向分量最大的态和因子化基矢之间具有如下关系：

$$\left|J^{\mathrm{Max}}, J_z^{\mathrm{Max}}\right\rangle \equiv |j, j\rangle = |j_1 + j_2, j_1 + j_2\rangle = |j_1, j_1\rangle \otimes |j_2, j_2\rangle$$

$$\begin{aligned} J_{\pm} &\equiv J_x \pm iJ_y = (J_{1x} + J_{2x}) \pm i(J_{1y} + J_{2y}) \\ &= (J_{1x} \pm iJ_{1y}) \otimes I_2 + I_1 \otimes (J_{2x} \pm iJ_{2y}) \\ &= J_{1-} \otimes I_2 + I_1 \otimes J_{2-} \end{aligned}$$

$$J_- |j, j\rangle = (J_{1-} + J_{2-}) |j_1, j_1\rangle \otimes |j_2, j_2\rangle$$

利用 $J_{\pm}|jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle$

和 $J_{-}|j,j\rangle \quad = \quad (J_{1-}+J_{2-})\,|j_1,j_1\rangle\otimes|j_2,j_2\rangle$

有 $J_{-}|j,j\rangle = \sqrt{2j}\,|j,j-1\rangle$

$$J_{1-}|j_1,j_1\rangle\otimes|j_2,j_2\rangle + |j_1,j_1\rangle\otimes J_{2-}|j_2,j_2\rangle$$

$$= \sqrt{2j_1}\,|j_1,j_1-1\rangle\,|j_2,j_2\rangle + \sqrt{2j_2}\,|j_1,j_1\rangle\,|j_2,j_2-1\rangle$$

故

$$|j,j-1\rangle = \sqrt{\frac{j_1}{j}}\,|j_1,j_1-1\rangle\,|j_2,j_2\rangle + \sqrt{\frac{j_2}{j}}\,|j_1,j_1\rangle\,|j_2,j_2-1\rangle$$

CG系数

$$|j,j\rangle \xrightarrow{J_-} |j,j-1\rangle \xrightarrow{J_-} |j,j-2\rangle \cdots |j,-j+1\rangle \xrightarrow{J_-} |j,-j\rangle$$

$$\hat{J}^2\,|j,m\rangle = j(j+1)\hbar^2\,|j,m\rangle \qquad (2j_1+2j_2+1)$$

下一个子空间的最高态是 $|j-1,j-1\rangle$。因为

$$\hat{J}^2|j-1,j-1\rangle = j(j-1)\hbar^2|j-1,j-1\rangle,$$

即，$|j-1,j-1\rangle$ 和 $|j,j-1\rangle$ 属于 $\hat{J}^2$ 算符的不同本征值的本征矢，所以 $|j-1,j-1\rangle$ 和 $|j,j-1\rangle$ 正交。从此可得

$$|j-1,j-1\rangle = \sqrt{\frac{j_2}{j_1}}|j_1,j_1-1\rangle|j_2,j_2\rangle - \sqrt{\frac{j_1}{j}}|j_1,j_1\rangle|j_2,j_2-1\rangle.$$

可验证

$$\langle j,j-1\,|j-1,j-1\rangle = \sqrt{\frac{j_1 j_2}{j^2}} - \sqrt{\frac{j_1 j_2}{j^2}} = 0.$$

在利用角动量降算符 $\hat{J}_-$，

$$|j-1,j-1\rangle \xrightarrow{J_-} |j-1,j-2\rangle \xrightarrow{J_-} |j-1,j-3\rangle \cdots |j-1,-j\rangle \xrightarrow{J_-} \big|j-1,-(j-1)\big\rangle$$

下面我们验证希尔伯特的独立基矢个数。因子化基矢的独立基矢个数是 $(2j_1+1)(2j_2+1)$，所以耦合基矢的对立基矢个数也一定是 $(2j_1+1)(2j_2+1)$。在耦合基矢中，设 $j_1>j_2$，我们发现总角动量取值和其独立基矢个数是

	$\hat{J}$本征值	独立基矢个数
总计 $2j_2+1$ 行	$j_1+j_2,$	$2(j_1+j_2)+1$
	$j_1+j_2-1,$	$2(j_1+j_2)+1-2$
	$j_1+j_2-2,$	$2(j_1+j_2)+1-2\times 2$
	$\vdots$	$\vdots$
	$j_1-j_2,$	$2(j_1+j_2)+1-2\times(2j_2)$

$\hat{J}$本征值	独立基矢个数	颠倒独立基矢个数
$j_1+j_2,$	$2(j_1+j_2)+1$	$2(j_1+j_2)+1-4j_2$
$j_1+j_2-1,$	$2(j_1+j_2)+1-2$	$2(j_1+j_2)+1-4j_2+2$
$j_1+j_2-2,$	$2(j_1+j_2)+1-2\times 2$	$2(j_1+j_2)+1-4j_2+4$
$\vdots$	$\vdots$	$\vdots$
$j_1-j_2,$	$2(j_1+j_2)+1-2\times(2j_2)$	$2(j_1+j_2)+1$

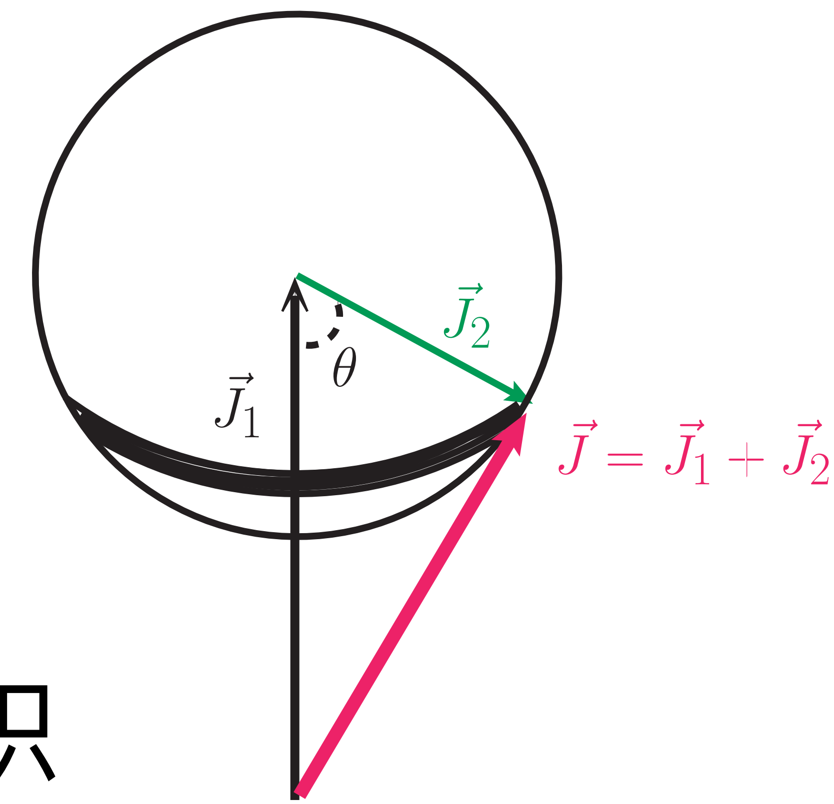
独立基矢个数是

$$\frac{1}{2}(4j_1+2)(2j_2+1)=(2j_1+1)(2j_2+1)$$

$$f=\frac{2j'+1}{(2j_1+1)(2j_2+1)}\xrightarrow[j'\gg 1 \quad j_{1,2}\gg 1]{\text{经典极限}}\frac{j'}{2j_1j_2}$$

角动量耦合的经典极限

$$\vec{J}^2 = \left(\vec{J}_1 + \vec{J}_2\right)^2 = J_1^2 + J_2^2 + 2J_1 J_2 \cos\theta$$



总角动量出现在 $\theta \;-\; \theta + d\theta$

(图中所示黑色环形带)

范围内的几率正比于此环形带的面积

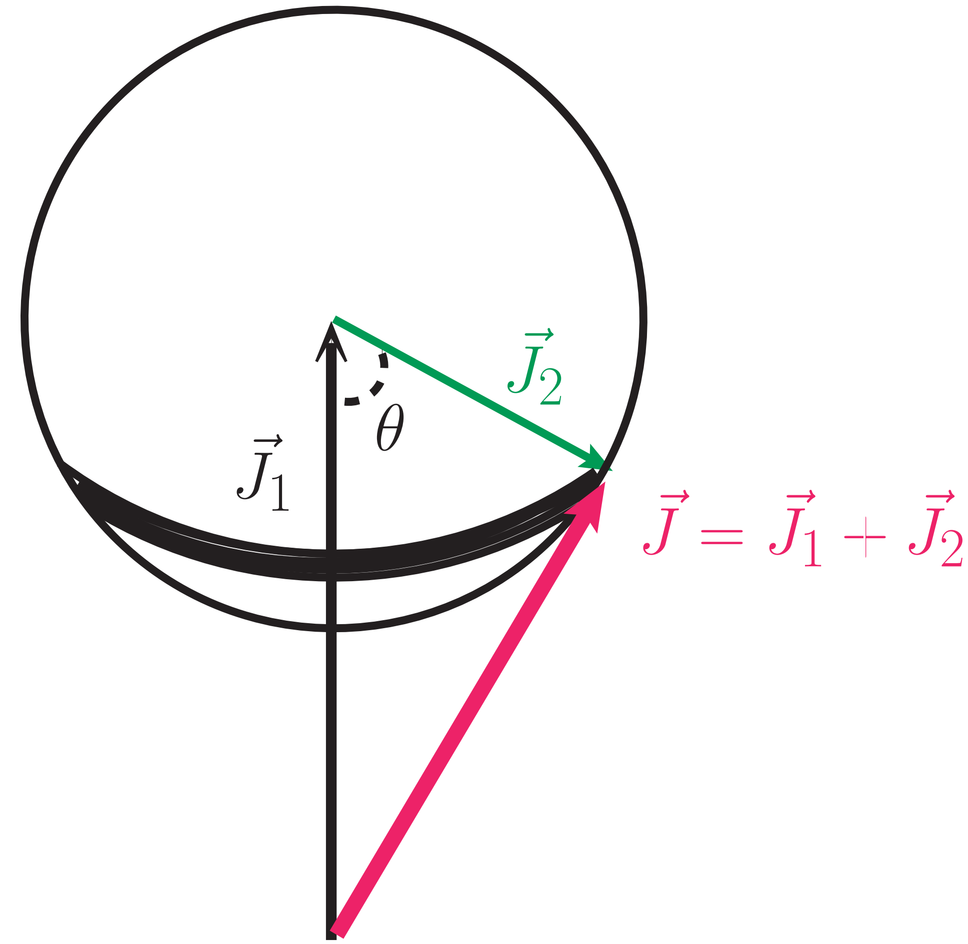
$$dP(\theta) = A2\pi(J_2 \sin\theta)d\theta$$

$$1 = \int dP(\theta) = A2\pi J_2(-\cos\theta)\Big|_{+1}^{-1} = A4\pi J_2$$

归一化的几率密度为 $dP(\theta) = \frac{1}{2}\sin\theta d\theta$ ➜ $\frac{dP}{dJ} = ?$

角动量耦合的经典极限

归一化的几率密度为



$$dP(\theta) = \frac{1}{2}\sin\theta d\theta$$

$$\vec{J}^2 = \left(\vec{J}_1 + \vec{J}_2\right)^2 = J_1^2 + J_2^2 + 2J_1J_2\cos\theta$$

$$dJ^2 = 2JdJ = -2J_1J_2\sin\theta d\theta$$

$$JdJ = -J_1J_2\sin\theta d\theta$$

总角动量在$(J, J+dJ)$范围内的几率是

$$dP = \frac{J}{2J_1J_2}dJ$$

自旋1/2的两个粒子的自旋角动量耦合

考虑两个自旋为 1/2 的粒子组成的系统（例如氢原子中的质子和电子或氦原子中的双电子等）的自旋波函数。设总自旋角动量 $\vec{S}=\vec{S}_1+\vec{S}_2$。体系的希尔伯特空间是

$$\mathcal{H}_{12}=\mathcal{H}^1_{\text{ext}}\otimes\mathcal{H}^1_{\text{spin}}\otimes\mathcal{H}^2_{\text{ext}}\otimes\mathcal{H}^2_{\text{spin}}.$$

定义自旋空间的张量积是

$$\mathcal{H}_{\text{spin}}=\mathcal{H}^1_{\text{spin}}\otimes\mathcal{H}^1_{\text{spin}}.$$

因为每个自旋 1/2 粒子的自旋空间维度是 2，所以两个粒子的自旋空间维度是 4。

因子化基矢

$$|\sigma_1;\sigma_2\rangle\equiv|\sigma_1\rangle\otimes|\sigma_2\rangle,$$

记号

$$|+;+\rangle\equiv\left|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle$$

$$|-;+\rangle\equiv\left|\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle$$

具体形式如下：

$$\Big\{|+;+\rangle,\quad|+;-\rangle,\quad|-;+\rangle,\quad|-;-\rangle\Big\}.$$

在 $\sigma_z^1\otimes\sigma_z^2$ 表象中，

$$|+;+\rangle=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad|+;-\rangle=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad|-;+\rangle=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad|-;-\rangle=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

两粒子的总自旋

选取力学量完备集是 $\{\hat{\vec{S}}^2, \hat{S}_z\}$。令 $|S, M\rangle$ 是 $\hat{\vec{S}}^2, \hat{S}_z$ 的共同本征态，

$$\begin{aligned}\hat{\vec{S}}^2|S,M\rangle &= S(S+1)\hbar^2|S,M\rangle \\ \hat{S}_z|S,M\rangle &= M\hbar|S,M\rangle.\end{aligned}$$

因为

$$\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z},$$

所以 $\hat{S}_z$ 算符的最大（最小）本征值和相应本征态是

$$M_{\max} = +\frac{1}{2} + \frac{1}{2} = +1, \qquad \text{本征态: } |+;+\rangle$$

$$M_{\min} = -\frac{1}{2} - \frac{1}{2} = -1, \qquad \text{本征态: } |-;-\rangle$$

将 $\hat{\vec{S}}^2$ 作用在 $|+;+\rangle$ 上就可得到本征值 S,

$$
\begin{aligned}
\hat{\vec{S}}^2|+;+\rangle &= \left(\hat{S}_1^2+\hat{S}_2^2+2\vec{S}_1\cdot\vec{S}_2\right)|+;+\rangle \\
&= \left[\frac{3}{4}\hbar^2+\frac{3}{4}\hbar^2+2\frac{\hbar}{2}\frac{\hbar}{2}\left(\hat{\sigma}_{1x}\hat{\sigma}_{2x}+\hat{\sigma}_{1y}\hat{\sigma}_{2y}+\hat{\sigma}_{1z}\hat{\sigma}_{2z}\right)\right]|+;+\rangle \\
&= \left[\frac{3}{2}\hbar^2+2\frac{\hbar}{2}\frac{\hbar}{2}\right]|+;+\rangle \\
&= 2\hbar^2|+;+\rangle.
\end{aligned}
$$

推导见下页

同理,

$$\hat{\vec{S}}^2|-;-\rangle=2\hbar^2|-;-\rangle,$$

这说明, $|+;+\rangle$ 和 $|-;-\rangle$ 都是 $\hat{\vec{S}}^2$ 算符的本征值 $S=1$ 的本征态。在耦合基矢中可以表示为

记号 $|j_1,j_2,J,J_z\rangle$

$$
\begin{aligned}
\left|\frac{1}{2},\frac{1}{2},1,1\right\rangle_{\text{耦合基矢}} &= \left|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle_{\text{因子化基矢}}, \\
\left|\frac{1}{2},\frac{1}{2},1,-1\right\rangle_{\text{耦合基矢}} &= \left|\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right\rangle_{\text{因子化基矢}}.
\end{aligned}
$$

记号 $|j_1,j_{1z};j_2,j_{2z}\rangle$

$$\left|\frac{1}{2},\frac{1}{2},1,0\right\rangle=?$$

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma_x |+\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-\rangle$$

$$\sigma_y |+\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i |-\rangle$$

$$\sigma_z |+\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+\rangle$$

$$\sigma_x |-\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+\rangle$$

$$\sigma_y |-\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i |+\rangle$$

$$\sigma_z |-\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = - |-\rangle$$

$$\hat{\Sigma} = \sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z}$$

$$\hat{\Sigma} |+;+\rangle = |-;-\rangle + (i |-\rangle_1)(i |-\rangle_2) + |+;+\rangle = |+;+\rangle$$

$$\hat{\Sigma} |-;-\rangle = |+;+\rangle + (i |+\rangle_1)(i |+\rangle_2) + |-;-\rangle = |-;-\rangle$$

$$\hat{S}_- \left|\tfrac{1}{2},\tfrac{1}{2},1,1\right\rangle = \sqrt{2}\left|\tfrac{1}{2},\tfrac{1}{2},1,0\right\rangle$$

$$\begin{aligned} &\left(\hat{S}_{1-}+\hat{S}_{2-}\right)|+;+\rangle = \left(\hat{S}_{1-}+\hat{S}_{2-}\right)|1+\rangle\otimes|2+\rangle \\ = \quad & \hat{S}_{1-}|1+\rangle\otimes|2+\rangle + |1+\rangle\otimes\hat{S}_{2-}|2+\rangle \\ = \quad & |1-\rangle\otimes|2+\rangle \;+\; |1+\rangle\otimes|2-\rangle \\ = \quad & |-;+\rangle \;+\; |+;-\rangle . \end{aligned}$$

$$J_\pm\,|jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle$$

$$\hat{S}_-\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{2}\times 2}\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$\Rightarrow\quad \left|\tfrac{1}{2},\tfrac{1}{2},1,0\right\rangle = \frac{1}{\sqrt{2}}\Big(|-;+\rangle + |+;-\rangle\Big)$$

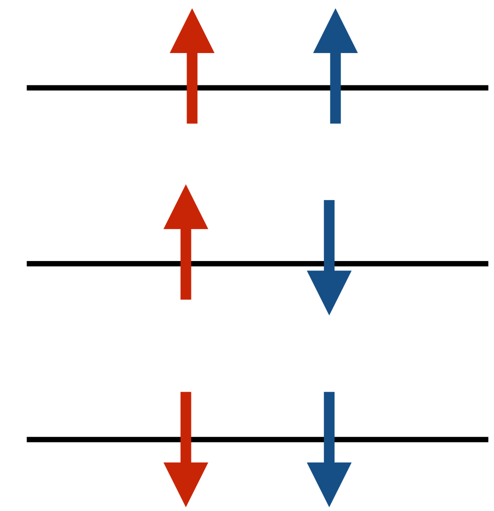
最后一个态矢量可以利用与 $\left|\frac{1}{2},\frac{1}{2},1,0\right\rangle$ 的正交性得到

$$\frac{1}{\sqrt{2}}\left(|+;-\rangle - |-;+\rangle\right)$$

$$S=0, S_z=0$$

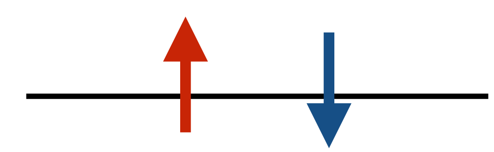
$$\left|\frac{1}{2},\frac{1}{2},0,0\right\rangle$$

$$S=1: \qquad \begin{pmatrix} |+;+\rangle \\ \frac{1}{\sqrt{2}}\left(|+;-\rangle+|-;+\rangle\right) \\ |-;-\rangle \end{pmatrix}$$



三重态，粒子1和2地位是对称的

$$S=0: \qquad \frac{1}{\sqrt{2}}\left(|+;-\rangle-|-;+\rangle\right).$$



单态，粒子1和2的地位是反对称的

$$2\otimes 2=3\oplus 1$$

矩阵表示

在 $\sigma_z^1 \otimes \sigma_z^2$ 表象中，

$$|+;+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |+;-\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |-;+\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |-;-\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\hat{S}_{1x} = \frac{\hbar}{2}\hat{\sigma}_x \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & \hat{I} \\ \hat{I} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\hat{S}_{2x} = \hat{I}_2 \otimes \frac{\hbar}{2}\hat{\sigma}_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{\hbar}{2}\hat{\sigma}_x = \frac{\hbar}{2}\begin{pmatrix} \hat{\sigma}_x & 0 \\ 0 & \hat{\sigma}_x \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$
\begin{aligned}
\hat{S}_{1y} &= \frac{\hbar}{2}\hat{\sigma}_y \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -i\hat{I} \\ i\hat{I} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \\
\hat{S}_{2y} &= \hat{I}_2 \otimes \frac{\hbar}{2}\hat{\sigma}_y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{\hbar}{2}\hat{\sigma}_y = \frac{\hbar}{2}\begin{pmatrix} \hat{\sigma}_y & 0 \\ 0 & \hat{\sigma}_y \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \\
\hat{S}_{1z} &= \frac{\hbar}{2}\hat{\sigma}_z \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} \hat{I} & 0 \\ 0 & -\hat{I} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
\hat{S}_{2z} &= \hat{I}_2 \otimes \frac{\hbar}{2}\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{\hbar}{2}\hat{\sigma}_z = \frac{\hbar}{2}\begin{pmatrix} \hat{\sigma}_z & 0 \\ 0 & \hat{\sigma}_z \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{aligned}
$$

总自旋角动量算符在 $\sigma_z^1 \otimes \sigma_z^2$ 表象中的矩阵表示是

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix}$$

$$\hat{S}_z = \hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \hat{S}^2 = \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

总角动量的升降算符

$$\hat{S}_+ = \hbar \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{S}_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\hat{S}^2|+;+\rangle = \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 2\hbar^2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\hat{S}^2|-;-\rangle = \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 2\hbar^2\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\hat{S}_{-}\left|\frac{1}{2},\frac{1}{2},1,1\right\rangle = \sqrt{2}\left|\frac{1}{2},\frac{1}{2},1,0\right\rangle = \hat{S}_{-}|+;+\rangle = \hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \left|\frac{1}{2},\frac{1}{2},1,0\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

最后一个态矢量 $|?\rangle$ 可以利用和 $\left|\frac{1}{2},\frac{1}{2},1,0\right\rangle$ 的正交性得到

$$|?\rangle = \frac{1}{\sqrt{2}}\Big(|-;+\rangle \;-\; |+;-\rangle\Big) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

$$\hat{S}^2|?\rangle \;=\; \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0,$$

$$\hat{S}_z|?\rangle \;=\; \hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0.$$

$$\left|\frac{1}{2},\frac{1}{2},0,0\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

表象变换

从耦合基矢到因子化基矢之间的变换矩阵是

$$\mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

从因子化基矢到耦合基矢的变换矩阵为

$$\mathcal{S}' = \mathcal{S}^{\dagger} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

总自旋角动量算符在耦合基矢空间的矩阵形式是

$$\begin{aligned}
\hat{\vec{S}}^2_{\text{耦合基矢}} &= \langle SM|\hat{\vec{S}}^2\Big|S'M'\rangle \\
&= \langle SM\,|s_{1z};s_{2z}\rangle\Big\langle s_{1z};s_{2z}\Big|\hat{\vec{S}}^2\Big|s'_{1z};s'_{2z}\Big\rangle\langle s'_{1z};s'_{2z}\,\Big|S'M'\rangle \\
&= \mathcal{S}'\hat{\vec{S}}^2_{\text{因子化基矢}}\mathcal{S}'^{\dagger} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \\
&= \hbar^2 \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right).
\end{aligned}$$

$$2 \otimes 2 = 3 \oplus 1$$

$$\hat{S}_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \hbar \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right).$$

$$2 \otimes 2 = 3 \oplus 1$$

C-G系数定义了一个幺正变换，将角动量的张量积空间化简为不可约化表示直和。

超精细结构——氢原子的21cm线

氢原子的原子核和电子之间的自旋磁矩相互作用

$$\hat{\boldsymbol{\mu}}_{\rm e} = \gamma_{\rm e}\hat{\boldsymbol{S}}_{\rm e}\,, \qquad \gamma_{\rm e} = -q/m_{\rm e}\,;$$

自旋-自旋相互作用

$$\hat{\boldsymbol{\mu}}_{\rm p} = \gamma_{\rm p}\hat{\boldsymbol{S}}_{\rm p}\,, \qquad \gamma_{\rm p} \simeq 2.79\; q/m_{\rm p}$$

将氢原子核视作为点粒子，则位于原点的质子磁偶极矩产生磁场为

$$\boldsymbol{B}(\boldsymbol{r}) = -\frac{\mu_0}{4\pi r^3}\left(\boldsymbol{\mu}_{\rm p} - \frac{3(\boldsymbol{\mu}_{\rm p}\cdot\boldsymbol{r})\,\boldsymbol{r}}{r^2}\right) + \frac{2\mu_0}{3}\,\boldsymbol{\mu}_{\rm p}\,\delta(\boldsymbol{r})$$

电子磁矩与磁场间的相互作用为

$$\hat{W}_{\rm cont} = -\frac{2\mu_0}{3}\hat{\boldsymbol{\mu}}_{\rm e}\cdot\hat{\boldsymbol{\mu}}_{\rm p}\,\delta(\hat{\boldsymbol{r}})$$

点粒子假设
适用于质子

$$\hat{W}_{\rm dip} = \frac{\mu_0}{4\pi\hat{r}^3}\left(\hat{\boldsymbol{\mu}}_{\rm e}\cdot\hat{\boldsymbol{\mu}}_{\rm p} - \frac{3(\hat{\boldsymbol{\mu}}_{\rm e}\cdot\hat{\boldsymbol{r}})(\hat{\boldsymbol{\mu}}_{\rm p}\cdot\hat{\boldsymbol{r}})}{\hat{r}^2}\right) \qquad r \neq 0$$

对于r=0处非奇异函数f(r)，对角度积分后W_{dip}贡献为零。

超精细结构——氢原子的21cm线

考虑处于基态的氢原子，采用微扰论计算

$$\hat{H}_1 = \int \psi_{100}^*(\boldsymbol{r})\ \hat{W}\ \psi_{100}(\boldsymbol{r})\ \mathrm{d}^3 r \qquad \psi_{100} = \frac{1}{\sqrt{\pi a_1^3}} e^{-\frac{r}{a_1}}$$

$$\hat{H}_1 = -\frac{2\mu_0}{3} \hat{\boldsymbol{\mu}}_{\mathrm{e}} \cdot \hat{\boldsymbol{\mu}}_{\mathrm{p}} \, |\psi_{100}(0)|^2$$

H_I算符仅作用在自旋波函数上

$$\hat{H}_1 = \frac{A}{\hbar^2} \hat{\boldsymbol{S}}_{\mathrm{e}} \cdot \hat{\boldsymbol{S}}_{\mathrm{p}}$$

$$A = -\frac{2}{3}\ \frac{\mu_0}{4\pi}\ \frac{4}{a_1^3}\ \gamma_{\mathrm{e}}\gamma_{\mathrm{p}}\hbar^2 = \frac{16}{3} \times 2.79\ \frac{m_{\mathrm{e}}}{m_{\mathrm{p}}} \alpha^2 E_{\mathrm{I}}$$

超精细结构——氢原子的21cm线

$$\hat{H}_1 = \frac{A}{\hbar^2}\hat{\boldsymbol{S}}_{\rm e}\cdot\hat{\boldsymbol{S}}_{\rm p}$$

令 $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_{\rm e} + \hat{\boldsymbol{S}}_{\rm p}$ ，则 $\hat{\boldsymbol{S}}_{\rm e}\cdot\hat{\boldsymbol{S}}_{\rm p} = \frac{1}{2}\left(\hat{S}^2 - \hat{S}_{\rm e}^2 - \hat{S}_{\rm p}^2\right)$

力学量完备集是 $\left\{\vec{S}^2, \vec{S}_e^2, \vec{S}_p^2, \vec{S}_z\right\}$，本征矢为耦合基矢$|S, S_z\rangle$

S=0或1

$\hat{\boldsymbol{S}}_{\rm e}\cdot\hat{\boldsymbol{S}}_{\rm p}$ 的本征值：$\frac{\hbar^2}{2}\left[S(S+1) - \frac{3}{2}\right]$

氢原子基态发生劈裂：

$$\begin{aligned} E_+ &= E_0 + A/4\,, &\quad \text{triplet state} \quad |1, M\rangle\,; \\ E_- &= E_0 - 3A/4\,, &\quad \text{singlet state} \quad |0, 0\rangle\,. \end{aligned}$$

能级差为

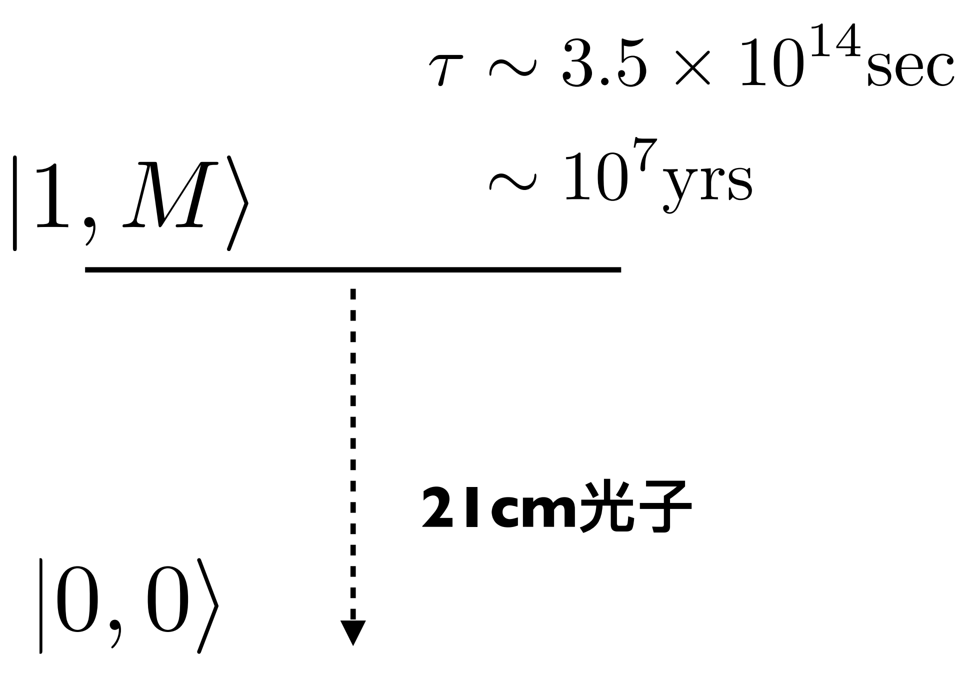
$$E_+ - E_- = A \simeq 5.87\times10^{-6}\text{eV}$$

$$\nu = \frac{A}{h} \simeq 1417\text{MHz} \qquad \lambda = \frac{c}{\nu} \simeq 21\text{cm}$$

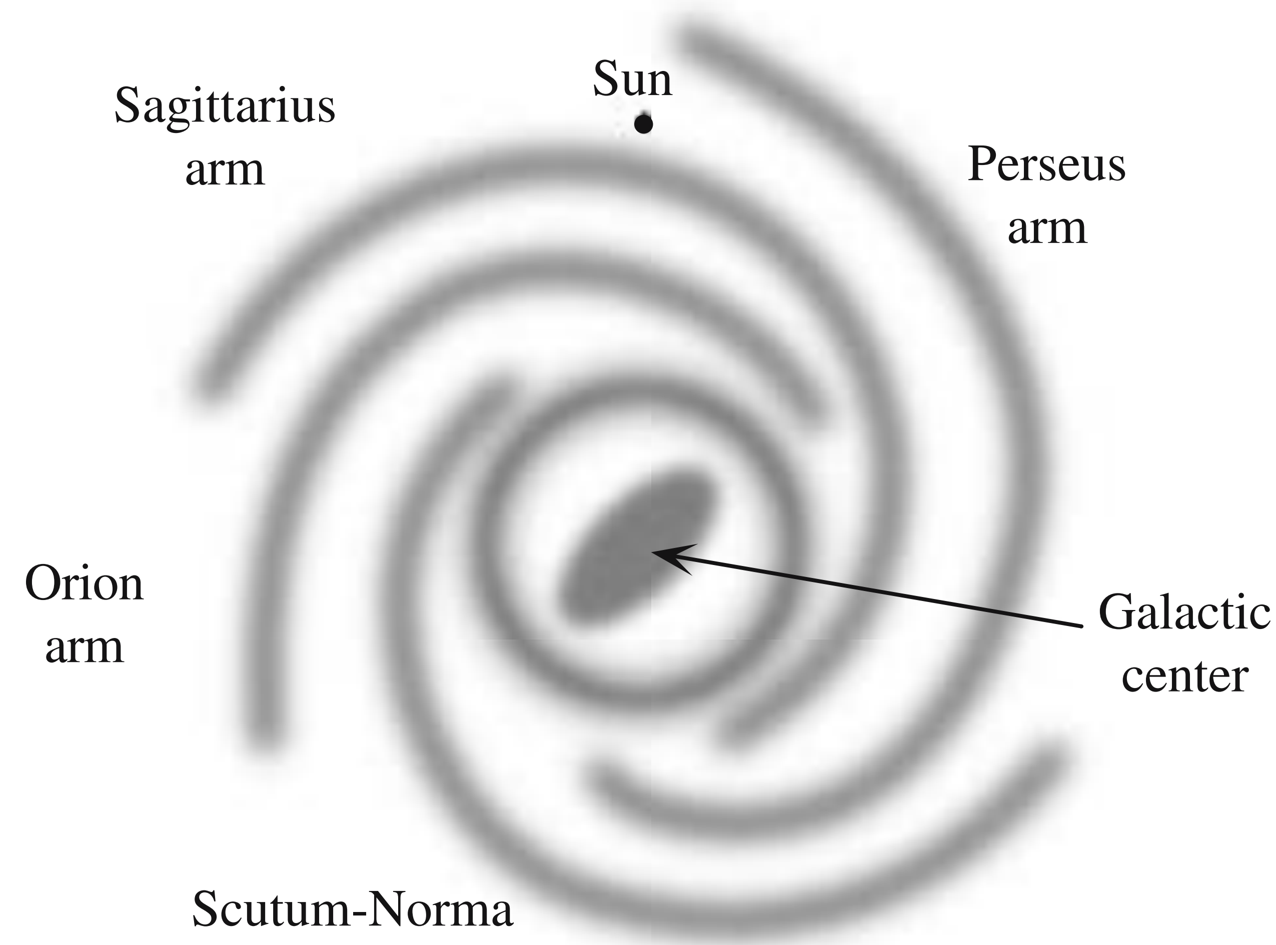
超精细结构——氢原子的21cm线

$$\hat{H}_1 = \frac{A}{\hbar^2}\hat{\boldsymbol{S}}_{\rm e}\cdot\hat{\boldsymbol{S}}_{\rm p} \qquad \begin{aligned} E_+ &= E_0 + A/4\,, \qquad \text{triplet state} \quad |1,M\rangle\,; \\ E_- &= E_0 - 3A/4\,, \qquad \text{singlet state} \quad |0,0\rangle\,. \end{aligned}$$

星际云的主要组分是氢原子，其温度~100K，相应的热动能是KT~10^{-2} eV，远小于氢原子能级差，无法激发氢原子发出Lyman谱，但可以在氢原子的超精细态S=1和S=0态之间跃迁。

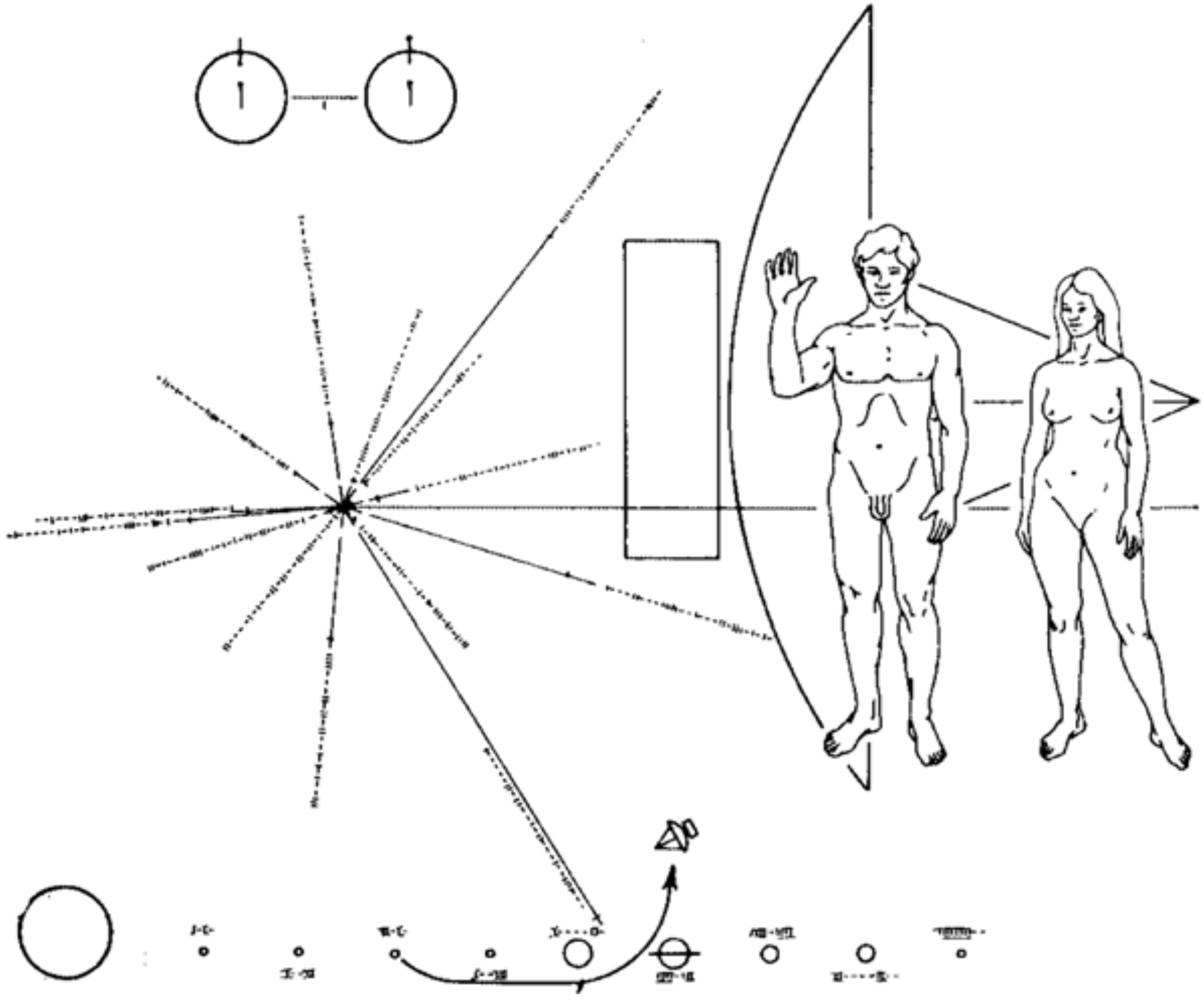


银河系螺旋结构



半径50000光年，
太阳系离银心30000光年

NASA发给外星智慧生物的信件



轨道-自旋角动量耦合

轨道-自旋角动量耦合

$$s = 1/2 \text{ and } l = 1$$

$$|\frac{1}{2},\frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\frac{1}{2},-\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$S_+ \otimes I = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), I \otimes L_+ = \hbar \left(\begin{array}{ccc|ccc} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$S_- \otimes I = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right), I \otimes L_- = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{array}\right),$$

$$S_z \otimes I = \frac{\hbar}{2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right), I \otimes L_z = \hbar \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right).$$

$$J_{\pm,z} = S_{\pm,z} \otimes I + I \otimes L_{\pm,z}$$

$$J_+ = \hbar \left(\begin{array}{ccc|ccc} 0 & \sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$J_- = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{2} & 0 \end{array}\right), \quad J_z = \hbar \left(\begin{array}{ccc|ccc} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \end{array}\right).$$

$$|\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right), \qquad |\frac{3}{2},-\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{array}\right).$$

$$J_+|\frac{3}{2},\frac{3}{2}\rangle \;=\; J_-|\frac{3}{2},-\frac{3}{2}\rangle \;=\; 0$$

$$J_+ = \hbar\left(\begin{array}{ccc|ccc} 0 & \sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad J_- = \hbar\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{2} & 0 \end{array}\right),$$

$$J_{\pm}|j,m\rangle = \sqrt{j(j+1) - m(m \pm 1)}|j, m \pm 1\rangle$$

$$J_{-}|\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ \sqrt{2} \\ 0 \\ \hline 1 \\ 0 \\ 0 \end{array}\right) = \sqrt{3}|\frac{3}{2},\frac{1}{2}\rangle, \qquad J_{+}|\frac{3}{2},-\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ \sqrt{2} \\ 0 \end{array}\right) = \sqrt{3}|\frac{3}{2},-\frac{1}{2}\rangle.$$

$$J_{-} = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{2} & 0 \end{array}\right), \qquad |\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right)$$

$$J_\pm|j,m\rangle = \sqrt{j(j+1)-m(m\pm 1)}|j,m\pm 1\rangle$$

$$J_-|\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ \sqrt{2} \\ 0 \\ \hline 1 \\ 0 \\ 0 \end{array}\right) = \sqrt{3}|\frac{3}{2},\frac{1}{2}\rangle, \qquad J_+|\frac{3}{2},-\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ \sqrt{2} \\ 0 \end{array}\right) = \sqrt{3}|\frac{3}{2},-\frac{1}{2}\rangle$$

$$J_+ = \hbar\left(\begin{array}{ccc|ccc} 0 & \sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \qquad |\frac{3}{2},-\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{array}\right)$$

$$J_-|\frac{3}{2},\frac{3}{2}\rangle = J_-|\frac{1}{2},\frac{1}{2}\rangle \otimes |1,1\rangle$$
$$= |\frac{1}{2},-\frac{1}{2}\rangle \otimes |1,1\rangle + \sqrt{2}|\frac{1}{2},\frac{1}{2}\rangle \otimes |1,0\rangle = \sqrt{3}|\frac{3}{2},\frac{1}{2}\rangle,$$
$$J_+|\frac{3}{2},-\frac{3}{2}\rangle = J_+|\frac{1}{2},-\frac{1}{2}\rangle \otimes |1,-1\rangle$$
$$= |\frac{1}{2},\frac{1}{2}\rangle \otimes |1,-1\rangle + \sqrt{2}|\frac{1}{2},-\frac{1}{2}\rangle \otimes |1,0\rangle = \sqrt{3}|\frac{3}{2},-\frac{1}{2}\rangle$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ \hline 0 \\ \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{pmatrix} + \sqrt{2}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \\ 0 \end{pmatrix} = \left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,1\rangle + \sqrt{2}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,0\rangle$$

$$J_+|\frac{3}{2},-\frac{3}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \hline 0 \\ \sqrt{2} \\ 0 \end{pmatrix} = \sqrt{3}|\frac{3}{2},-\frac{1}{2}\rangle$$

Their orthogonal linear combinations belong to the $j = 1/2$ representation

$$|\frac{1}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \qquad |\frac{1}{2},-\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$J_+|\tfrac{1}{2},\tfrac{1}{2}\rangle = J_-|\tfrac{1}{2},-\tfrac{1}{2}\rangle = 0, \text{ and } J_-|\tfrac{1}{2},\tfrac{1}{2}\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$J_- = \hbar\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{2} & 0 \end{array}\right),$$

Define a unitary rotation

$$U = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 \end{array}\right)$$

$$U|\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right), U|\frac{3}{2},\frac{1}{2}\rangle = \left(\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right), U|\frac{3}{2},-\frac{1}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right), U|\frac{3}{2},-\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \end{array}\right),$$

$$U|\frac{1}{2},\frac{1}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \end{array}\right), U|\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \end{array}\right).$$

Generators in new basis

$$V_{1/2} \otimes V_1 = V_{3/2} \oplus V_{1/2}$$

$$2 \otimes 3 = 4 \oplus 2$$

$$UJ_zU^\dagger = \hbar \left(\begin{array}{cccc|cc} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{array}\right),$$

$$UJ_+U^\dagger = \hbar \left(\begin{array}{cccc|cc} 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$UJ_-U^\dagger = \hbar \left(\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

C-G系数定义了一个幺正变换，
将角动量的张量积空间化简为不可约化表示直和。

纠缠态

可因子化态和纠缠态

由两个粒子组成的复合体系的量子态，如果能够表示为每个粒子的量子态的乘积，则称为分离态（或可因子化态）。反之，称为纠缠态。

因子化基矢：

$$|\uparrow\rangle_1 |\uparrow\rangle_2$$
$$|\uparrow\rangle_1 |\downarrow\rangle_2$$
$$|\downarrow\rangle_1 |\uparrow\rangle_2$$
$$|\downarrow\rangle_1 |\downarrow\rangle_2$$

耦合基矢

分离态 ← $|\uparrow\rangle_1 |\uparrow\rangle_2 \quad |\downarrow\rangle_1 |\downarrow\rangle_2$

纠缠态 ←

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$

$(\hat{s}_{1z}, \hat{s}_{2z})$ 都是单体算符

$\hat{\vec{S}}^2 = (\hat{\vec{s}}_1 + \hat{\vec{s}}_2)^2$ 是两体算符

自旋为 $\hbar/2$ 的二粒子体系的 4 个归一化的纠缠态可以如下构成：

$$\chi_{00}=1/\sqrt{2}\,[|\uparrow\rangle_1|\downarrow\rangle_2-|\downarrow\rangle_1|\uparrow\rangle_2],$$

$$\chi_{10}=1/\sqrt{2}\,[|\uparrow\rangle_1|\downarrow\rangle_2+|\downarrow\rangle_1|\uparrow\rangle_2],$$

$$1/\sqrt{2}\,(\chi_{11}-\chi_{1\text{-}1})=1/\sqrt{2}\,[|\uparrow\rangle_1|\uparrow\rangle_2-|\downarrow\rangle_1|\downarrow\rangle_2],$$

$$1/\sqrt{2}\,(\chi_{11}+\chi_{1\text{-}1})=1/\sqrt{2}\,[|\uparrow\rangle_1|\uparrow\rangle_2+|\downarrow\rangle_1|\downarrow\rangle_2]$$

可以证明，它们是二体算符完全集的共同本征态，称为Bell基

Bell基

Bell 基	$\sigma_{1z}\sigma_{2z}$	$\sigma_{1x}\sigma_{2x}$
$\lvert\psi^-\rangle_{12}=1/\sqrt{2}[\,\lvert\uparrow\rangle_1\lvert\downarrow\rangle_2-\lvert\downarrow\rangle_1\lvert\uparrow\rangle_2]$	-1	-1
$\lvert\psi^+\rangle_{12}=1/\sqrt{2}[\,\lvert\uparrow\rangle_1\lvert\downarrow\rangle_2+\lvert\downarrow\rangle_1\lvert\uparrow\rangle_2]$	-1	$+1$
$\lvert\phi^-\rangle_{12}=1/\sqrt{2}[\,\lvert\uparrow\rangle_1\lvert\uparrow\rangle_2-\lvert\downarrow\rangle_1\lvert\downarrow\rangle_2]$	$+1$	-1
$\lvert\phi^+\rangle_{12}=1/\sqrt{2}[\,\lvert\uparrow\rangle_1\lvert\uparrow\rangle_2+\lvert\downarrow\rangle_1\lvert\downarrow\rangle_2]$	$+1$	$+1$

注意，二粒子体系的自旋二体算符的完全集可以有多种选择。例如这四个Bell基是（$\sigma_{1x}\sigma_{2x}$，$\sigma_{1y}\sigma_{2y},\sigma_{1z}\sigma_{2z}$）中任何两个算符的共同本征态，或等价地是（$S_x^2,S_y^2,S_z^2$）中任何两个算符的共同本征态。