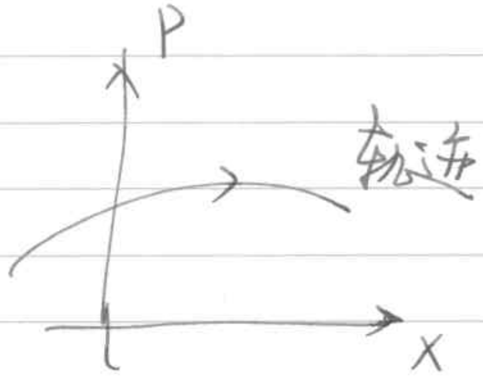
相干态

1) 动机:

经典物理中



QM中

$\Delta x \cdot \Delta p \geq \frac{\hbar}{2} \Longrightarrow$ 放弃"轨道" —几率波

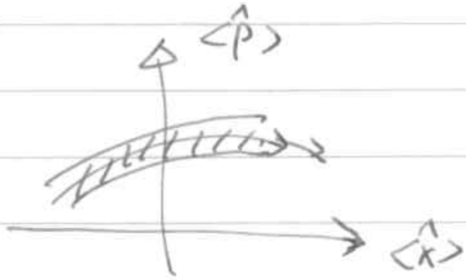
问题: 是否有量子态满足 $\Delta x \cdot \Delta p = \frac{\hbar}{2}$ (最小不确定关系)

同时在此不确定范围内, 其平均值 $\langle \hat{x} \rangle$ 和 $\langle \hat{p} \rangle$

仍可用经典轨道来描述?

(1) $\Delta x \cdot \Delta p = \frac{\hbar}{2}$

(2) $\langle \hat{x} \rangle$ versus $\langle \hat{p} \rangle$



$\Rightarrow$ 薛定谔的相干态

2) 最小不确定态 $\Delta x\cdot\Delta p=\frac{\hbar}{2}$

我们推导过

$$\psi(x)=C\,e^{\frac{i\langle\hat{p}\rangle x}{\hbar}}\,e^{-\frac{(x-\langle\hat{x}\rangle)^2}{4(\Delta x)^2}}$$

$$\Delta x=\Delta p=\sqrt{\frac{\hbar}{2}}\ \Rightarrow\ \Delta x^2=\frac{\hbar}{2}$$

$$\psi(x)=C\,e^{\frac{i\langle\hat{p}\rangle x}{\hbar}}\,e^{-\frac{(x-\langle\hat{x}\rangle)^2}{2\hbar}m\omega}$$

（高斯波包）

① 自由粒子的高斯波包

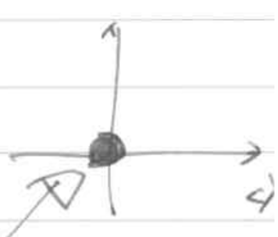
$t=0\longrightarrow t$ 时刻 扩散　Δp 不变，Δx 变大

$\Rightarrow$ 不再满足 $\Delta x\Delta p=\frac{\hbar}{2}$

② 简谐振子基态

$$\langle\hat{p}\rangle=\langle\hat{x}\rangle=0$$

$$\psi_0(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}$$



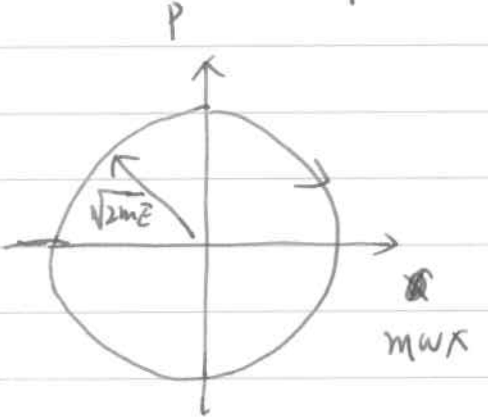
定态 始终满足 $\Delta x\cdot\Delta p=\frac{\hbar}{2}$

3) 运动的最小不确定波包

简谐振子无疑是个非常好的研究对象.

经典振子. $E = \frac{p^2}{2m} + \frac{1}{2}kx^2$

$$p^2 + (m\omega x)^2 = 2mE, \quad \omega = \sqrt{\frac{k}{m}}$$



$$z = \frac{m\omega x + ip}{\sqrt{2mE}}$$

$$z(t) = z_0 e^{-i\omega t}$$

*) 如何实现运动的波包? $\boxed{\text{经典单摆}}$

小角度单摆,

初始条件: 偏离平衡位置

*) 量子力学中, 对简谐振子基态做一个平移

t=0时刻 $\langle \hat{x} \rangle = x_0 \neq 0$

$$\langle \hat{p} \rangle = p_0 \neq 0$$

$$\psi(x, t=0) = c\, e^{\frac{ip_0 x}{\hbar}} e^{-\frac{m\omega (x-x_0)^2}{2\hbar}}$$

不是$\hat{H}$本征态
随时间变化

4) 偏移波包仍是 $\hat{a}$ 的本征态 —— 满足最小不确定关系

$$\psi(x,t=0) = C e^{\frac{i p_0 x}{\hbar}} e^{-\frac{m\omega (x-x_0)^2}{2\hbar}}$$

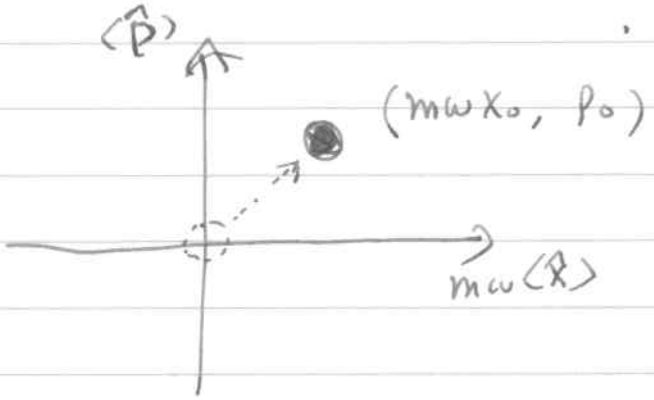
满足: $(\hat{p} - p_0)\psi(x,0) = i m\omega (x - x_0)\psi(x,0)$

即 $(m\omega\hat{x} + i\hat{p})\psi(x,0) = (m\omega x_0 + i p_0)\psi(x,0)$

因为 $\hat{a} = \dfrac{m\omega\hat{x} + i\hat{p}}{\sqrt{2\pi\hbar\omega}}$,

$\Rightarrow \hat{a}\psi(x,0) = z_0\psi(x,0) = \dfrac{m\omega x_0 + i p_0}{\sqrt{2\hbar m\omega}}$

$\Longrightarrow$ 满足 $\Delta x \cdot \Delta p = \dfrac{\hbar}{2}$



5) 偏移波包在简谐振子势中随时间的变化行为

$\Rightarrow$ 是否仍处于最小不确定态 $\left(\Delta x \cdot \Delta p = \frac{\hbar}{2}\right)$

或是否仍是 $\hat{a}$ 的本征函数？

设 $\hat{H}\phi_n = E_n\phi_n = \hbar\omega(n+\frac{1}{2})\phi_n$.

$$\psi(x, t=0) \equiv \psi(x_0, p_0) = \sum_n C_n \phi_n(x)$$

$$\psi(x, t) = e^{-\frac{i\hat{H}t}{\hbar}}\psi(x_0, p_0)$$

*) 下面我们检验 $\hat{a}\psi(x,t)$

$$\hat{a}\psi(x,t) = \hat{a}\, e^{-\frac{i\hat{H}t}{\hbar}}\psi(x_0, p_0)$$

$$= e^{-\frac{i\hat{H}t}{\hbar}} \underbrace{e^{\frac{i\hat{H}t}{\hbar}}\hat{a}\, e^{-\frac{i\hat{H}t}{\hbar}}\psi(x_0, p_0)}$$

其中

$$e^{\frac{i\hat{H}t}{\hbar}}\hat{a}\, e^{-\frac{i\hat{H}t}{\hbar}}\psi(x_0, p_0)$$

$$= e^{\frac{i\hat{H}t}{\hbar}}\hat{a}\, e^{-\frac{i\hat{H}t}{\hbar}}\sum_n C_n\phi_n$$

$$= \sum_n C_n\left(e^{\frac{i\hat{H}t}{\hbar}}\hat{a}\, e^{-\frac{i\hat{H}t}{\hbar}}\right)\phi_n$$

$$= \sum_n C_n e^{-i(n+\frac{1}{2})\omega t}\left(e^{\frac{i\hat{H}t}{\hbar}}\hat{a}\,\phi_n\right)$$

$$= \sum_n C_n e^{-i(n+\frac{1}{2})\omega t} e^{i(n-\frac{1}{2})\omega t}\hat{a}\phi_n$$

$$= \sum_n C_n e^{-i\omega t} \hat{a}\phi_n = e^{-i\omega t} \hat{a} \underbrace{\sum_n C_n \phi_n}_{\psi(x_0, p_0)}$$

$$= e^{-i\omega t} \hat{a} \psi(x_0, p_0)$$

$$= e^{-i\omega t} z_0 \psi(x_0, p_0)$$

故，$\hat{a}\psi(x,t) = e^{-\frac{i\hat{H}t}{\hbar}} \left(e^{\frac{i\hat{H}t}{\hbar}} \hat{a} e^{-\frac{i\hat{H}t}{\hbar}} \right) \psi(x_0, p_0)$

$$= e^{-\frac{i\hat{H}t}{\hbar}} \left[e^{-i\omega t} z_0 \psi(x_0, p_0) \right]$$

$$= e^{-i\omega t} z_0 \, e^{-\frac{i\hat{H}t}{\hbar}} \psi(x_0, p_0)$$

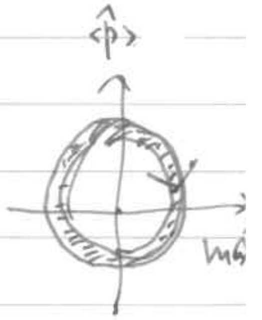
$$= z_0 e^{-i\omega t} \psi(x, t)$$

$\Rightarrow$ $\psi(x,t)$仍然是 $\hat{a}$ 的本征函数，

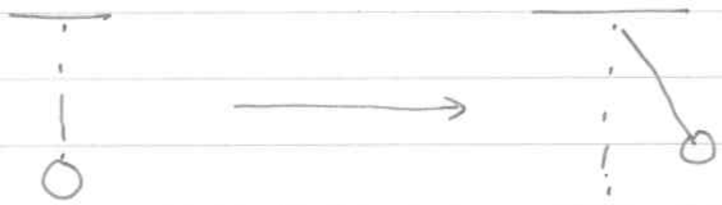
本征值为 $z(t) = z_0 e^{-i\omega t}$

即 $$\frac{m\omega \langle \hat{x} \rangle_t + i \langle \hat{p} \rangle_t}{\sqrt{2\hbar m\omega}} = \frac{m\omega x_0 + i p_0}{\sqrt{2\hbar m\omega}} e^{-i\omega t}$$

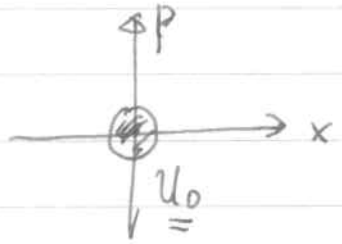
$$\Rightarrow \begin{cases} \langle \hat{x} \rangle_t = x_0 \cos\omega t + \dfrac{p_0}{m\omega} \sin\omega t \\ \langle \hat{p} \rangle_t = p_0 \cos\omega t - m\omega x_0 \sin\omega t \end{cases}$$



相干态小结:　(上节课内容简介)



$$\psi(x_0=0,\ p_0=0) \longrightarrow \psi(x_0\neq 0,\ p_0\neq 0)$$



$$\psi(x) = c\, e^{\frac{i p_0 x}{\hbar}} e^{-\frac{m\omega(x-x_0)^2}{2\hbar}}$$

$$\hat{a}\,\psi(x_0,p_0) = z_0\,\psi(x_0,p_0) = \frac{m\omega x_0 + i p_0}{\sqrt{2\hbar m\omega}}\,\psi(x_0,p_0)$$

$$\psi(x_0,p_0) = \sum_n C_n u_n(x)$$

$$\psi(x_0,p_0,t) = e^{-i\frac{\hat{H}t}{\hbar}}\,\psi(x_0,p_0)$$

$$\hat{a}\,\psi(x_0,p_0,t) = \hat{a}\,e^{-i\frac{\hat{H}t}{\hbar}}\,\psi(x_0,p_0)$$

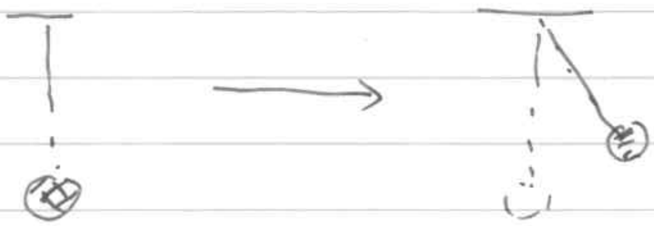
$$= e^{-\frac{i\hat{H}t}{\hbar}}\,e^{\frac{i\hat{H}t}{\hbar}}\,\hat{a}\,e^{-i\frac{\hat{H}t}{\hbar}}\,\psi(x_0,p_0)$$

$$= e^{-\frac{i\hat{H}t}{\hbar}}\,e^{-i\omega t}\,\hat{a}\,\psi(x_0,p_0)$$

$$= z_0 e^{-i\omega t}\,e^{-i\frac{\hat{H}t}{\hbar}}\,\psi(x_0,p_0) \qquad \boxed{z(t) = z_0 e^{-i\omega t}}$$

$$= z_0 e^{-i\omega t}\,\psi(x_0,p_0,t)$$

位移算符 (translation Operator)



$$\psi(x_0=0,\ p_0=0) \longrightarrow \psi(x_0\neq 0,\ p_0\neq 0)$$

$$\underline{\psi(x) = C e^{\frac{i p_0 x}{\hbar}} e^{-\frac{m\omega(x-x_0)^2}{2\hbar}}}$$

简记为 $\psi(0.0) \xrightarrow{\hat{D}} \psi(x_0, p_0)$

$$\psi(x_0, p_0) = \underline{\underline{\hat{D}}}\ \psi(0.0)$$

→ 位移算符

下面我们推导 $\hat{D}$ 算符的表达式.

1) 仅有 $\underline{\underline{x_0 \neq 0}}$, $p_0 = 0$

$$\psi(x_0, 0) = C^{-\frac{m\omega(x-x_0)^2}{2\hbar}}$$

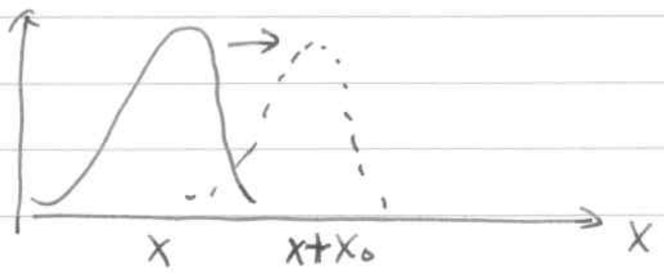
考虑 $\hat{O} = e^{-a\frac{d}{dx}}$ 算符.

$$\hat{O}\psi(x) = e^{-a\frac{d}{dx}}\psi(x)$$

$$= \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \frac{d^n}{dx^n}\psi(x)$$

$$= \psi(x-a)$$

故 $\psi(x_0, 0) = e^{-x_0\frac{d}{dx}}\psi(0,0)$



$P_x = -i\hbar\frac{\partial}{\partial x}$, $\Rightarrow$ $\underline{\underline{\hat{O} = e^{-ia\hat{P}_x/\hbar}}}$ 位移算符

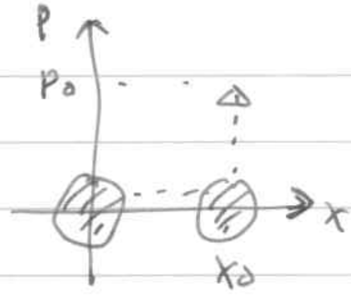
又因为 $\hat{P}_x = -i\hbar\frac{d}{dx} = i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^+ - \hat{a})$

故 $\psi(x_0, 0) = e^{-\frac{ix_0\hat{P}_x}{\hbar}}\psi(0,0)$

$$= e^{x_0\sqrt{\frac{m\omega}{2\hbar}}(\hat{a}^+ - \hat{a})}\psi(0,0)$$

$$= e^{z_0(\hat{a}^+ - \hat{a})}\psi(0,0)$$

$$z_0 = x_0\sqrt{\frac{m\omega}{2\hbar}}$$



2) $\psi(x_0,0) \longrightarrow \psi(x_0,p_0)$

(在动量方向平移)

$$Ce^{-\frac{m\omega(x-x_0)^2}{2\hbar}} \longrightarrow \underline{\underline{Ce^{\frac{ip_0\hat{x}}{\hbar}}}}\, e^{-\frac{m\omega(x-x_0)^2}{2\hbar}}$$

类似于 $e^{-ia\hat{p}_x/\hbar}$

方向

$e^{\frac{ip_0\hat{x}}{\hbar}}$ 是沿着动量平移算符

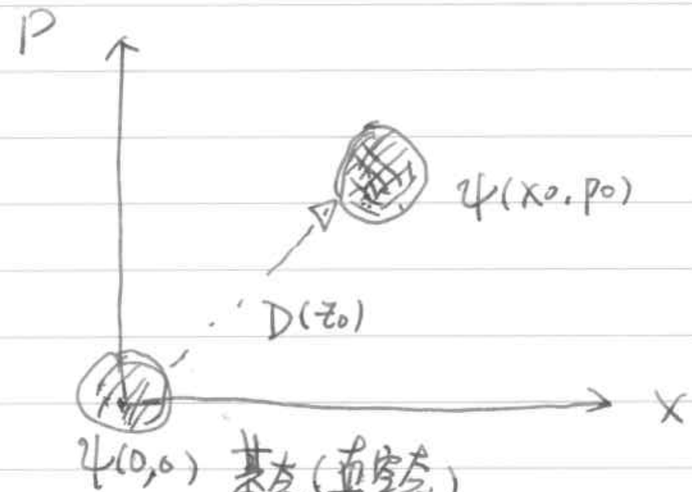
$(\hat{x} = i\hbar\frac{\partial}{\partial p})$

将 $\hat{p}$ 和 $\hat{x}$ 算符用 $\hat{a}$ 和 $\hat{a}^+$ 表示，

$$\Rightarrow \psi(x_0,p_0) = \hat{D}\,\psi(0,0)$$

$$= e^{(z_0\hat{a}^+ - z_0^*\hat{a})}\,\psi(0,0)$$

$$z_0 = \frac{m\omega x_0 + ip_0}{\sqrt{2\hbar m\omega}}$$



$$z_0 \hat{D}(z_0)\phi = [\hat{a}, \hat{D}(z_0)]\phi$$

$$= \hat{a}\hat{D}(z_0)\phi - \hat{D}(z_0)\hat{a}\phi$$

$$\Rightarrow \hat{D}(z_0)\left(z_0\phi + \hat{a}\phi\right) = \hat{a}\hat{D}(z_0)\phi$$

取 $\phi = u_0(x)$ 简谐振子基态，

$$\hat{a}\,u_0(x) = 0$$

则有 $\hat{D}(z_0) z_0 u_0 = \hat{a}\hat{D}(z_0) u_0$

$$\Rightarrow \hat{a}\left(\underline{\underline{\hat{D}(z_0)u_0}}\right) = z_0\left(\hat{D}(z_0)u_0\right)$$

$\hookrightarrow$ 相干态

*) 下面我们验证 $\hat{D}\psi(0,0)$ 是相干态

$$\psi(x_0,p_0)=\hat{D}(z_0)\psi(0,0)=e^{(z_0\hat{a}^+-z_0^*\hat{a})}\psi(0,0)$$

证明: $[\hat{a},\hat{a}^+]=1$

$$\Rightarrow [\hat{a},z_0\hat{a}^+-z_0^*\hat{a}]=z_0[\hat{a},\hat{a}^+]=z_0$$

$$[\hat{a},(z_0\hat{a}^+-z_0^*\hat{a})^n]=[\hat{a},(z_0\hat{a}^+-z_0^*\hat{a})]\frac{\partial(z_0\hat{a}^+-z_0^*\hat{a})^n}{\partial(z_0\hat{a}^+-z_0^*\hat{a})}$$

$$=z_0\cdot n(z_0\hat{a}^+-z_0^*\hat{a})^{n-1}$$

$$\Rightarrow [\hat{a},\hat{D}(z_0)]=[\hat{a},e^{(z_0\hat{a}^+-z_0^*\hat{a})}]$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}[\hat{a},(z_0\hat{a}^+-z_0^*\hat{a})^n]$$

$$=\sum_{n=0}^{\infty}z_0\frac{(z_0\hat{a}^+-z_0^*\hat{a})^{n-1}}{(n-1)!}$$

$$=z_0e^{(z_0\hat{a}^+-z_0^*\hat{a})}$$

$$=z_0\hat{D}(z_0)$$

将上式作用在任意态上可得.

相干态和简谐振子能量本征态间关系

① $$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}$$

$$\hat{A} = z_0\hat{a}^+,\quad \hat{B} = z_0^*\hat{a},\quad [\hat{A},\hat{B}] = [z\hat{a}^+, z^*\hat{a}]$$
$$= -|z|^2$$

$$\Rightarrow e^{(z_0\hat{a}^+ - z_0^*\hat{a})} = e^{z_0\hat{a}^+} e^{-z_0^*\hat{a}} e^{-\frac{1}{2}|z_0|^2}$$

$$\psi(x_0,p_0) = e^{(z_0\hat{a}^+ - z_0^*\hat{a})} u_0$$
$$= e^{-\frac{1}{2}|z_0|^2} e^{z_0\hat{a}^+} e^{-z_0^*\hat{a}} u_0$$
$$= e^{-\frac{1}{2}|z_0|^2} e^{z_0\hat{a}^+} \left(1 + (\hat{a})^1 + \hat{a}^2 + \dots\right) u_0$$
$$= e^{-\frac{1}{2}|z_0|^2} e^{z_0\hat{a}^+} u_0$$
$$= e^{-\frac{1}{2}|z_0|^2} \left(1 + z_0\hat{a}^+ + \frac{(z_0\hat{a}^+)^2}{2!} + \cdots\right) u_0$$

$\because$ $$u_n = \frac{(\hat{a}^+)^n}{\sqrt{n!}} u_0$$

$\therefore$ $$\psi(x_0,p_0) = e^{-\frac{1}{2}|z_0|^2}\left(u_0 + z_0 u_1 + \frac{z_0^2}{\sqrt{2!}} u_2 + \frac{z_0^3}{\sqrt{3!}} u_3 + \cdots\right)$$
$$= e^{-\frac{1}{2}|z_0|^2} \sum_{n=0}^{\infty} \frac{z_0^n}{\sqrt{n!}} u_n$$

$\because \quad u_n(t) = e^{-in\omega t} u_n$

$$\therefore \quad \psi(x_0,p_0,t) = e^{-\frac{|z_0|^2}{2}}\left(u_0 + z_0 e^{-i\omega t} u_1 + \frac{z_0^2 e^{-2i\omega t}}{\sqrt{2!}} u_2 + \frac{z_0^3 e^{-3i\omega t}}{\sqrt{3!}} u_3 + \cdots\right)$$

$$= e^{-\frac{|z_0|^2}{2}} e^{z_0 e^{-i\omega t}\hat{a}^+} u_0$$

和 $\psi(x_0,p_0) = e^{-\frac{|z_0|^2}{2}} e^{z_0 \hat{a}^+} u_0$ 相比

$$\Rightarrow \boxed{z_0(t) = z_0 e^{-i\omega t}}$$

*) 相干态彼此是不正交的，但这组相干态是完备的.

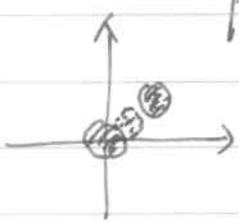
QM中：物理可观测量 $\longleftrightarrow$ 厄米算符 $\longrightarrow$ 本征值为实数

$\longrightarrow$ 本征函数正交

本征函数组是完备集合，

$\hat{a}$ 非厄米，$\Rightarrow$ 本征值为复数，是 $(\langle x\rangle, \langle p\rangle)$ 平面中所有复数，

属于不同本征值的本征函数是不正交的.



例如：$\psi(x_0,p_0)$ 和 $\psi(0,0) = u_0$

证明：相干态是非正交的.

$$\psi(\alpha) = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} u_n$$

$$\psi(\beta) = e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} u_n$$

$$(\beta,\alpha) = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\beta^*)^m}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} \underbrace{(u_m, u_n)}_{\delta_{mn}}$$

$$= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} \sum_{n=0}^{\infty} \frac{(\alpha\beta^*)^n}{n!}$$

$$= e^{\left[-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha\beta^*\right]}$$

$$\neq 0$$

$\Rightarrow$ $\psi(\alpha)$ 和 $\psi(\beta)$ 不正交

$$|(\beta,\alpha)|^2 = e^{-|\alpha-\beta|^2}$$

$\Rightarrow$ 相干态随 $|\alpha-\beta|$ 的增大而趋于正交。

*) 相干态是完备的

$$\frac{1}{\pi}\int \psi^*(\alpha)\,\psi(\alpha)\,d\alpha = 1$$

α 是复数，积分遍及整个复平面

压缩态 (Squeezed states)

相干态是一个最小不确定态

在满足 $\Delta x \cdot \Delta p = \frac{\hbar}{2}$ 前提下，如果 $\Delta x < \sqrt{\frac{\hbar}{2}}$ 或 $\Delta p < \sqrt{\frac{\hbar}{2}}$

$\Rightarrow$ 压缩态 $\Delta x < \sqrt{\frac{\hbar}{2}} < \Delta p$

或 $\Delta p < \sqrt{\frac{\hbar}{2}} < \Delta x$